S-Duality in Gauge Theories as a Canonical Transformation

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Abstract

We show that S-duality in four dimensional non-supersymmetric gauge theories can be formulated as a canonical transformation in the phase space of the theory. It is shown that in phase space the modular anomaly emerges as the result of integrating out the momenta degrees of freedom. In the case of non-abelian gauge theories the canonical transformation yields also a dual theory with $\tilde{\tau} = -1/\tau$, $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$, and gauge group the dual of the original one. The generalization to d dimensional abelian gauge theories of p-forms is also considered.

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1 Introduction

A lot of progress has been made in the last few years in the understanding of S-duality as a symmetry of four dimensional gauge theories. The conjecture of Montonen and Olive [1] that $N = 4$ supersymmetric Yang-Mills theories were invariant under strong-weak coupling with the exchange of the gauge group by its dual was tested in [2], were it was shown that in fact the partition function transformed as a modular form. Some progress has been also made for $N = 2$ and $N = 1$ supersymmetric Yang-Mills theories [7, 4]. However a path integral derivation of S-duality is in general still unknown. In [8] Witten showed that S-duality in four dimensional abelian gauge theories [5, 6, 7] can be implemented at the level of the path integral in a very similar way to T-duality in non-linear sigma models in String Theory [11]. The idea is to consider a global isometry of the Lagrangian which can be expressed as translations of a given coordinate (the adapted coordinate), gauge this isometry by introducing a fake gauge field and impose the constraint that the curvature tensor associated to this gauge field is zero so that the gauge field is non-propagating. Integrating the Lagrange multiplier and fixing the gauge field to zero the original theory is recovered and integrating the gauge field and fixing the adapted coordinate to zero the new dual theory is obtained. In the case of T-duality the initial variables are 0-forms and the global isometry that is gauged is $\theta \to \theta + \epsilon$ where θ is the adapted coordinate. In the case of abelian gauge theories the initial variables are 1-forms and the isometry which is gauged is $A \to A + \epsilon$ where now the ϵ parameter is a 1-form. Then the gauge field which has to be introduced is a 2-form and its field strength a 3-form. In 4 dimensions the Lagrange multiplier imposing that the field strength vanishes is a 1-form, like the original gauge field, and the dual theory is expressed also in terms of 1-forms. Also for this non-supersymmetric case the partition function transforms as a modular function with a modular weight proportional to the Euler characteristic and the signature of the manifold [8, 9].

Given the analogy with T-duality a canonical transformation must be beyond this path integral manipulation, since this is the case in T-duality [13, 14, 15]. In section 2 we present the explicit generating functional producing this transformation and show that it is the generalization of the functional in 2-dimensional non-linear sigma models to 4 dimensions and 1-forms. Under this transformation electric and magnetic degrees of freedom get interchanged (with the minus relative sign) as shown in abelian lattice gauge theories in [6]. The canonical transformation approach is the simplest in order to obtain the dual theory, also in this case in which in the Hamiltonian formulation one has to be careful with the constraints. It is easy to show that both the initial and the dual theory are defined in the same subspace of the phase space after the canonical transformation is performed. We show that in phase space the partition functions of the initial and dual theories coincide and that only after integrating out the momenta degrees of freedom the modular anomaly [8, 9] appears.

In the canonical transformation approach the generalization to Yang-Mills theories is straightforward. The dual theory is also a Yang-Mills theory with gauge group the dual of the original one (as conjectured by Montonen and Olive in the $N = 4$ supersymmetric case) and inverted coupling. We show this in section 3. We also exhibit a path integral derivation in the style presented in [8] for the abelian case.

The results presented in section 2 can be easily generalized to the case of d dimensional abelian gauge theories of p-forms, as it is explained in section 4. The modular anomaly in the transformation of the partition function is obtained. The implementation at the level of the path integral using a coset construction was presented in [10].

2 The abelian case

In this section we construct the explicit canonical transformation which produces the change

$$\tau \to -1/\tau \tag{2. 1}$$

with $\tau = \theta/2\pi + 4\pi i/g^2$, for $U(1)$ four dimensional euclidean gauge theories.

Let us consider the Lagrangian

$$\begin{aligned} L &= \frac{1}{8\pi}(\frac{4\pi}{g^2}F_{mn}F^{mn} + \frac{i\theta}{4\pi}\epsilon_{mnpq}F^{mn}F^{pq}) \\ &= \frac{i}{8\pi}(\bar\tau F^+_{mn}F^{+mn} - \tau F^-_{mn}F^{-mn}) \end{aligned} \tag{2. 2}$$

where

$$\begin{aligned} F^+_{mn} &= \frac{1}{2}(F_{mn} + {}^*F_{mn}) = \frac{1}{2}(F_{mn} + \frac{1}{2}\epsilon_{mnpq}F^{pq}), \\ F^-_{mn} &= \frac{1}{2}(F_{mn} - {}^*F_{mn}) = \frac{1}{2}(F_{mn} - \frac{1}{2}\epsilon_{mnpq}F^{pq}) \end{aligned} \tag{2. 3}$$

and $F_{mn} = \partial_m A_n - \partial_n A_m$. It was shown in [8] that the transformation (2. 1) could be derived at the level of the path integral by the usual Rocek and Verlinde's procedure [12] one follows to construct abelian T-duals of two dimensional sigma models in String Theory. In this case given a global abelian continuous isometry of the sigma model one can turn it local by introducing a fake gauge field in the Lagrangian by minimal coupling and imposing the constraint that this gauge field is non-dynamical. Solving this constraint and fixing the gauge field to be zero one recovers the original theory. If instead the gauge field is integrated and the gauge is fixed in the original variables a sigma model written in terms of the Lagrange multiplier introduced to impose the constraint is obtained. This is the dual sigma model. In [8] the same construction is applied to obtain the dual of the abelian gauge theory. The global continuous abelian isometry in this theory is

$$A \to A + \epsilon \tag{2. 4}$$

where now the isometry parameter is a 1-form. This global isometry can be gauged by introducing a gauge field G, 2-form, which is imposed to be non-dynamical with the term

$$\int_M d^4x \tilde{A} dG \tag{2. 5}$$

where the Lagrange multiplier $\tilde{A}$ is a 1-form. Integrating $\tilde{A}$ the constraint $dG = 0$ is obtained, ie. G pure gauge, and we can recover (2. 2) by either fixing $A = 0$ or $G = 0$. On the other hand by integrating out G and then fixing $A = 0$ the following Lagrangian is gotten:

$$\tilde{L} = \frac{i}{8\pi}(-\frac{1}{\bar\tau}\tilde{F}^+_{mn}\tilde{F}^{+mn} + \frac{1}{\tau}\tilde{F}^-_{mn}\tilde{F}^{-mn}) \tag{2. 6}$$

with $\tilde{F}^{\pm}$ the self- and antiself-dual components of $\tilde{F}_{mn} \equiv \partial_m \tilde{A}_n - \partial_n \tilde{A}_m$. This is the S-dual of the initial electromagnetic theory since in the particular case $\theta = 0$ it corresponds to the inversion of the coupling constant g.

In this procedure we have made an integration by parts in the Lagrange multipliers term and neglected a total derivative[1]. However this total derivative contains some information, in particular it implies that the initial and dual Lagrangians are equal up to a total time derivative, exactly what happens when two theories are related by a canonical transformation. To be more precise, the generating functional of a canonical transformation from $\{q^i, p_i\}$ to $\{Q^i, P_i\}$ is such that

$$p_i \dot{q}^i - H(q^i, p_i) = P_i \dot{Q}^i - \tilde{H}(Q^i, P_i) + \frac{dF}{dt}. \tag{2. 7}$$

If F is a type I generating functional (depending only on coordinates) $H = \tilde{H}$ if and only if[2]

$$\begin{aligned} \frac{\partial F}{\partial q^i} &= p_i \\ \frac{\partial F}{\partial Q^i} &= -P_i \end{aligned} \tag{2. 8}$$

Under duality:

$$\tilde{L}(\tilde{A}) = L(A) + d\tilde{A} \wedge dA \tag{2. 9}$$

which implies[3]

$$\epsilon^{mnpq}(\partial_m \tilde{A}_n - \partial_n \tilde{A}_m)(\partial_p A_q - \partial_q A_p) = -(\frac{\delta F}{\delta \tilde{A}_m}\dot{\tilde{A}}_m + \frac{\delta F}{\delta A_m}\dot{A}_m) \tag{2. 10}$$

This produces the canonical transformation

$$\begin{aligned} \Pi^\alpha &= \frac{\delta F}{\delta A_\alpha} = -4\, {}^*\tilde{F}^{0\alpha}, \qquad \Pi^0 = 0, \\ \tilde{\Pi}^\alpha &= -\frac{\delta F}{\delta \tilde{A}_\alpha} = 4\, {}^*F^{0\alpha}, \qquad \tilde{\Pi}^0 = 0 \end{aligned} \tag{2. 11}$$

plus a constraint

$$\Pi^\alpha \partial_\alpha A_0 = \tilde{\Pi}^\alpha \partial_\alpha \tilde{A}_0, \tag{2. 12}$$

where greek indices run over spatial coordinates.

The generating functional producing this canonical transformation is

$$F = -2 \int_{M, t\, fixed} d^3x (\tilde{A}_\alpha {}^*F^{0\alpha} + A_\alpha {}^*\tilde{F}^{0\alpha}) = -\frac{1}{2}\int_M d^4x \tilde{F} \wedge F. \tag{2. 13}$$

[1]This term is seen in the gauge $A = 0$.

[2]We assume F does not depend explicitly on time.

[3]Our convention for the product of forms is: $\tilde{F} \wedge F = \epsilon^{mnpq}\tilde{F}_{mn}F_{pq}$.

This is the result one would expect a priori from what is known in two-dimensional sigma-models, where the generating functional is given in terms of the adapted coordinate to the isometry θ and the Lagrange multiplier $\tilde{\theta}$ by [13, 14]

$$F = -\frac{1}{2}\int_{M_2} d\tilde{\theta}\wedge d\theta. \tag{2. 14}$$

The Hamiltonian associated to (2. 2) is given by[4]:

$$H = \frac{1}{4(\bar{\tau}-\tau)}\Pi_\alpha\Pi^\alpha + \partial_\alpha A_0\Pi^\alpha - \frac{\bar{\tau}+\tau}{\bar{\tau}-\tau}\Pi_\alpha\,{}^*F^{0\alpha} + \frac{4\bar{\tau}\tau}{\bar{\tau}-\tau}\,{}^*F^{0\alpha}\,{}^*F_{0\alpha} \tag{2. 15}$$

plus the constraints

$$\Pi_0 = 0, \qquad \partial_\alpha\Pi^\alpha = 0, \tag{2. 16}$$

where

$$\Pi^\alpha = \frac{\delta F}{\delta\dot{A}_\alpha} = 4\bar{\tau}F^{+0\alpha} - 4\tau F^{-0\alpha}. \tag{2. 17}$$

Π_0 is a primary constraint and $\partial_\alpha\Pi^\alpha = 0$ is the secondary constraint emerging from the equation of motion for Π_0. They imply that the theory is defined in the reduced phase space given by $\Pi_0 = 0$, $\partial_\alpha\Pi^\alpha = 0$. These constraints are also satisfied in the dual theory, since they are obtained directly from the canonical transformation. Then the dual theory is defined in the same reduced phase space than the original one. The relation (2. 12) is trivial in this subspace. However we need to consider it in order to recover the dual Lagrangian from the canonically transformed Hamiltonian, since for that we need the naive Hamiltonian without taking into account the constraints. Our purpose is to show that the canonically transformed Lagrangian is the dual Lagrangian and for that we do not need to study in detail the way the theory gets defined in the Hamiltonian formalism [16], it is enough to show that both the initial and dual theories are defined in the same reduced phase space.

The canonically transformed Hamiltonian reads:

$$\tilde{H} = \frac{1}{4}\frac{\bar{\tau}\tau}{\bar{\tau}-\tau}\tilde{\Pi}_\alpha\tilde{\Pi}^\alpha + \partial_\alpha V_0\tilde{\Pi}^\alpha + \frac{\bar{\tau}+\tau}{\bar{\tau}-\tau}\tilde{\Pi}_\alpha\,{}^*\tilde{F}^{0\alpha} + \frac{4}{\bar{\tau}-\tau}\,{}^*\tilde{F}_{0\alpha}\,{}^*\tilde{F}^{0\alpha}. \tag{2. 18}$$

The corresponding Lagrangian is given by the dual Lagrangian (2. 6). Recall that (2. 11):

$$\begin{aligned}\Pi^\alpha &= -4\,{}^*\tilde{F}^{0\alpha},\\ \tilde{\Pi}^\alpha &= 4\,{}^*F^{0\alpha}\end{aligned}$$

corresponds to the usual interchange between electric and magnetic degrees of freedom when there is no θ-term.

Some useful information can be obtained within this approach. The generating functional (2. 13) is linear in both the original and dual variables. Then the following relation holds:

$$He^{\frac{iF}{8\pi}} = \tilde{H}e^{\frac{iF}{8\pi}} \tag{2. 19}$$

[4] We have droped the global $i/8\pi$ factor. It will then appear when exponentiating these quantities.

which implies:

$$\psi_k[\tilde{A}] = N(k)\int \mathcal{D}A(x^\alpha)e^{\frac{i}{8\pi}F[\tilde{A},A(x^\alpha)]}\phi_k[A(x^\alpha)] \tag{2. 20}$$

with $\phi_k[A]$ and $\psi_k[\tilde{A}]$ eigenfunctions of the initial and dual Hamiltonians respectively with the same eigenvalue and $N(k)$ a normalization factor [17]. From this relation global properties can be easily worked out. The Dirac quantization condition:

$$\int_\Sigma F = 2\pi n, \quad n \in Z, \tag{2. 21}$$

for Σ any closed two-surface in the manifold, implies for $\tilde{F}$:

$$\int_\Sigma \tilde{F} = 2\pi m, \quad m \in Z \tag{2. 22}$$

and $\tilde{F}$ must live in the dual lattice. Also from (2. 20) the transformation applies to any four dimensional manifold M since $\phi_k[A]$ can be the result of integrating the theory in an arbitrary manifold with boundary.

We can obtain in phase space the modular anomaly emerging in the transformation of the partition function [8, 9]. The argument goes as follows. In phase space the partition function is given by[5]:

$$Z_{ps} = \int \mathcal{D}A_\alpha \mathcal{D}\Pi^\alpha e^{-\frac{i}{8\pi}\int d^4x(\dot{A}_\alpha\Pi^\alpha - H)} \tag{2. 23}$$

Under (2. 11)

$$\mathcal{D}A_\alpha\mathcal{D}\Pi^\alpha = \mathcal{D}\tilde{A}_\alpha\mathcal{D}\tilde{\Pi}^\alpha. \tag{2. 24}$$

Then the dual phase space partition function is given by:

$$\tilde{Z}_{ps} = \int \mathcal{D}\tilde{A}_\alpha\mathcal{D}\tilde{\Pi}^\alpha e^{-\frac{i}{8\pi}\int d^4x(\dot{\tilde{A}}_\alpha\tilde{\Pi}^\alpha - \tilde{H})} = Z_{ps} \tag{2. 25}$$

showing that in phase space the partition function is invariant under duality. Integration on momenta in (2. 23) gives:

$$Z_{ps} = \int \mathcal{D}A_\alpha(\mathrm{Im}\tau)^{B_2/2}e^{-\int d^4xL} \tag{2. 26}$$

with L given by (2. 2). The factor $(\mathrm{Im}\tau)^{B_2/2}$ in the measure is the regularized $(\det \mathrm{Im}\tau)^{1/2}$ coming from the gaussian integration over the momenta. B_2 is the dimension of the space of 2-forms in the four dimensional manifold M (regularized on a lattice) and emerges because the momenta are 2-forms.

The same calculation in the dual phase space partition function gives:

$$\tilde{Z}_{ps} = \int \mathcal{D}\tilde{A}_\alpha(\det(\mathrm{Im} - \frac{1}{\tau}))^{1/2}e^{-\int d^4x\tilde{L}} \tag{2. 27}$$

with $\tilde{L}$ given by (2. 6). We regularize the factor

$$\det(\mathrm{Im} - \frac{1}{\tau}) = \det(\mathrm{Im}\tau/(\tau\bar{\tau})) \tag{2. 28}$$

[5]In order to have a well-defined partition function we have to fix the gauge symmetry. The following arguments are in this sense formal.

by

$$(\mathrm{Im}\tau)^{B_2/2}\bar{\tau}^{-B_2^+/2}\tau^{-B_2^-/2} \tag{2. 29}$$

where B_2^+ and B_2^- are respectively the dimensions of the spaces of self-dual and anti-self-dual 2-forms. In configuration space the partition function is defined by [8]:

$$Z = (\mathrm{Im}\tau)^{(B_1-B_0)/2}\int \mathcal{D}A_\alpha e^{-S} = (\mathrm{Im}\tau)^{(B_1-B_0-B_2)/2} Z_{ps} \tag{2. 30}$$

and in the dual model

$$\tilde{Z} = (\frac{\mathrm{Im}\tau}{\tau\bar{\tau}})^{(B_1-B_0)/2}\int \mathcal{D}\tilde{A}_\alpha e^{-\tilde{S}} \tag{2. 31}$$

From $Z_{ps} = \tilde{Z}_{ps}$ we arrive to

$$Z = \tau^{-(\chi-\sigma)/4}\bar{\tau}^{-(\chi+\sigma)/4}\tilde{Z} \tag{2. 32}$$

where $\chi = 2(B_0 - B_1) + B_2$ is the Euler number (the regularization is such that $B_p = B_{4-p}$) and $\sigma = B_2^+ - B_2^-$ is the signature of the manifold. This is the modular factor appearing in [8, 9]. In phase space the partition function is simply defined as the integration over coordinates and momenta and it transforms as a scalar with modular weight equal to zero. Is only when going to the configuration space that the integrations over the momenta produce some determinants which after being regularized yield the modular factor found in [8, 9]. A very similar argument applies to the transformation of the dilaton in two-dimensional non-linear sigma-models.

3 The non-abelian case

The canonical transformation approach can be straightforwardly generalized to the case of non-abelian gauge theories with arbitrary compact group G. The initial Lagrangian is given by:

$$\begin{aligned} L &= \frac{1}{8\pi}(\frac{4\pi}{g^2}F^{(a)}_{mn}F^{(a)mn} + \frac{i\theta}{4\pi}\epsilon^{mnpq}F^{(a)}_{mn}F^{(a)}_{pq}) \\ &= \frac{i}{8\pi}(\bar{\tau}F^{(a)+}_{mn}F^{(a)+mn} - \tau F^{(a)-}_{mn}F^{(a)-mn}) \end{aligned} \tag{3. 1}$$

where $F = dA - A\wedge A$ and we have chosen $Tr(T^aT^b) = \delta^{ab}$ (T^a are the generators of the Lie algebra). The conjugate momenta and the Hamiltonian are:

$$\begin{aligned} \Pi^a_\alpha &= \frac{\delta L}{\delta \dot{A}^a_\alpha} = 2(\bar{\tau}-\tau)F^{(a)}_{0\alpha} + 2(\bar{\tau}+\tau)\,{}^*F^{(a)}_{0\alpha} \\ \Pi^a_0 &= 0 \end{aligned} \tag{3. 2}$$

$$H = \frac{1}{4}\frac{1}{\bar{\tau}-\tau}\Pi^a_\alpha\Pi^{a\alpha} + (\partial_\alpha A^a_0 + f_{abc}A^b_0A^c_\alpha)\Pi^{a\alpha} - \frac{\bar{\tau}+\tau}{\bar{\tau}-\tau}\Pi^{a\alpha}\,{}^*F^{(a)}_{0\alpha} + \frac{4\bar{\tau}\tau}{\bar{\tau}-\tau}\,{}^*F^{(a)}_{0\alpha}\,{}^*F^{(a)0\alpha}. \tag{3. 3}$$

In the non-abelian case it proves more useful to use $\{{}^*F_{0\alpha}, \Pi^\alpha\}$ as the coordinates in phase space and look for a canonical transformation

$$\{{}^*F_{0\alpha}, \Pi^\alpha\} \rightarrow \{{}^*\tilde{F}_{0\alpha}, \tilde{\Pi}^\alpha\}. \tag{3. 4}$$

Then as in the abelian case the canonical transformation is the usual interchange between electric and magnetic degrees of freedom:

$$\begin{aligned} \Pi^{a\alpha} &= -4\,{}^*\tilde{F}^{0\alpha} \\ \tilde{\Pi}^{a\alpha} &= 4\,{}^*F^{0\alpha}. \end{aligned} \tag{3. 5}$$

This leads to the dual Lagrangian:

$$\tilde{L} = \frac{i}{8\pi} Tr(-\frac{1}{\bar{\tau}}\tilde{F}^{+}_{mn}\tilde{F}^{+mn} + \frac{1}{\tau}\tilde{F}^{-}_{mn}\tilde{F}^{-mn}). \tag{3. 6}$$

We have used the constraint:

$$\Pi^{a\alpha}(\partial_\alpha A^a_0 + f_{abc}A^b_0 A^c_\alpha) = \tilde{\Pi}^{a\alpha}(\partial_\alpha \tilde{A}^a_\alpha + f_{abc}\tilde{A}^b_0\tilde{A}^c_\alpha), \tag{3. 7}$$

trivial in the phase space of the theory since

$$\partial_\alpha \Pi^{a\alpha} - f_{abc}A^b_\alpha \Pi^{c\alpha} = 0 \tag{3. 8}$$

are the secondary constraints that result from the equations of motion of the primary constraints $\Pi^a_\alpha = 0$. As in the abelian case the original and canonically transformed Hamiltonians are defined in the same restricted phase space.

We must stress that in order to define correctly the phase space of the theory as parametrized by $\{{}^*F_{0\alpha}, \Pi^\alpha\}$ we have to introduce first order formalism for the initial Lagrangian. The detailed description is given in [16]. The idea is to introduce a Lagrangian $L[F, A]$, where now F are arbitrary two-forms in the manifold, arranged to give $F = dA - A \wedge A$ from the equation of motion for F. Now the F have no dynamical meaning since they have no time derivative, and the momenta are conjugate to the A-variables:

$$\Pi^{am} = \frac{\delta L[F, A]}{\delta \dot{A}^a_m} \tag{3. 9}$$

In this way the phase space variables Π_α and ${}^*F^{0\alpha}$ are not conjugate variables and are only related through the equations of motion.

The corresponding generating functional in the first order formalism is:

$$\mathcal{F} = -4\int_{M,t fixed} d^3x Tr(\tilde{A}_\alpha\,{}^*F^{0\alpha} + A_\alpha\,{}^*\tilde{F}^{0\alpha}) \tag{3. 10}$$

Within this approach strong-weak coupling duality is straightforwardly generalized to non-abelian gauge theories. The same arguments that we applied in the abelian case in order to obtain global properties can be applied in this case. For instance the dual variables must live in the dual lattice. This implies that for the case of a $SU(N)$ gauge group the dual theory is invariant under $SU(N)/Z_N$ gauge transformations, as was conjectured by Montonen and Olive for $N = 4$ Yang-Mills theories. Also, the same arguments yielding the modular factor in the transformation of the partition function in configuration space apply in this case.

The dual Lagrangian can also be obtained by manipulating the path integral in a very similar way to the abelian case (see [18, 19] for alternative derivations). Let us consider the following intermediate Lagrangian (in the sense that the initial and dual theories will be obtained from it):

$$L_I = \frac{i}{8\pi} Tr(\bar{\tau} F^+_{mn} F^{+mn} - \tau F^-_{mn} F^{-mn} + \epsilon^{mnpq} G_{mn} G_{pq}) \qquad (3.\ 11)$$

where F is an arbitrary 2-form in the manifold and $G \equiv dA - A \wedge A$. Integration over A produces the constraint:

$$\epsilon^{mnpq}(\partial_n F^{(a)}_{pq} - f_{abc} F^b_n F^{(c)}_{pq}) = 0 \qquad (3.\ 12)$$

In manifolds without non-trivial homology two-cycles[6] F must be the curvature tensor $F = dA - A \wedge A$. Substituting in (3. 11) we obtain

$$L = \frac{i}{8\pi} Tr((\bar{\tau}+2) F^+_{mn} F^{+mn} - (\tau+2) F^-_{mn} F^{-mn}). \qquad (3.\ 13)$$

Since the theory is invariant under $\tau \to \tau + 2$ (for even lattices it is invariant under $\tau \to \tau + 1$) we recover the original theory. If instead we integrate out the F-fields it is easy to see that the dual Lagrangian (3. 6) with $\tilde{F} = G$ is gotten. The dual gauge group is the dual of the original group (in the sense that the metric defined by its weight vectors is the inverse of the one in the original gauge group). In order to see this we have to proceed more carefully in our previous derivation. We take the gauge fields in the fundamental representation of the gauge Lie algebra and the metric defined by the weight vectors $g_{ab} \equiv d_{ab}$. Following the steps explained above from the Lagrangian:

$$L_I = \frac{i}{8\pi} (\bar{\tau} d_{ab} F^{(a)+}_{mn} F^{(b)+mn} - \tau d_{ab} F^{(a)-}_{mn} F^{(b)-mn} + \epsilon^{mnpq} G^{(a)}_{mn} F^{(a)}_{pq}) \qquad (3.\ 14)$$

we arrive to the dual Lagrangian (3. 6) with metric $\tilde{g}_{ab} = \tilde{d}_{ab}$ where $\tilde{d}_{ab} d^{bc} \equiv \delta^c_a$. Then the dual theory is defined on the dual Lie algebra. For instance in the case of $SU(2)$ gauge theories the dual group is $SO(3)$. The theory defined in $SU(2)$ is invariant under $\theta \to \theta + 2\pi$ ($\tau \to \tau + 1$) since in $SU(2)$:

$$\frac{1}{16\pi^2} \int Tr(F_{mn}\, {}^*F^{mn}) = n \qquad (3.\ 15)$$

with n an integer and the trace in the fundamental representation. For $SO(3)$ the instanton number is $n/4$ which implies that the dual theory is invariant under $\tilde{\theta} \to \tilde{\theta} + 8\pi$ or $\tilde{\tau} \to \tilde{\tau} + 4$. Then the dual theory is not invariant under the whole $SL(2, Z)$ but only under the subgroup generated by

$$\tau \to -\frac{1}{\tau}, \qquad \tau \to \tau + 4. \qquad (3.\ 16)$$

Similar considerations apply to $SU(N)$.

4 Generalization to p-forms abelian gauge theories

The generalization to p-forms abelian gauge theories in d dimensions is direct from what we have studied in section 2. We are going to consider the case $d = 2(p+1)$ which is the

[6] We have not studied in detail the case of manifolds with non-trivial homology two-cycles.

one in which both the initial and dual theories are expressed as functions of $(p+1)$-forms[7]. The generalized S-duality transformation is implemented in the path integral by gauging the global isometry

$$A \to A + \epsilon \tag{4. 1}$$

where now A and the gauge parameter are p-forms. The total derivative term that gives information about the generating functional of the canonical transformation is $d\tilde{A}\wedge dA$, with $\tilde{A}$, the Lagrange multiplier, also a p-form.

It is immediate to show that the canonical transformation is generated by the type-I generating functional[8]:

$$F = -\frac{1}{(p+1)!}\int d^d x d\tilde{A}\wedge dA \tag{4. 2}$$

which produces:

$$\Pi^{\alpha_1\ldots\alpha_p} = \frac{\delta F}{\delta A_{\alpha_1\ldots\alpha_p}} = -((p+1)!)^2\, {}^*\tilde{F}^{0\alpha_1\ldots\alpha_p} \tag{4. 3}$$

$$\tilde{\Pi}^{\alpha_1\ldots\alpha_p} = -\frac{\delta F}{\delta \tilde{A}_{\alpha_1\ldots\alpha_p}} = ((p+1)!)^2\, {}^*F^{0\alpha_1\ldots\alpha_p} \tag{4. 4}$$

The same relation (2. 20) for the wave functions holds in this case since the generating functional is linear in the initial and dual variables. From it we can obtain global information about the dual variables. We can also obtain the modular weight appearing in the transformation of the partition function [10].

Let us consider first the case p odd. $p+1$ is even and then the theory allows for a θ-term. In phase space (we omit the p indices):

$$Z_{ps} = \int \mathcal{D}A\mathcal{D}\Pi e^{-\frac{i}{8\pi}\int d^d x(\dot{A}\Pi - H)} = (\mathrm{Im}\tau)^{B_{p+1}/2}\int \mathcal{D}A e^{-S}, \tag{4. 5}$$

after regularizing the determinant coming from the gaussian integration on the momenta, $(p+1)$-forms in this case. The dual phase space partition function coincides with the initial one and it is given by:

$$\tilde{Z}_{ps} = \int \mathcal{D}\tilde{\Pi}\mathcal{D}\tilde{A} e^{-\frac{i}{8\pi}\int d^d x(\dot{\tilde{A}}\tilde{\Pi} - \tilde{H})} = (\mathrm{Im}\tau)^{B_{p+1}/2}\tau^{-B^-_{p+1}/2}\bar{\tau}^{-B^+_{p+1}/2}\int \mathcal{D}\tilde{A} e^{-\tilde{S}} \tag{4. 6}$$

The configuration space partition function is:

$$Z = (\mathrm{Im}\tau)^{N_p/2}\int \mathcal{D}A e^{-S} \tag{4. 7}$$

where we have followed the notation in [10], N_p being the dimension of the space of p forms after substracting all the gauge invariances (see [10] for the detailed analysis). In the dual model the partition function is the same with $\tau \to -1/\tau$. Then we have:

$$Z = \tau^{-\frac{\chi-\sigma}{4}}\bar{\tau}^{-\frac{\chi+\sigma}{4}}\tilde{Z} \tag{4. 8}$$

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[7] In the arbitrary case the dual theory would depend on $(d-p-1)$ forms

[8] Our convetions are: ${}^*F^{i_1\ldots i_{p+1}} = \frac{1}{(p+1)!}\epsilon^{i_1\ldots i_d}F_{i_{p+2}\ldots i_d}$ and $\tilde{F}\wedge F = \epsilon^{i_1\ldots i_d}\tilde{F}_{i_1\ldots i_{p+1}}F_{i_{p+2}\ldots i_d}$.

where $\chi = 2(-1)^p N_p + (-1)^{p+1} B_{p+1}$ is the Euler number and $\sigma = B^+_{p+1} - B^-_{p+1}$ the signature of the manifold.

In the case p even a θ-term does not exist. Similar arguments to the ones above yield:

$$Z = (\frac{4\pi}{g^2})^{\chi/2} \tilde{Z} \tag{4. 9}$$

All these results agree with the ones presented in [10].

5 Conclusions

We have seen that for non-supersymmetric abelian four dimensional gauge theories S-duality can be implemented as a canonical transformation in the phase space of the theory. This is easily generalized to the case of non-abelian gauge theories with arbitrary compact group. In this case the dual non-abelian gauge group is the dual of the original gauge group, as conjectured by Montonen and Olive [1] and tested by Vafa and Witten [2] for $N = 4$ Yang-Mills theories. In the non-abelian case we have also presented a way of implementing non-abelian S-duality in the path integral in configuration space.

We have seen that in phase space the partition function is invariant under S-duality and it is only after integrating out the momenta degrees of freedom that a modular factor appears and the partition function in configuration space transforms as a modular function. This applies for abelian and non-abelian gauge theories.

We have generalized the canonical transformation approach to arbitrary d-dimensional abelian gauge theories defined with p forms and obtained the corresponding modular weights appearing in the transformation of the partition function.

It could be very interesting to generalize the results presented in this paper to the case of supersymmetric gauge theories.

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