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An axially-symmetric Newtonian Boson Star

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Abstract

A new solution to the coupled gravitational and scalar field equations for a condensed boson field is found in Newtonian approximation. The solution is axially symmetric, but not spherically symmetric. For N particles the mass of the object is given by $M = Nm - 0.02298 N^3 G_N^2 m^5$, to be compared with $M = Nm - 0.05426 N^3 G_N^2 m^5$ for the spherically symmetric case.

Recent developments in particle physics and cosmology suggest that evolving scalar fields may have played an important role in the evolution of the early universe, for instance in primordial phase transitions, and that they may make up the missing dark matter. Models for galaxy formation using cold dark matter and the inflationary scenario suggest that the ratio of baryonic (luminous) matter to dark matter can be of the order of 10 %. These facts naturally raise the question whether cold gravitational equilibrium configurations of massive scalar fields - Bose stars - may exist and whether such configurations are dynamically stable. For bosonic fields interacting only via gravity spherically symmetric equilibrium solutions were studied by Kaup [1] as well as Ruffini and Bonazzola [2] by solving the coupled Einstein-Klein-Gordon equations. They analyzed only the zero-node solutions, corresponding to the lowest energy state. The results of ref.[1,2] have been confirmed and extended later on. For reviews we refer to ref.[3,4]. Recently the suggestion has been made, that the halo of galaxies is itself a condensed bosonic object[5,6]. This model was studied in the Newtonian approximation, where reasonable agreement with experimental rotation curves was found. All studies sofar have been restricted to non-rotating objects, i.e. spherically symmetric solutions. In this letter we make a first approach in studying axially symmetric solutions. For simplicity we restrict ourselves here to the Newtonian approximation.

The Newtonian treatment of self-gravitating bosons of mass m interacting only gravitationally has been studied in ref.[2,7]. For condensed bosons at T=0 the equations reduce to two coupled equations for the gravitational potential and the Schrödinger field. The gravitational potential V satisfies the Poisson equation

$$\Delta V = 4\pi G_N \rho \tag{1}$$

where the mass density is given by

$$\rho = N m \psi^* \psi \tag{2}$$

The one particle wave function ψ is determined by the Schrödinger equation

$$-\Delta\psi + 2m(E + mV)\psi = 0 \tag{3}$$

together with the normalization

$$\int d^3r \psi^* \psi = 1 \tag{4}$$

After rescaling

$$\hat{x} = 2m^3 G_N x \tag{5}$$

$$\phi = \sqrt{4\pi}(2m^3 G_N N)^{-3/2}\psi \tag{6}$$

$$\hat{V} = (2G_N^2 N^2 m^4)^{-1} V \tag{7}$$

$$\hat{E} = (2G_N^2 N^2 m^5)^{-1} E \tag{8}$$

the system of equations reduces to the simple form

$$\Delta\hat{V} = \phi^2 \tag{9}$$

and

$$\Delta\phi - \hat{V}\phi = \hat{E}\phi \tag{10}$$

The search for a solution is then reduced to looking for an eigenfunction with the correct norm

$$\int_0^\infty \phi_{\hat{E}}^2 d^3r = 4\pi \tag{11}$$

In the case of a spherically symmetric solution the equations can be reduced to a set of ordinary differential equations. These can be solved relatively simply. In our case we are also interested in solutions, that have only an axial symmetry. Here the system of equations cannot be reduced to ordinary differential equations. To solve the resulting equations, we used the method of finite elements [8,9]. In practice the following procedure was adopted. The equations were rewritten in cylindrical coordinates. As boundary condition we took

$$\phi(r,z) = 0 \ \text{ for } \ r^2 + z^2 = R \tag{12}$$

When R is large enough this gives a sufficient approximation for the condition $\phi \to 0$ at infinity. Furthermore there is the condition

$$\partial_r\phi(r=0,z) = 0 \tag{13}$$

It is to be noticed that the equations are invariant under the parity transformation $z \to -z$, $\hat{V} \to \hat{V}$, $\phi \to \pm\phi$. The solutions therefore fall into positive and negative parity classes. Using this symmetry it is therefore sufficient to solve the equations in a quadrant in the r,z plane with the following boundary conditions:
Dirichlet for ϕ and $\hat{V}$ at $r^2 + z^2 = R$,
Neumann for ϕ and $\hat{V}$ at $r = 0$ and for $\hat{V}$ at $z = 0$.
For positive parity one has then the Neumann condition for ϕ at $z = 0$, while

for negative parity one has the Dirichlet condition. With these conditions the problem is well defined in a finite domain and can be solved with the finite element method. Because the equations are partial differential equations one can typically find this way the lowest energy solutions corresponding to the given boundary conditions. The solution for positive parity is given in the figures 1 and 2 . The corresponding eigenvalue is given by $\hat{E} = 0.081385$, which is in perfect agreement with the results in the literature and the integration after imposing spherical symmetry. This good agreement proves that the finite element method can be satisfactorily applied to this problem. The corresponding results for the negative parity solution is given in figures 3 and 4. We found an energy value of $\hat{E} = 0.034465$. These numbers correspond to the following mass formulae,

$$M = Nm - 0.05426 N^3 G_N^2 m^5 \tag{14}$$

for the spherically symmetric case and

$$M = Nm - 0.02298 N^3 G_N^2 m^5 \tag{15}$$

for the axially symmetric case. The maximum masses and particle numbers are given by

$$M = 1.6524 m_{Pl}^2/m \ \ and \ \ N = 2.4786 m_{Pl}^2/m^2 \tag{16}$$

respectively

$$M = 2.5391 m_{Pl}^2/m \ \ and \ \ N = 3.8086 m_{Pl}^2/m^2 \tag{17}$$

References

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Figure Captions

1. Presented is the behaviour of the funktion $\phi(r, z)$ for the spherically symmetric case. The straight line corresponds to the z-axis. The semi-circle corresponds to $r^2 + z^2 = 50$. Outside the plotted area $\phi \approx 0$.

2. Presented is the behaviour of the gravitational potential. Outside the plotted region one has asymptotically $\hat{V} = (r^2 + z^2)^{-1/2}$

3. The same as fig.(1), but for the axially-symmetric case. Here the semi-circle is at R=100.

4. The gravitational potential for the axially-symmetric case.