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LATTICE ENERGY SUM RULE AND THE TRACE ANOMALY

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Abstract

We show that the additional contribution to the Michael lattice energy sum rule pointed out recently, can be identified with the contribution to the field energy of a quark-antiquark pair arising from the trace anomaly of the energy momentum tensor.

Recently, there has been a renewed interest in the action and energy lattice Michael sum rules [1], which relate the static $q\bar{q}$ potential in an SU(N) gauge theory to the action and energy stored in the colour electric and magnetic fields, after it had been pointed out in [2], that an important contribution to the action sum rule had not been taken into account in [1]. Subsequently it was found [3] that also the energy sum rule obtained in [1] is modified in an essential way by an additional contribution to the colour electric and magnetic field energy which, in the case of a confining $q\bar{q}$-potential, accounts for half of the energy in the flux tube matching the interquark potential. That the derivation of the sum rule in [1] was incomplete has also been noticed in [4], where the authors however speculate that the additional contribution is merely an artefact of an approximation. In [3] we had not given a physical interpretation of the additional contribution appearing in the modified energy sum rule. This note is intended to fill the gap. In particular we will show that the additional term is nothing but the contribution to the field energy coming from the trace anomaly [5] of the energy momentum tensor. That the trace anomaly plays an important role in constructing phenomenological hadron models, and accounts, for example, for 1/4 of the nucleon mass, has been recently stressed in [6]. This paper has been the trigger for the present note. In lattice perturbation theory the trace anomaly has been computed in one loop order by Caracciolo, Menotti and Pelisetto [7]. Starting from a traceless, non-conserved tree-level energy momentum tensor on the lattice, and correcting it by imposing the Ward identities for its conservation, the authors show that the corrected energy momentum tensor reproduces the correct anomaly in the continuum limit. This anomaly must contribute to the field energy [6] matching the $q\bar{q}$-potential, and should manifest itself in the lattice energy sum rule. This we now show.

Starting from the expression for the expectation value of the Wilson loop on an anisotropic lattice with temporal lattice spacing a_τ and spatial lattice spacing a, and requiring that for $a \to 0$, $a_\tau \to 0$, with $\xi = a/a_\tau$ fixed, the potential be independent of the anisotropy parameter ξ, we arrived in [3] at the following relation connecting the interquark potential to correlators of plaquette variables with the Wilson loop,

$$\hat{V}(\hat{R},\hat{\beta}) = \eta_- < -\mathcal{P}'_\tau + \mathcal{P}'_s >_{q\bar{q}-0} - \eta_+ < \mathcal{P}'_\tau + \mathcal{P}'_s >_{q\bar{q}-0} \ , \tag{1}$$

where a subtraction at some reference quark-antiquark separation is implied, in order to elliminate the self energy contributions. Here $\mathcal{P}'_\tau$ and $\mathcal{P}'_s$ stand for the time-like and space-like plaquette contributions to the action with base on a fixed time slice (taken to be the $x_4 = 0$ plane), and $< \mathcal{O} >_{q\bar{q}-0}$ denotes generically the following correlator with the Wilson loop, $W(\hat{R},\hat{T})$:

$$< O >_{q\bar{q}-0} = \frac{< W(\hat{R},\hat{T}) O >}{< W(\hat{R},\hat{T}) >} - < O > . \tag{2}$$

In the temporal direction the loop is taken to extend from $-\frac{\hat{T}}{2}$ to $\frac{\hat{T}}{2}$, with $\hat{T}$ very large. $\hat{R}$ is the separation of the quark-antiquark pair. Quantities measured in lattice units are denoted with a "hat". The coefficients $\eta_\pm$ are related as follows [3] to the ξ- derivatives of the couplings $\beta_\tau(a,\xi)$ and $\beta_s(a,\xi)$ associated with the time-like and space-like plaquette contributions, $\mathcal{P}_\tau$ and $\mathcal{P}_s$, to the action [8] on an anisotropic lattice ($S = \beta_\tau \mathcal{P}_\tau + \beta_s \mathcal{P}_s$),

$$\eta_\pm = \frac{1}{2}\left[\frac{\partial \beta_\tau}{\partial \xi} \pm \frac{\partial \beta_s}{\partial \xi}\right]_{\xi=1} , \tag{3}$$

where $\beta_\tau(a,1) = \beta_s(a,1) = \hat{\beta}(a)$, with $\hat{\beta} = 2N/g_0^2$ the usual SU(N) gauge coupling on an isotropic lattice. We have denoted it with a "hat", in order to avoid any confusion with the β-function. The correlators in (1) and (2) are calculated with the Wilson action on an isotropic lattice. The physical $q\bar{q}$-potential is given by

$$V(R) = \lim_{a \to 0} \frac{1}{a} \hat{V}(\frac{R}{a}, \hat{\beta}(a)) \ , \tag{4}$$

with the dependence of $\hat{\beta}(a)$ on the lattice spacing beeing determined by the usual renormalization group relation. In one loop order it is given by $a = \Lambda_L^{-1} \exp(-\hat{\beta}/4Nb_0)$, with $b_0 = 11N/48\pi^2$.

Consider the second term appearing on the rhs of (1), which is the essential modification of the sum rule in [1], referred to above. The coefficient η_+ is given by $\eta_+ = N[(\partial g_\tau^{-2}/\partial\xi) + (\partial g_s^{-2}/\partial\xi)]_{\xi=1}$, where $g_\sigma^{-2}(a,\xi)$, $(\sigma = s,\tau)$, are related [8] to $\beta_\sigma(a,\xi)$ by $\beta_\tau = 2Ng_\tau^{-2}\xi$ and $\beta_s = 2Ng_s^{-2}\xi^{-1}$. It has been calculated non-perturbatively by Karsch [8],

$$\eta_+ = -\frac{\beta_L(g_0)}{2g_0}\hat{\beta} \tag{5}$$

where $\beta_L(g_0) = -a\frac{\partial g_0}{\partial a}$ is the lattice β-function. Hence the contribution of the second term appearing on the rhs of (1) is given by

$$-\eta_+ < \mathcal{P}'_\tau + \mathcal{P}'_s >_{q\bar{q}-0} = \frac{1}{4}\left(\frac{2\beta_L}{g_0} < \hat{L} >_{q\bar{q}-0}\right) \tag{6a}$$

where

$$\hat{L} = \hat{\beta}(\mathcal{P}'_\tau + \mathcal{P}'_s) \tag{6b}$$

is the (dimensionless) lattice version of the (euclidean) continuum Lagrangian density integrated over all space at a fixed time. The quantity $(2\beta_L/g_0)\hat{L}$ in (6a) has presicely the form of the trace anomaly computed in lattice perturbation theory in ref. [7], summed over the spatial lattice sites on a fixed time slice. Since the (euclidean) energy momentum tensor can be decomposed into a traceless-and trace part, i.e., $T_{\mu\nu} = (T_{\mu\nu} - \frac{1}{4}\delta_{\mu\nu}T) + \frac{1}{4}\delta_{\mu\nu}T$, where T denotes the (euclidean) trace of $T_{\mu\nu}$, we conclude that the second term appearing on the rhs of (1) is the contribution to the field energy of a $q\bar{q}$-pair (determined by the space integral of T_{44}) arising from the trace anomaly. In the weak coupling limit, corresponding to vanishing lattice spacing, the anomalous contribution to the potential (4) takes the form

$$V_a(R, g_0(a), a) = \frac{\beta_L(g_0)}{g_0} < \int d^3x \frac{1}{4}[E^2(x) + B^2(x)] >_{q\bar{q}-0} , \tag{7}$$

where E^2 and B^2 denote square of the (euclidean) colour electric and magnetic fields. A summation over colours is understood. The subscript "a" on V_a stands for "anomalous".

This is the lattice regularized version of the contribution of the trace anomaly to the gluon field energy [6], generalized to the presence of a quark-antiquark pair. For $a \to 0$, the expression (7) is a finite, renormalization group invariant expression. By following a similar line of arguments as in [2], it can be expressed in terms of a renormalized coupling constant g, and renormalized squared colour electric and magnetic fields. The form of the rhs of (7) remains the same, except that $\beta_L(g_0)/g_0$ is replaced by $\beta(g)/g$, where $\beta(g) = \mu\partial g/\partial\mu$ is the continuum beta-function, with μ the renormalization scale. The corresponding anomalous contribution to the potential is thus given by 1/4 of the space integral of the trace anomaly obtained in [5].

The contribution of the traceless part of the energy momentum tensor to the field energy is expected to be given by the first term in (1). This is consistent with the observation that in the weak coupling limit ($g_0 \to 0$), $\eta_- \to \hat{\beta}$ [8], and that for $a \to 0$, with $R = \hat{R}a$ fixed and the dependence of $\hat{\beta}(a)$ given by the usual renormalization group relation,

$$\frac{1}{a}\hat{\beta} < \mathcal{P}'_\tau + \mathcal{P}'_s >_{q\bar{q}-0} \to < \int d^3x \frac{1}{2}(-E^2 + B^2) >_{q\bar{q}-0} \quad . \tag{8}$$

The quantity appearing within brackets is the usual expression for the colour electric and magnetic field energy in the absence of a trace anomaly, expressed in terms of the euclidean fields.

Finally we remark that the lattice action sum rule obtained in [3,4] (derived in [2] in the continuum formulation) can be written in the form

$$\hat{V}(\hat{R},\hat{\beta}) + \hat{R}\frac{\partial\hat{V}(\hat{R},\hat{\beta})}{\hat{R}} = \frac{2\beta_L}{g_0} < \hat{L} >_{q\bar{q}-0} \quad . \tag{9}$$

The rhs of this equation is just the trace of the energy momentum tensor summed over the spatial lattice sites at fixed (euclidean) time. This is similar to the observation made by Michael [1], that the action sum rule for the lowest glueball mass relates this mass to the trace anomaly. For a confining potential $\hat{V} = \hat{\sigma}\hat{R}$, with $\hat{\sigma}$ the string tension in lattice units, the lhs of (9) is just twice the potential. Hence the contribution to the field energy in (1) arising from the trace anomaly is one-half of the interquark potential.

Concluding, we have seen that by extracting the $q\bar{q}$-potential from the expectation value of the Wilson loop on an anisotropic lattice, as described in [3], one is led in a straight forward way to an energy sum rule, in which the contribution to the field energy arising from the traceless and trace parts of the energy momentum tensor can be readily identified.

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References

[1] C. Michael, Nucl. Phys. **B280** [FS18] (1987) 13

[2] H.G. Dosch, O. Nachtmann and M. Rueter, Heidelberg preprint HD-THEP- 95-12, hep-ph 9503386

[3] H.J. Rothe, Phys. Lett. **B355** (1995) 101

[4] G.S. Bali, K. Schilling and Ch. Schlichter, Phys. Rev. **D51** (1995) 5165

[5] J.C. Collins, A. Duncan and S.D. Joglekar, Phys. Rev. **D16** (1977) 438;

[6] X. Ji, "Breakup of Hadron Masses and Energy-Momentum Tensor of QCD", MIT preprint, MIT-CTP-2407 (1995)

[7] S. Caracciolo, P. Menotti and A. Pelissetto, Nucl. Phys. **B375** (1992) 195

[8] F. Karsch, Nucl. Phys. **B205** [FS5] (1982) 285