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Alternative derivation of the correspondence between Rindler and Minkowski particles

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Abstract

We develop an alternative derivation of Unruh and Wald's seminal result that the absorption of a Rindler particle by a detector as described by uniformly accelerated observers corresponds to the emission of a Minkowski particle as described by inertial observers. Actually, we present it in an inverted version, namely, that the emission of a Minkowski particle corresponds in general to either the emission or the absorption of a Rindler particle.

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We present a brute force, but straightforward method to reobtain Unruh and Wald's seminal result [1] that the *absorption* of a Rindler particle by a detector as described by uniformly accelerated observers corresponds to the *emission* of a Minkowski particle as described by inertial observers. Actually, we present it in an inverted version, namely, that the emission of a Minkowski particle in the inertial vacuum will correspond in general to a thermal emission or absorption of a Rindler particle. In the particular case where the source is a uniformly accelerated Unruh–DeWitt detector [2,3], the emission of a Minkowski particle will uniquely correspond to the absorption of a Rindler particle. We believe that our approach may be particularly useful in understanding the behavior of realistic sources following arbitrary worldlines. We assume natural units ($\hbar = c = k_B = 1$), and an n-dimensional Minkowski spacetime with signature $(+ - -...-)$ for sake of generality.

In order to capture only the essential features of a quantum device without making use of some particular detector, let us motivate the use of complex currents. The current describing the excitation of a DeWitt–like detector [3] can be defined as

$$j(\tau) = \langle E|\hat{m}(\tau)|E_0\rangle, \tag{1}$$

where $\hat{m}(\tau)$ is the monopole which represents the detector, τ is its proper time, and $\Delta E = E - E_0$ is the energy gap of the detector. In the Heisenberg picture the monopole is time–evolved as $\hat{m}(\tau) = e^{iH_0\tau}\hat{m}(0)e^{-iH_0\tau}$, where H_0 is the free Hamiltonian of the detector. Thus, current (1) is clearly complex since $j(\tau) \propto e^{i\Delta E\tau}$. This is the reason why we will consider here arbitrary complex currents. These currents should be interpreted as describing the transition of a non–necessarily pointlike quantum device following an arbitrary worldline. For sake of simplicity, we couple our complex current to a real massless scalar field through the interaction operator

$$\hat{S} = \int d^n x\sqrt{|g(x)|}\; j(x)\hat{\phi}(x). \tag{2}$$

Concluded the preliminaries, let us begin by computing the emission rate of Minkowski particles in the inertial frame. The emission amplitude of a Minkowski particle as calculated in the inertial frame is

$$ {}^{M}\mathcal{A}_{\rm em}(k, \mathbf{k}) = {}_{M}\langle \mathbf{k}|\hat{\mathcal{S}}|0\rangle_{M}, \tag{3} $$

where $k = |\mathbf{k}|$. Expanding the scalar field $\hat{\phi}(x)$ in terms of positive and negative energy modes with respect to inertial observers [4]

$$ \hat{\phi}(x^\mu) = \int \frac{d^{n-1}\mathbf{k}}{\sqrt{2k(2\pi)^{n-1}}} (\hat{a}^M_{\mathbf{k}} e^{-ik_\mu x^\mu} + \text{H.c.}), \tag{4} $$

we express from (2) the emission amplitude (3) as

$$ {}^{M}\mathcal{A}_{\rm em}(k, \mathbf{k}) = \left[2k(2\pi)^{n-1}\right]^{-1/2} \int d^n x j(x) e^{ik_\mu x^\mu}, \tag{5} $$

where $x^\mu \equiv (t, x, y^{2\leq i \leq n-1})$. Thus, the emission probability of a Minkowski particle as described by inertial observers is

$$ {}^{M}\mathcal{P}_{\rm em} = \int d^{n-1}\mathbf{k}\, |{}^{M}\mathcal{A}_{\rm em}(k, \mathbf{k})|^2, \tag{6} $$

where ${}^{M}\mathcal{A}_{em}$ is given in (5).

Next, we aim to express the emission rate of Minkowski particles (6) in terms of the emission and absorption of Rindler particles. For this purpose, it is convenient to introduce Rindler coordinates $(\tau, \xi, y^{2\leq i\leq n-1})$. These coordinates are related with Minkowski coordinates $(t, x, y^{2\leq i\leq n-1})$ by

$$ t = \frac{e^{a\xi}}{a} \sinh a\tau, \;\; x = \frac{e^{a\xi}}{a} \cosh a\tau. \tag{7} $$

A worldline defined by $\xi, y^{2\leq i\leq n-1} = \text{const}$ describes an observer with constant proper acceleration $ae^{-a\xi}$. We will denominate these observers *Rindler* observers. The natural manifold to describe Rindler observers is the Rindler wedge, *i.e.* the portion of the Minkowski space defined by $x > |t|$. It is crucial for our purposes that the current $j(x^\mu)$ be confined inside this wedge. The Rindler wedge is a globally hyperbolic spacetime in its own right, with a Killing horizon at $x = \pm t$ ($\tau = \pm\infty$) associated with the boost Killing field ∂_τ at its boundary. Using (7), the Minkowski line element restricted to the Rindler wedge is

$$ ds^2 = e^{2a\xi}(d\tau^2 - d\xi^2) - \sum_{i=2}^{n-1}(dy^i)^2. \tag{8} $$

Solving the massless Klein-Gordon equation $\Box\phi = 0$ in the Rindler wedge, we obtain a complete set of Klein-Gordon orthonormalized functions [5]

$$u_{\omega\mathbf{k}_\perp}(x^\mu) = \left[\frac{2\sinh \pi\omega/a}{(2\pi)^{n-1}\pi a}\right]^{\frac{1}{2}} K_{i\omega/a}\left(\frac{k_\perp}{a}e^{a\xi}\right) e^{i\mathbf{k}_\perp \mathbf{y} - i\omega\tau}, \tag{9}$$

where $k_\perp \equiv |\mathbf{k}_\perp| = \sqrt{\sum_{i=2}^{n-1}(k_{y^i})^2}$, ω is the frequency of the Rindler mode, and $K_{i\lambda}(x)$ is the MacDonald function. Using (9), we express the scalar field in the uniformly accelerated frame in terms of positive and negative frequency modes with respect to the boost Killing field ∂_τ

$$\hat{\phi}(x^\mu) = \int d^{n-2}\mathbf{k}_\perp \int_0^{+\infty} d\omega \left\{\hat{a}^R_{\omega\mathbf{k}_\perp} u_{\omega\mathbf{k}_\perp}(x^\mu) + \text{H.c.}\right\}, \tag{10}$$

where $\hat{a}^R_{\omega\mathbf{k}_\perp}$ is the annihilation operator of Rindler particles, and obeys the usual commutation relation

$$\left[\hat{a}^R_{\omega\mathbf{k}_\perp}, \hat{a}^{R\dagger}_{\omega'\mathbf{k}'_\perp}\right] = \delta(\omega - \omega')\delta(\mathbf{k}_\perp - \mathbf{k}'_\perp). \tag{11}$$

It is possible now to Fourier analyze the current $j(x^\mu)$ in terms of Rindler modes. We define its *Rindler–Fourier* transform as

$$\tilde{\jmath}_R(\omega_0, \omega, k_\perp) \equiv \int d^n x \sqrt{|g(x)|}\, j(x^\mu) u^*_{\omega\mathbf{k}_\perp} e^{-i(\omega-\omega_0)\tau}. \tag{12}$$

This relation can be easily inverted

$$j(x^\mu) = \frac{e^{-2a\xi}}{\pi} \int d^{n-2}\mathbf{k}_\perp \int_0^{+\infty} d\omega\omega \int_{-\infty}^{+\infty} d\omega_0 \tilde{\jmath}_R(\omega_0, \omega, k_\perp) u_{\omega\mathbf{k}_\perp}(x^\mu) e^{i(\omega-\omega_0)\tau} \tag{13}$$

by using the completeness relation [6]

$$\int_0^{+\infty} d\omega\omega \sinh\frac{\pi\omega}{a}\, K_{i\omega/a}\left(\frac{k_\perp}{a}e^{a\xi}\right) K_{i\omega/a}\left(\frac{k_\perp}{a}e^{a\xi'}\right) = \frac{\pi^2 a}{2}\delta(\xi - \xi'). \tag{14}$$

In order to relate the particle emission and absorption rates in both reference frames, we substitute (13) and (9) in (5). The y^i and $\mathbf{k}_\perp$ integrals are easy to perform, while the τ and ξ integrals can be solved by noting that (use the change of variables $\eta = e^{a\tau}$ in conjunction with Eq. 3.471.10 of Ref. [7])

$$\int_{-\infty}^{+\infty} d\tau\, e^{i(kt-k_x x-\omega_0\tau)} = \frac{2}{a}\left[\frac{k+k_x}{k-k_x}\right]^{-i\omega_0/2a} e^{\pi\omega_0/2a} K_{i\omega_0/a}\left(\frac{k_\perp}{a}e^{a\xi}\right), \tag{15}$$

and by using the following orthonormality relation [8,9]:

$$\int_{-\infty}^{+\infty} d\xi\; K_{i\omega/a}\left(\frac{k_\perp}{a}e^{a\xi}\right) K_{i\omega'/a}\left(\frac{k_\perp}{a}e^{a\xi}\right) = \frac{\pi^2 a}{2\omega\sinh(\pi\omega/a)}\delta(\omega-\omega') \tag{16}$$

for $\omega, \omega' \in \mathbf{R}_+$. With this procedure we reduce the emission amplitude (3) to

$$\begin{aligned} {}^{M}\!\mathcal{A}_{\rm em}(k,\mathbf{k}) = \frac{1}{(4\pi a k)^{1/2}}\int_0^{+\infty}\frac{d\omega}{\sinh^{1/2}(\pi\omega/a)} & \left\{\tilde{\jmath}_R(\omega,\omega,\mathbf{k}_\perp)\left(\frac{k+k_x}{k-k_x}\right)^{-i\omega/2a} e^{\pi\omega/2a}\right. \\ & \left. +\tilde{\jmath}_R(-\omega,\omega,\mathbf{k}_\perp)\left(\frac{k+k_x}{k-k_x}\right)^{+i\omega/2a} e^{-\pi\omega/2a}\right\}. \end{aligned} \tag{17}$$

Our next task will be to interpret $\tilde{\jmath}_R(\pm\omega,\omega,\mathbf{k}_\perp)$ in terms of the emission and absorption amplitudes of Rindler particles

$${}^{R}\!\mathcal{A}_{\rm em} = {}_R\langle \omega\mathbf{k}_\perp|\,\hat{\mathcal{S}}|0\rangle_R, \;\; {}^{R}\!\mathcal{A}_{\rm abs} = {}_R\langle 0|\hat{\mathcal{S}}|\omega\mathbf{k}_\perp\rangle_R \tag{18}$$

respectively, where $\hat{\mathcal{S}}$ is given in (2). Using explicitly (2) and (10) in (18) we obtain

$$\tilde{\jmath}_R(\omega,\omega,\mathbf{k}_\perp) = {}^{R}\!\mathcal{A}_{\rm em}(\omega,\mathbf{k}_\perp), \quad \tilde{\jmath}_R(-\omega,\omega,\mathbf{k}_\perp) = {}^{R}\!\mathcal{A}_{\rm abs}(\omega,-\mathbf{k}_\perp). \tag{19}$$

As a consequence, we can express the Minkowski particle emission amplitude (17) as

$$\begin{aligned} {}^{M}\!\mathcal{A}_{\rm em}(k,\mathbf{k}) = \frac{1}{(4\pi a k)^{1/2}}\int_0^{+\infty}\frac{d\omega}{\sinh^{1/2}(\pi\omega/a)} & \left\{{}^{R}\!\mathcal{A}_{\rm em}(\omega,\mathbf{k}_\perp)\left(\frac{k+k_x}{k-k_x}\right)^{-i\omega/2a} e^{\pi\omega/2a}\right. \\ & \left. +{}^{R}\!\mathcal{A}_{\rm abs}(\omega,-\mathbf{k}_\perp)\left(\frac{k+k_x}{k-k_x}\right)^{+i\omega/2a} e^{-\pi\omega/2a}\right\}. \end{aligned} \tag{20}$$

Now, it is useful to introduce the following representation of the delta function

$$\frac{1}{2\pi a}\int_{-\infty}^{+\infty}\frac{dk_x}{k}\left[\frac{k+k_x}{k-k_x}\right]^{-i(\omega-\omega')/2a} = \delta(\omega-\omega'), \tag{21}$$

where $k=\sqrt{k_x^2+k_\perp^2}$. (This delta function representation can be cast in the more familiar form $\int_{-\infty}^{+\infty} dK e^{-iK(\omega-\omega')} = 2\pi\delta(\omega-\omega')$ after the change of variables $k_x \to K = \ln[(k+k_x)/(k-k_x)]$.) Finally, introducing (20) in (6), and using (21) to perform the integral in k_x we are able to express the total emission rate of Minkowski particles in its final form as

$$ {}^{M}\mathcal{P}_{\rm em} = {}^{R}\mathcal{P}_{\rm em} + {}^{R}\mathcal{P}_{\rm abs}, \tag{22} $$

where

$$ {}^{R}\mathcal{P}_{\rm em} = \int d^{n-2}\mathbf{k}_\perp \int_0^{+\infty} d\omega |{}^{R}\mathcal{A}_{\rm em}(\omega, \mathbf{k}_\perp)|^2 \left(1 + \frac{1}{e^{2\pi\omega/a} - 1}\right), \tag{23} $$

and

$$ {}^{R}\mathcal{P}_{\rm abs} = \int d^{n-2}\mathbf{k}_\perp \int_0^{+\infty} d\omega |{}^{R}\mathcal{A}_{\rm abs}(\omega, -\mathbf{k}_\perp)|^2 \frac{1}{e^{2\pi\omega/a} - 1}. \tag{24} $$

The thermal factors which appear in (23) and (24) are in agreement with the fact that the Minkowski vacuum corresponds to a thermal state with respect to uniformly accelerated observers [2]. The physical content of (22) combined with (23) and (24) can be summarized as follows: *The emission of a Minkowski particle in the vacuum with some fixed transverse momentum* $\mathbf{k}_\perp$ *as described by an inertial observer* (6) *will correspond either to the emission of a Rindler particle with the same transverse momentum* $\mathbf{k}_\perp$*, or to the absorption of a Rindler particle with transverse momentum* $-\mathbf{k}_\perp$ *from the Davies-Unruh thermal bath as described by uniformly accelerated observers.* Notice that the conservation of transverse momentum in both frames appears naturally enclosed in this result. This is mandatory since the transverse momentum is invariant under boosts. In the particular case where the source is a uniformly accelerated Unruh-DeWitt detector, ${}^{R}\mathcal{A}_{\rm em} = 0$ implying that in this case the absorption of a Rindler particle corresponds uniquely to the emission of a Minkowski particle. However, as seen above, in more general situations where the detector is switched on/off [10] or follows some arbitrary worldline, the excitation of the detector usually associated with the *absorption* of a Rindler particle can be also associated with the *emission* of a Rindler particle. This is also in agreement with energy conservation arguments.

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