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The Kazhdan-Lusztig conjecture for $\mathcal{W}$-algebras

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ABSTRACT

The main result in this paper is the character formula for arbitrary irreducible highest weight modules of $\mathcal{W}$ algebras. The key ingredient is the functor provided by quantum Hamiltonian reduction, that constructs the $\mathcal{W}$ algebras from affine Kac-Moody algebras and in a similar fashion $\mathcal{W}$ modules from KM modules. Assuming certain properties of this functor, the $\mathcal{W}$ characters are subsequently derived from the Kazhdan-Lusztig conjecture for KM algebras. The result can be formulated in terms of a double coset of the Weyl group of the KM algebra: the Hasse diagrams give the embedding diagrams of the Verma modules and the Kazhdan-Lusztig polynomials give the multiplicities in the characters.

1 Introduction

$\mathcal{W}$ algebras were introduced more than a decade ago as (higher spin) extensions of the Virasoro algebra in the context of two dimensional conformal field theory [1]. Analogous to the Virasoro algebra, one expects that the representation theory of $\mathcal{W}$ algebras plays a crucial role in applications such as in conformal field theories with $\mathcal{W}$ symmetry, and in theories where the $\mathcal{W}$ symmetry is gauged ($\mathcal{W}$ strings and $\mathcal{W}$ gravity), see [2, 3] for reviews. For these applications, the relevant representations are highest weight modules. A basic goal is therefore to describe the irreducible modules, and more specifically to compute their characters.
There exists a general approach to find the irreducible characters from the characters of Verma modules. Any Verma module $M(x)$ can be decomposed into irreducible highest weight modules $L(y)$ (local composition series). This gives rise to character formulas of the form

$$\mathrm{ch}\, M(x) = \sum_y m_{xy}\, \mathrm{ch}\, L(y), \tag{1.1}$$

where m is a matrix whose entries m_{xy} count the number of times that $L(y)$ appears in the decomposition of $M(x)$. Doing this for all $M(x)$ such that m can be inverted gives

$$\mathrm{ch}\, L(x) = \sum_y m^{-1}_{xy}\, \mathrm{ch}\, M(y). \tag{1.2}$$

The characters of Verma modules are in general easy to compute, hence the computation of the characters of the irreducible modules boils down to determining the multiplicities m_{xy}.
This general programme has been applied successfully to the Virasoro algebra [4]. The key ingredient there is that every submodule of a Verma module is a sum of Verma modules. Since there is at most one embedding between Verma modules this implies that the multiplicities m_{xy} are 0 or 1, and the irreducible characters follow directly from the embedding pattern of the Verma modules. These embedding patterns are completely classified, and consequently for the Virasoro algebra, the characters of *all* irreducible highest weight modules are known.
For $\mathcal{W}$ algebras the submodule structure of Verma modules is much more complicated: in general submodules are not sums of Verma modules. Therefore the embedding patterns of the Verma modules do not determine the irreducible characters. This is directly related to the occurence of multiplicities $m_{xy} > 1$.
There are of course also other approaches. For instance, for the $\mathcal{W}_N$ minimal models, the irreducible character $\mathrm{ch}\, L(x)$ (for x inside the Kac-table) has been determined directly, using free field methods [5]. In terms of the multiplicities this amounts to having computed a single row of m^{-1}. It does not appear to be possible to apply these methods to compute the other rows, which is necessary to determine the characters of all irreducible highest weight modules (*i.e.* also for x outside or on the boundary of the Kac-table). In a way, the results of [5] for $\mathcal{W}_N$ algebras amount to having the $\mathcal{W}$ analogue of the Weyl-Kac character formula for affine Kac-Moody algebras.
For affine KM algebras the characters are known beyond the Weyl-Kac character formula. For $k + h^\vee \neq 0$ the programme described above has been fully completed. The result can be summarized as follows:
(1) The weights y appearing in the decomposition (1.1) are determined by a subgroup of the affine Weyl group, and the associated Bruhat ordering (the Kac-Kazhdan condition [6])

(2) The multiplicities m_{xy} are given in terms of the Kazhdan-Lusztig polynomials associated to the affine Weyl group (the Kazhdan-Lusztig conjecture [7, 8])

The main ingredient in the proof of (1) is the Jantzen filtration, whereas (2) has been proven using the intersection cohomology of Schubert varieties (only for integral weights, for other weights it is still a conjecture). Neither of these concepts seems to have been worked out for $\mathcal{W}$ algebras.

It is now interesting to note that $\mathcal{W}$ algebras and KM algebras are intimately related. In particular, a large class of $\mathcal{W}$ algebras can be obtained from affine KM algebras by (quantum) Hamiltonian reduction, where one imposes certain constraints on the KM generators (see [9] for a review). In this way a $\mathcal{W}$ algebra can be constructed for every embedding of sl_2 into the simple Lie algebra underlying the affine KM algebra [10]. The quantum construction naturally allows for a BRST formulation, in which the $\mathcal{W}$ algebra arises as the BRST cohomology of a complex involving the KM algebra [11, 12, 13]. Of course, given an sl_2 embedding, one can also compute the cohomology of a KM module. By construction, the result will be a module of the corresponding $\mathcal{W}$ algebra. Thus, one obtains in a natural way a functor from KM modules to $\mathcal{W}$ modules. The action of this 'reduction functor' is in general hard to compute. In [12], the action on (resolutions of) admissible KM modules was computed for principal sl_2 embeddings, assuming certain properties of the reduction functor. This way the characters of the $\mathcal{W}_N$ minimal models are recovered.

The main new idea in this paper is to apply the reduction functor to 'arbitrary' KM modules, to find the analogues of the general results (1) and (2) for $\mathcal{W}$ algebras. The result is a natural generalization of the KL conjecture to $\mathcal{W}$ algebras associated to arbitrary sl_2 embeddings. We show how this 'KL conjecture for $\mathcal{W}$ algebras' can be derived from the KL conjecture for KM algebras, assuming similar properties as in [12] of the reduction functor. These assumptions are motivated by the results [14] for finite $\mathcal{W}$ algebras. The upshot is that *all* irreducible characters for such $\mathcal{W}$ algebras are thereby determined. We verified the conjecture for a nontrivial set of $\mathcal{W}_3$ modules.

The setup of this paper is as follows. In section 2, we review the representation theory and KL conjectures of affine KM algebras, including a discussion of the translation functor that serves as a helpful analogy with the reduction functor. Then in section 3, after some remarks on the representation theory of general $\mathcal{W}$ algebras, we present the main result of this paper in section 3.2, the KL conjecture for $\mathcal{W}$ algebras. We also give an idea of how it can be derived using the reduction functor. Several applications of the conjecture are discussed in section 4. Properties of Coxeter groups and their KL polynomials are given in the appendix.

2 The Kazhdan-Lusztig conjectures for affine Kac-Moody algebras

We first present a collection of results concerning affine KM algebras and their highest weight modules, leading to the KL conjectures. Virtually everything stated here can be found somewhere in the mathematical literature on the subject, or can be concluded directly from it. We have avoided a rigorous presentation, but instead focussed on the line of thought, and made clear what is well-established and what is conjectured. For background and explanations on KM algebras and the structure of the highest weight modules we refer to [15, 16], and for Weyl groups and KL polynomials to [17].

2.1 Composition series and character formulae

Let g be an affine KM algebra, and fix a triangular decomposition $g = n_+ \oplus h \oplus n_-$ in positive root generators, Cartan subalgebra and negative root generators. A singular vector v_λ is an eigenvector of the generators of the Cartan subalgebra h with weight $\lambda \in h^*$, and is annihilated by the positive root generators. A highest weight module is a module that is generated from a singular vector, the highest weight vector, by the action of the negative root generators. There are two important examples of highest weight modules. The first is the Verma module $M(\lambda)$, which is uniquely defined by the property that it is generated freely from v_λ. The second is the quotient of $M(\lambda)$ by its maximal proper submodule, which gives the unique irreducible highest weight module $L(\lambda)$.

Highest weight modules themselves are special examples of modules in the so-called category $\mathcal{O}$ [18, 8]. In general this category consists of modules V which have a weight space decomposition

$$V = \oplus_{\mu \leq \lambda} V_\mu \ , \tag{2.1}$$

where the μ's satisfy $\mu \leq \lambda$ for λ in some finite subset of h^* (recall that $\mu \leq \lambda$ iff $\lambda - \mu$ is on the positive root lattice Q_+ of g) and $\dim V_\mu < \infty$. The category $\mathcal{O}$ contains highest weight modules, tensor products, submodules, quotients, etc.

For every module V in $\mathcal{O}$, one can define a (formal) character $\mathrm{ch}\, V$,

$$\mathrm{ch}\, V = \sum_\mu \dim V_\mu \, e^\mu \ , \tag{2.2}$$

where the formal exponentials satisfy $e^\lambda e^\mu = e^{\lambda+\mu}$ and $e^0 = 1$. The character of the Verma module $M(\lambda)$ is given by

$$\mathrm{ch}\, M(\lambda) = e^\lambda \sum_{\gamma \in Q_+} P(\gamma) e^{-\gamma} = e^\lambda \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{-\dim g_\alpha} \ , \tag{2.3}$$

where g_α is the root space of root α, Δ_+ is the set of positive roots and $P(\gamma)$ is the (generalized) Kostant partition function. One of the central problems of representation theory is to find the characters of the irreducible highest weight modules $L(\lambda)$. The strategy is to relate these to the explicit characters of Verma modules (2.3). This is possible due to the following general structure theorem, which also illustrates that $\mathcal{O}$ is natural in the context of highest weight modules (in particular, the $L(\lambda)$'s are the only irreducibles in $\mathcal{O}$). Every module V in the category $\mathcal{O}$ has a local composition series at any weight λ of V. A local composition series for V at λ is a sequence of submodules of V, $V = V_0 \supset V_1 \supset \ldots \supset V_{n-1} \supset V_n = 0$, such that either $V_i/V_{i+1} \cong L(\mu)$ for some $\mu \geq \lambda$, or $(V_i/V_{i+1})_\mu = 0$ for all $\mu \geq \lambda$. One denotes by $[V : L(\mu)]$ the number of times that $L(\mu)$ appears in the local composition series of V at λ, it is called the multiplicity of $L(\mu)$ in V. It is independent of the particular sequence of submodules one chooses. We stress that $[V : L(\mu)]$ does not count the number of singular vectors at weight μ in V: the statement that $V_i/V_{i+1} \cong L(\mu)$ only requires that there is a vector v_μ which is singular in the quotient V_i/V_{i+1} but not necessarily singular in V_i, let alone V. A vector that is singular in a quotient of submodules is called primitive, and the corresponding weight is called a primitive weight. Obviously, a singular vector is also primitive, but it is important to realize that there are also other types of primitive vectors. We also stress that the multiplicities $[M : L]$ can be larger than 1, contrary to what was initially thought based

on the known trivial multiplicities of the simple Lie algebras $\bar{g} = \overline{sl}_2, \overline{sl}_3$ and the affine KM algebra $g = sl_2$.
At the level of characters, the local composition series implies that (2.2) is given by a sum over the irreducible characters: $\mathrm{ch}\, V = \sum_\mu [V : L(\mu)]\, \mathrm{ch}\, L(\mu)$ where the sum runs over the weights of V (of course, only the primitive weights give a non-vanishing contribution). This applies in particular to Verma modules, leading to

$$\mathrm{ch}\, M(\lambda) = \sum_{\mu \leq \lambda} [M(\lambda) : L(\mu)]\, \mathrm{ch}\, L(\mu)\ . \tag{2.4}$$

Note that the composition series starts with $[M(\lambda) : L(\lambda)] = 1$, since $\dim M(\lambda)_\lambda = 1$. Ordering the set of weights $\mu \leq \lambda$ as $\lambda = \mu_0,\ \mu_1,\ \mu_2,\ \ldots$ such that $j \geq i$ whenever $\mu_j \leq \mu_i$, one has the following set of equations,

$$\mathrm{ch}\, M(\mu_i) = \sum_{\mu_j \leq \mu_i} [M(\mu_i) : L(\mu_j)]\, \mathrm{ch}\, L(\mu_j)\ .$$

The matrix $[M(\mu_i) : L(\mu_j)]$, called the Jantzen matrix (for λ), is upper triangular with ones on the main diagonal. Therefore, it can be inverted. Denoting the inverse matrix elements by $(L(\mu_i) : M(\mu_j))$ (which are possibly negative integers), one finds

$$\mathrm{ch}\, L(\mu_i) = \sum_{\mu_j \leq \mu_i} (L(\mu_i) : M(\mu_j))\, \mathrm{ch}\, M(\mu_j).$$

In conclusion, from (2.4) one finds the character formula

$$\mathrm{ch}\, L(\lambda) = \sum_{\mu \leq \lambda} (L(\lambda) : M(\mu))\, \mathrm{ch}\, M(\mu)\ . \tag{2.5}$$

Here $\mathrm{ch}\, M(\mu)$ is given through (2.3). Computing $\mathrm{ch}\, L(\lambda)$ boils down to computing the numbers $(L(\lambda) : M(\mu))$ for all $\mu \leq \lambda$, or equivalently, the Jantzen matrix $[M(\mu_i) : L(\mu_j)]$ for λ.

2.2 The Kac-Kazhdan conditions

The first step in determining the multiplicities $[M(\lambda) : L(\mu)]$ is to find all pairs λ, μ such that $[M(\lambda) : L(\mu)] \neq 0$. The general solution to this problem has been given by Kac and Kazhdan [6], using the generalized Casimir of g and the Jantzen-filtration of Verma modules [15]. For the purposes of this paper it is sufficient to consider only weights λ with $\langle \lambda + \rho, \delta \rangle = k + h^\vee \neq 0$. In that case the result of [6] can be rephrased in terms of properties of the affine Weyl group [19].
The affine Weyl group W is a Coxeter group, generated by the simple reflections s_i where $s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$ are the reflections in the simple roots α_i of g. Arbitrary elements $w \in W$ correspond to expressions $w = s_{i_1} s_{i_2} \ldots s_{i_k}$. The minimal number of simple reflections needed to generate w is called the length $\ell(w)$ of w. An expression of minimal length is called reduced. If $w, w' \in W$ are two reduced expressions, then we denote $w < w'$ if the reduced expression for w can be obtained by dropping simple reflections from a reduced expression for w'. The resulting relation $w \leq w'$ is a partial ordering of W, called the Bruhat ordering.

An important ingredient in what follows is the subgroup $W_\lambda \subset W$: it is the group generated by reflections $r_{\hat\alpha}$ with $\hat\alpha \in \Delta^{re}_{\lambda,+} = \{\alpha \in \Delta^{re}_+ | \langle \lambda+\rho, \alpha^\vee \rangle \in \mathbf{Z}\}$. Clearly, only if λ is integral, $W_\lambda = W$ (recall that $\rho \in h^*$ satisfies $\langle \rho, \alpha_i^\vee \rangle = 1$), otherwise W_λ will be a proper subgroup of W (which for affine W may be isomorphic to W). It can be shown that W_λ is again a Weyl group, it is generated by simple reflections $\hat s_i = r_{\hat\alpha_i}$ (simple in W_λ) where $\hat\alpha_i$ are the simple roots of the rootsystem $\Delta^{re}_{\lambda,+}$. The length function on W_λ is denoted $\ell_\lambda(w)$. Obviously, the relation between λ and W_λ is many-to-one, for instance $W_\lambda = W_{\lambda+\mu}$ for arbitrary integral weight μ. In fact, up to isomorphisms, there is only a finite number of W_λ [20].
The groups W_λ organize the non-vanishing multiplicities in the following way: the primitive weights of $M(\lambda)$ are on the shifted Weyl orbit $W_\lambda.\lambda$, where

$$w.\lambda \equiv w(\lambda+\rho) - \rho, \tag{2.6}$$

and vice versa: only the lower weights (with respect to Bruhat ordering) on the orbit are primitive weights of $M(\lambda)$. In the remainder of this section we describe this in more detail.

First consider $k + h^\vee > 0$. Then every orbit $W_\mu.\mu$ has precisely one maximal element λ, the dominant weight, such that $w.\lambda \leq \lambda$ for all $w \in W_\mu$. Using (2.6) it is easy to see that such a dominant weight λ is characterised by

$$\langle \lambda + \rho, \hat\alpha_i^\vee \rangle \geq 0. \tag{2.7}$$

Clearly, there is a one-to-one correspondence between dominant weights λ and orbits $W_\lambda.\lambda$. There may not be a one-to-one correspondence between elements of W_λ and the weights on the orbit $W_\lambda.\lambda$. This happens precisely if there is a subgroup W^0_λ of W_λ which leaves λ invariant. W^0_λ is a finite parabolic subgroup of W_λ, generated by the simple reflections $r_{\hat\alpha_i}$ with $\hat\alpha_i$ satisfying $\langle \lambda+\rho, \hat\alpha_i^\vee \rangle = 0$. A dominant weight is called regular if W^0_λ is trivial, and it is called singular otherwise. Thus, weights on the orbit $W_\lambda.\lambda$ of a dominant weight are in one-to-one correpondence with elements of the coset

$$W_\lambda / W^0_\lambda, \tag{2.8}$$

i.e. any weight μ can be written uniquely as $\mu = w.\lambda$ with λ dominant and $w \in W_\lambda/W^0_\lambda$. This coset will be crucial in what follows: in particular *the multiplicities depend on λ only through the coset W_λ/W^0_λ!*
Denote $M_w = M(w.\lambda)$ and $L_w = L(w.\lambda)$, then the Kac-Kazhdan condition for $k + h^\vee > 0$ can be described as follows

$$[M_w : L_{w'}] \neq 0 \quad \text{iff} \quad w \leq w' \quad \text{with } w, w' \in W_\lambda/W^0_\lambda. \tag{2.9}$$

Here, the ordering on the coset W_λ/W^0_λ is induced from the Bruhat ordering on W_λ:

$$w \leq w' \quad \text{with}\, w, w' \in W_\lambda/W^0_\lambda \qquad \text{iff} \qquad \underline{w} \leq \underline{w}' \quad \text{with}\, \underline{w}, \underline{w}' \in W_\lambda. \tag{2.10}$$

(here $\underline{w}$ is the minimal coset representative of w in the coset, defined through $\ell(\underline{w}s) > \ell(\underline{w})$ for all $s \in W^0_\lambda$. Of course we could also have chosen the maximal representatives $\overline{w}$ which have $\ell(\overline{w}s) < \ell(\overline{w})$ for all $s \in W^0_\lambda$).

For the character formulas (2.4) and (2.5) the Kac-Kazhdan result implies the following. First of all, the sum over the weight space in (2.4) reduces to a sum over $w' \in W_\lambda/W_\lambda^0$

$$\mathrm{ch}\, M_w = \sum_{w' \geq w} [M_w : L_{w'}]\, \mathrm{ch}\, L_{w'}. \tag{2.11}$$

Secondly, using transitivity of the Bruhat order this can be inverted

$$\mathrm{ch}\, L_w = \sum_{w' \geq w} (L_w : M_{w'})\, \mathrm{ch}\, M_{w'}. \tag{2.12}$$

Unlike the sum in (2.11) not all terms in this sum have to be nonvanishing.

For weights with $k + h^\vee < 0$ the result can be rephrased analogously. We note that weights with $k + h^\vee < 0$ are the image of weights with $k + h^\vee > 0$ under the shifted inversion

$$\sigma.\lambda = -\lambda - 2\rho. \tag{2.13}$$

Clearly, $W_{\sigma.\lambda} = W_\lambda$, so σ is also a one-to-one map between the orbits on either side (orbits always belong to one side only as W leaves $k + h^\vee$ invariant). Since σ reverses the order of weights, every orbit now will have a minimal weight, called anti-dominant, which is of the form $\sigma.\lambda$ with λ dominant. In terms of these anti-dominant weights one has the analogue of (2.9) describing the full KK condition for $k + h^\vee < 0$

$$[M_w : L_{w'}] \neq 0 \quad \text{iff} \quad w \geq w' \quad \text{with } w, w' \in W_\lambda/W_\lambda^0. \tag{2.14}$$

Thus one finds the same character formulas (2.11) and (2.12) but with the sum over $w' \leq w$.

2.3 Embeddings of Verma modules

In the previous section we have discussed the role of the cosets W_λ/W_λ^0 in finding the primitive weights of a Verma module $M(\lambda)$. In this section we discuss how the same cosets also describe the embeddings between Verma modules.
This is based on the property of KM Verma modules that at every primitive weight there is at least one singular vector [6]. Since a singular vector v_μ in a Verma module $M(\lambda)$ give rise to a homomorphism $M(\mu) \hookrightarrow M(\lambda)$ (embedding) between Verma modules, this statement implies that there is a homomorphism iff the multiplicity $[M(\lambda) : L(\mu)]$ is nonvanishing. Hence

$$M_{w'} \hookrightarrow M_w \quad \text{iff} \quad w \leq w' \quad \text{with } w, w' \in W_\lambda/W_\lambda^0. \tag{2.15}$$

In other words: the diagram representing the embeddings of the Verma modules is given by the Hasse diagram of the coset W_λ/W_λ^0: the vertices of this diagram are the elements of the coset and the links between the vertices connect the adjacent elements (two coset elements x, y are called adjacent if there is no third coset element z such that $x < z < y$). Since one can classify the Hasse diagrams, this gives a classification of embedding diagrams.

In fact, if $k+h^\vee \neq 0$ the relation between embeddings and the Hasse diagram is even stronger, because in that case there is at most 1 singular vector at every primitive weight. This implies that the homomorphism $M(\mu) \hookrightarrow M(\lambda)$ is unique, or

$$\dim \mathrm{Hom}(M(\mu), M(\lambda)) \leq 1. \tag{2.16}$$

This can be argued as follows. If there is a sequence $M(\mu_1) \hookrightarrow M(\mu_2) \hookrightarrow M(\mu_3)$ of homomorphisms, the embedding property implies that

$$\dim \mathrm{Hom}(M(\mu_1), M(\mu_3)) \geq \dim \mathrm{Hom}(M(\mu_2), M(\mu_3)). \tag{2.17}$$

For $k + h^\vee < 0$ any Verma module contains always a lowest primitive weight (the anti-dominant weight). At this weight, there is precisely one singular vector (because any two embedded Verma modules necessarily overlap). This immediately implies (2.16).
For $k + h^\vee > 0$ there is no lowest primitive weight. In that case (2.16) follows from the result for $k + h^\vee < 0$ through the 'reflection principle' of semi-infinite homology [21],

$$\mathrm{Hom}(M(\mu), M(\lambda)) \simeq \mathrm{Hom}(M(\sigma.\lambda), M(\sigma.\mu)). \tag{2.18}$$

2.4 Jantzens translation functor

In this section we discuss how the multiplicities for *arbitrary* dominant weights follow from the multiplicities for *regular* dominant weights. The idea is to use Jantzens translation functor [15, 8] to map modules with regular weights (trivial W^0_λ) to modules with singular weights (nontrivial W^0_λ). The reason for highlighting this ingredient here is the striking similarity between this derivation and the derivation of $\mathcal{W}$ multiplicities from KM multiplicities using the reduction functor in section 3.2.
Let λ' be a singular dominant weight, and let λ be a regular dominant weight such that $\lambda - \lambda'$ is an integral weight. Clearly, $W_\lambda = W_{\lambda'}$, but W^0_λ is trivial whereas $W^0_{\lambda'}$ is not. The tensorproduct with the irreducible module associated with $\lambda' - \lambda$ gives rise to an exact functor [15, 8, 22] (the translation functor) that maps

$$M(w.\lambda) \overset{t}{\mapsto} M(w.\lambda'). \tag{2.19}$$

To obtain the action of the translation functor on irreducible modules, observe that for Verma modules $M(w'.\lambda) \hookrightarrow M(w.\lambda)$ with w, w' in the same coset, the functor maps the quotient $M(w.\lambda)\backslash M(w'.\lambda)$ (which contains $L(w.\lambda')$) to zero, so it immediately follows that

$$L(w.\lambda) \overset{t}{\mapsto} L(w.\lambda')\delta_{\overline{w},w}. \tag{2.20}$$

(with $\overline{w}$ is the maximal representative of w in the coset $W_\lambda/W^0_{\lambda'}$). The maps (2.19) and (2.20) determine the multiplicities for singular weights from the multiplicities of the regular weights:

$$[M(w.\lambda') : L(w'.\lambda')] = [M(\overline{w}.\lambda) : L(\overline{w}'.\lambda)]. \tag{2.21}$$

Another useful application of the translation functor is the computation of the character of the irreducible module L_e for regular dominant weights λ (without having to determine the full Jantzen matrix). For such weights namely, the sum in (2.12) runs over all the elements of W_λ. Applying (2.19) and (2.20) to it for a translation chosen such that $W^0_{\lambda'}$ contains just one reflection, gives that the coefficients are given by $\varepsilon_w = (-1)^{\ell_\lambda(w)}$ [22], hence

$$\operatorname{ch} L_e = \sum_{w\in W_\lambda} \varepsilon_w \operatorname{ch} M_w. \tag{2.22}$$

This is the generalisation of the Weyl-Kac formula [16] to arbitrary regular dominant weights. The same trick cannot be applied to obtain arbitrary characters (*i.e.* $\operatorname{ch} L_w$ or for λ singular). It is this particular character formula (for admissible λ) that forms the starting point of [12] for generalization to $\mathcal{W}$ algebras.

2.5 The KL conjectures

Now we are ready to describe the final step, *i.e.* to give the Kahdan-Lusztig formula for the multiplicities. In [7], Kazhdan and Lusztig defined a set of polynomials $P_{x,y}(q)$, labelled by pairs of elements x, y for an arbitrary Coxeter group W, and depending on a single variable q. For details and properties about the definition of these polynomials see the appendix, important for us is that they can be computed explicitly from a recursion relation (see (A.8))

$$P_{x,ys} = q^{1-c}P_{xs,y} + q^c P_{x,y} - q \sum_{\substack{x\le z<y\\ zs<z}} P_{x,z}\not{P}_{z,y}. \tag{2.23}$$

The simple reflection s is chosen such that $y < ys$, such that the polynomials $P_{x,y}$ are expressed in terms of polynomials $P_{x',y'}$ with $\ell(y') < \ell(y)$.
Similarly, one defines a set of inverse polynomials $Q_{x,y}(q)$ through

$$\sum_{x\le z\le w} P_{x,z}(q)Q_{z,y}(q)\varepsilon_z\varepsilon_y = \delta_{x,y} \tag{2.24}$$

which can also be computed directly from a recursion relation (see (A.18))

$$Q_{x,ys} = cQ_{xs,y} + (-q)^c Q_{x,y} + cq \sum_{\substack{x<z\le y\\ zs>z}} \not{Q}_{x,z}Q_{z,y} \tag{2.25}$$

Analogously, one may also associate KL polynomials P^I, Q^I to a coset W/W_I for W_I a parabolic subgroup of W. If W_I is finite these are related to the KL polynomials on W as follows

$$P^I_{x,y} = P_{\overline{x},\overline{y}}, \qquad Q^I_{x,y} = Q_{\underline{x},\underline{y}} \tag{2.26}$$

Here $\underline{z}$ and $\overline{z}$ are the minimal and maximal representatives of z in the coset $[z]$. In general, the polynomials P^I and Q^I are not each others inverse. The inverse polynomials of P^I, Q^I are denoted $\tilde{Q}^I, \tilde{P}^I$, they are defined through

$$\sum_{x\le z\le y} \tilde{Q}^I_{x,z}P^I_{z,y} = \sum_{x\le z\le y} Q^I_{x,z}\tilde{P}^I_{z,y} = \delta_{x,y}. \tag{2.27}$$

They can also be expressed in terms of the polynomials on W

$$\tilde{P}^I_{x,y} = \sum_{z\in[x]} P_{z,\underline{y}}\varepsilon_z\varepsilon_{\underline{y}}, \qquad \tilde{Q}^I_{x,y} = \sum_{z\in[y]} Q_{\overline{x},z}\varepsilon_{\overline{x}}\varepsilon_z \tag{2.28}$$

The KL conjectures relate the multiplicities in the character formulas to the value of these polynomials at $q=1$. Let λ be a dominant weight with coset W_λ/W^0_λ, $P_{w,w'}$ the KL polynomials for W_λ and $Q_{w,w'}$ the associated inverse KL polynomials. Then the multiplicities are given by [7, 8, 15, 23]

$$\begin{array}{lll} k+h^\vee>0: & [M_w:L_{w'}] = P^I_{w,w'}(1), & (L_w:M_{w'}) = \tilde{Q}^I_{w,w'}(1) \\ k+h^\vee<0: & [M_w:L_{w'}] = Q^I_{w',w}(1), & (L_w:M_{w'}) = \tilde{P}^I_{w',w}(1) \end{array} \tag{2.29}$$

(the superscript I refers to the subgroup W^0_λ) These conjectures have been proven for integral weights, in [25] for $k+h^\vee>0$, and [26] for $k+h^\vee<0$. It is not inconceivable that the conjectures for $k+h^\vee>0$ are related to the conjecture for $k+h^\vee<0$ through the semi-infinite cohomology of affine KM algebras.
The conjectures naturally fit in a circle of ideas generally referred to as Kazhdan-Lusztig theory. This theory interrelates many different problems, such as the classification of primitive ideals in enveloping algebras, the computation of the multiplicities in composition series and the intersection cohomology of Schubert varieties (see [24] for an overview). It applies in particular to simple Lie algebras, affine KM algebras and quantum groups. In the next section we show that it also applies to $\mathcal{W}$ algebras.

3 The KL conjectures for $\mathcal{W}$ algebras

Compared to the situation for affine Kac-Moody algebras, relatively little is known about the representation theory of $\mathcal{W}$ algebras. The fact that a classification of such algebras is still lacking makes it harder to give a general approach to this problem. We claim, however, that for the class of $\mathcal{W}$ algebras obtained through hamiltonian reduction of affine KM algebras, the analogue of most results described in the previous section exists. In particular, we formulate the KL conjecture for such $\mathcal{W}$ algebras.

3.1 Some generalities on $\mathcal{W}$ algebras and modules

Let us first consider a general $\mathcal{W}$ algebra, generated by the modes of a finite set of quasiprimary fields (for a precise definition see [2]). The $\mathcal{W}$ algebra will have a CSA h, *i.e.* a maximal abelian subalgebra of the zero modes. Unlike for KM algebras, the adjoint action of h on the generators of the $\mathcal{W}$ algebra is in general not diagonalizable (*e.g.* the zero-mode W_0 of the spin 3 field of $\mathcal{W}_3$); therefore the 'triangular' decomposition of $\mathcal{W} = \mathcal{W}_+ \oplus h \oplus \mathcal{W}_-$ in positive root generators, Cartan subalgebra and negative root generators is given with respect to a subalgebra $h' \subset h$

$$\mathcal{W} = \oplus_{a'}\mathcal{W}_{-a'} \oplus h \oplus_{a'} \mathcal{W}_{a'}, \tag{3.1}$$

where a' runs over the set of positive roots Δ'_+. By assumption $\mathcal{W}_{-a'} \cong \mathcal{W}_{a'}$ as vector spaces, paired by an involutive map $\sigma : \mathcal{W}_{-a'} \to \mathcal{W}_{a'}$.

The set-up of representation theory is similar to that of affine KM algebras, in the following sense. A singular vector v_a is an eigenvector of the generators of h with weight $a \in h^*$, and v_a is annihilated by all positive root generators. A highest weight module V is generated from v_a by the action of the negative root generators. Similarly, one introduces a category $\mathcal{O}$, which consists of modules V which have a weight space decomposition into a direct sum of weight spaces of the subalgebra h',

$$V = \oplus_{b' \le a'} V_{b'}, \tag{3.2}$$

where the sum is over weights b' satisfying $b' \le a'$ for a' in some finite subset of h'^*, and $\dim V_{b'} < \infty$ (note that $b' \le a'$ iff $a' - b'$ is on the positive root lattice Q'_+ of h').
The category $\mathcal{O}$ again contains Verma modules $M(a)$, irreducible quotients $L(a)$, submodules, etc (but no tensor products as in general the tensor product of two $\mathcal{W}$ modules is not a $\mathcal{W}$ module).
For every module V in $\mathcal{O}$ one can define a (formal) character $\mathrm{ch}\, V$,

$$\mathrm{ch}\, V = \sum_{b'} \dim V_{b'} e^{b'}. \tag{3.3}$$

The Verma module $M(a)$ has character formula

$$\mathrm{ch}\, M(a) = e^{a'} \sum_{b' \in Q'_+} P(b') e^{-b'} = e^{a'} \prod_{b' \in \Delta'_+} (1 - e^{-b'})^{-\dim \mathcal{W}_{b'}}, \tag{3.4}$$

where $P(b')$ is some generalized Kostant partition function.
The finite dimensionality of the weight spaces $V_{b'}$ implies that the action of the generators of the CSA h outside h' is reasonably well-behaved: every weight space $V_{b'}$ can be decomposed into a finite number of Jordan blocks U_b,

$$V_{b'} = \oplus_b U_b. \tag{3.5}$$

This implies that one can make local composition series in $\mathcal{O}$, where the irreducible quotients are again the highest weight modules $L(b)$, occurring with multiplicities $[V\!:\!L(b)]$. This leads to character formulas $\mathrm{ch}\, V = \sum_b [V : L(b)] \mathrm{ch}\, L(b)$, where of course b can only appear in the sum if b' is a weight of V. This applies in particular to a Verma module $M(a)$, leading to

$$\mathrm{ch}\, M(a) = \sum_{b' \le a'} [M(a)\!:\!L(b)] \mathrm{ch}\, L(b), \tag{3.6}$$

where clearly $[M(a)\!:\!L(a)] = 1$. Once again, this character formula can be inverted, such that the characters of irreducible modules can be expressed in characters of Verma modules

$$\mathrm{ch}\, L(a) = \sum_{b' \le a'} (L(a)\!:\!M(b)) \mathrm{ch}\, M(b). \tag{3.7}$$

To conclude: also for $\mathcal{W}$ algebras, the general strategy to find character formulas is to compute the multiplicities $[M(a)\!:\!L(b)]$. This is what we will do in the next section.

3.2 $\mathcal{W}$ modules from sl_2 reductions

A large class of $\mathcal{W}$ algebras can be obtained through a procedure of (quantum) Hamiltonian reduction of affine KM algebras [9]. A particularly nice set of reductions are those related to sl_2 embeddings [10]. For every sl_2 embedding into the simple Lie algebra underlying the untwisted affine KM algebra, one can define a BRST complex such that the associated cohomology is non-vanishing only in the zero-th term. This cohomology is a $\mathcal{W}$ algebra [11, 12, 13].
Similarly, on the level of the representation theory, the cohomology of a complex associated to a KM module gives a $\mathcal{W}$ module. This defines a functor from the category of KM modules to the category of $\mathcal{W}$ modules. We assume the following properties of this reduction functor [12, 14]: (1) the cohomology of the BRST complex associated to the KM module is non-vanishing only in the zero-th term, (2) KM Verma modules $M(\lambda)$ are mapped to $\mathcal{W}$ Verma modules $M(a(\lambda))$, and (3) a local composition series of a KM Verma module is mapped to a local composition series of the corresponding $\mathcal{W}$ Verma module.

From these assumptions it immediately follows that, when acting on KM irreducible modules, the reduction functor maps

$$L(\lambda) \to L(a(\lambda)) \quad \text{or} \quad L(\lambda) \to 0, \tag{3.8}$$

where at least one of the maps is to be non-trivial. If one knows which $L(\lambda)$ have vanishing or non-vanishing cohomology, then the multiplicities of $\mathcal{W}$ Verma modules are determined. The main result of this paper is an explicit formula for these multiplicities, in terms of KL polynomials associated to a double coset which is completely fixed by the reduction data. Note that the reduction only gives rise to a $\mathcal{W}$ algebra for $k + h^\vee \neq 0$, *i.e.* precisely those weights for which the KM multiplicities are given by the KL conjecture. This implies that one has the complete KL conjecture for this class of $\mathcal{W}$ algebras, so that the characters of all irreducible highest weight $\mathcal{W}$ modules are known.

Let us explain how this should work. Associated to the particular sl_2-reduction is a regular subalgebra g_r of the finite-dimensional simple Lie algebra $\bar{g}$ underlying the affine Kac-Moody algebra g [14]. The sl_2 subalgebra is principally embedded into g_r. This embedding determines a set of constraints which can be chosen in such a way that they involve only positive roots. This is necessary to get non-vanishing cohomology from KM Verma modules. In explicit examples it is possible to verify that this cohomology is given by a Verma module of the corresponding $\mathcal{W}$ algebra [12, 14]. We assume that this holds in general. From the results of [14] we expect that the parametrization $a(\lambda)$ of the $\mathcal{W}$ weight is invariant under the shifted action of the Weyl group W^r of g_r (which is a finite parabolic subgroup of W). More precisely

$$a(w.\lambda) = a(\lambda) \quad \text{iff} \quad w \in W^r, \tag{3.9}$$

so there is a one-to-one correspondence between the $\mathcal{W}$-weights and the invariants of the Weyl group W^r. Using this parametrization we will from now on denote Verma modules and irreducible modules for the $\mathcal{W}$ algebra by $M^r(\lambda)$ and $L^r(\lambda)$ with λ a weight of g. Up to W^r invariance, the labelling by g weights fixes the $\mathcal{W}$ weights uniquely.
Let λ be a dominant weight. From the existence of the composition series it follows that the set of primitive weights in a $\mathcal{W}$ Vermas module $M^r(\lambda)$ is contained in the orbit of the double coset

$$W^r_\lambda \backslash W_\lambda / W^0_\lambda, \tag{3.10}$$

where $W^r_\lambda = W^r \cap W_\lambda$. From the embedding property (2.15) of KM Verma modules it now follows that for each weight on this orbit there is an embedding of $\mathcal{W}$ Verma modules, thus there is a one-to-one correspondence between primitive weights and weights on the orbit of the double coset (3.10).
It is instructive to note the analogy with the translation functor discussed in section 2.4: the translation functor maps regular KM Verma modules $M(\lambda)$ to arbitrary KM Verma modules $M(\lambda')$, such that the relevant cosets W_λ are mapped to $W_{\lambda'}/W^0_{\lambda'}$. Similarly, the reduction functor maps arbitrary KM Verma modules $M(\lambda)$ to arbitrary $\mathcal{W}$ Verma modules $M^r(\lambda)$, such that the relevant cosets W_λ/W^0_λ are mapped to $W^r_\lambda\backslash W_\lambda/W^0_\lambda$. Indeed, the derivation of the $\mathcal{W}$ multiplicities from KM multiplicities from this point on goes completely analogous to the derivation in section 2.4.
The irreducible $\mathcal{W}$ module $L^r(\mu)$ may arise only as the cohomology of the KM modules $L(w.\mu)$ with $w \in W^r$. Oviously, the cohomology of the associated KM Verma modules $M(w.\mu)$ are identical. Therefore, the cohomology of $L(\mu)$ must vanish when there is a $w \in W^r_\mu$ such that $M(w.\mu) \subset M(\mu)$ with $w.\mu \neq \mu$. On every W^r_μ orbit of μ, only the lowest weight contributes therefore.
It follows that the reduction functor maps

$$M_w \to M^r_w, \qquad L_w \to L^r_w \delta_{w,\overline{w}}. \tag{3.11}$$

where again $M^r_w = M^r(w.\lambda), L^r_w = L^r(w.\lambda)$ and $\overline{w}$ is the maximal representative of w in the double coset (3.10).
Thus we observe that again the way to associate KL polynomials with the double coset (3.10) is to take maximal representatives.

To summarize, consider the $\mathcal{W}$ algebra associated with the regular subalgebra g_r. Let λ be a dominant weight, and let $w, w' \in W^r_\lambda\backslash W_\lambda/W^0_\lambda$. Denote the double coset of w by $[w]$, the minimal representatives by $\underline{w}$ and the maximal representative by $\overline{w}$, and define the following polynomials

$$\begin{array}{ll} P^{IJ}_{w,w'} = P_{\overline{w},\overline{w}'}, & Q^{IJ}_{w,w'} = Q_{\underline{w},\underline{w}'}. \\ \tilde{P}^{IJ}_{w,w'} = \sum_{x\in[w]} P_{x,\underline{w}'}\varepsilon_x\varepsilon_{\underline{w}'}, & \tilde{Q}^{IJ}_{w,w'} = \sum_{x\in[w']} Q_{\overline{w},x}\varepsilon_{\overline{w}}\varepsilon_x. \end{array} \tag{3.12}$$

Conjecture 1 (KL conjecture for $\mathcal{W}$ algebras) *The multiplicities in Verma modules are given by the KL polynomials associated with the double coset (3.10)*

$$\begin{array}{lll} k+h^\vee > 0: & [M^r_w : L^r_{w'}] = P^{IJ}_{w,w'}(1), & (L^r_w : M^r_{w'}) = \tilde{Q}^{IJ}_{w,w'}(1). \\ k+h^\vee < 0: & [M^r_w : L^r_{w'}] = Q^{IJ}_{w',w}(1), & (L^r_w : M^r_{w'}) = \tilde{P}^{IJ}_{w',w}(1). \end{array} \tag{3.13}$$

Hence the character formulas for irreducible $\mathcal{W}$ modules is given by

$$\begin{array}{ll} k+h^\vee > 0: & \mathrm{ch}\, L^r_w = \sum_{w'\geq w} \tilde{Q}^{IJ}_{w,w'}(1)\mathrm{ch}\, M^r_{w'}. \\ k+h^\vee < 0: & \mathrm{ch}\, L^r_w = \sum_{w'\leq w} \tilde{P}^{IJ}_{w',w}(1)\mathrm{ch}\, M^r_{w'}. \end{array} \tag{3.14}$$

Conjecture 2 *The embedding diagram of Verma modules corresponds to the Hasse diagram* *of the double coset (3.10)*

$$\left.\begin{array}{ll} k+h^\vee>0: & M^r_{w'}\hookrightarrow M^r_w \\ k+h^\vee<0: & M^r_w\hookrightarrow M^r_{w'} \end{array}\right\} \quad \textit{iff} \quad w\le w' \quad \textit{with } w,w'\in W^r_\lambda\backslash W_\lambda/W^0_\lambda. \tag{3.15}$$

Moreover, we expect that also for $\mathcal{W}$ algebras there is just one singular vector at any given weight. The KL conjecture supports this as follows. For $k+h^\vee<0$ there is an anti-dominant weight, so here the proof is identical to the case discussed in section 2.3. Barring a reflection principle for $\mathcal{W}$ algebras, a general proof for $k+h^\vee>0$ is lacking. However, in the examples we studied, the polynomials appear to have the property that at arbitrary length $\ell(w)$ one can always find a w such that polynomial $P_{e,w}=1$. This provides an upperbound for the number of singular vectors at that weight and consequently also at every primitive weight $w'.\lambda$ for any $w'\le w$ (see (2.17)). So we expect that also for $\mathcal{W}$ algebras, $\dim\mathrm{Hom}((M^r(\lambda),M^r(\mu))\le 1$.

4 Examples and Applications

In this section we discuss some examples that on the one hand provide evidence for the validity of the conjectures, and on the other are an illustration of their effectiveness for actual computations. In particular the (explicit) calculation of $\mathcal{W}$ characters for irreducible highest weight modules is now reduced to combinatorics on the Weyl group of g. For simplicity we restrict to $g=sl_N$. If $\lambda=\sum_{i=0}^l\lambda_i\Lambda_i$ is a dominant weight (where Λ_i are the fundamental weights of sl_N, $\langle\Lambda_i,\alpha_j^\vee\rangle=\delta_{ij}$), then the level $k=\sum_{i=0}^l\lambda_i$, P_+^k is the set of dominant integral weights of level k, and P_{++}^k are the regular weights in P_+^k. The finite part $\bar\lambda\in\overline{h}^*$ is $\bar\lambda=\sum_{i=1}^l\lambda_i\Lambda_i$ and finally $h^\vee=N$.

4.1 Comparison with known results

The first check is provided by the Virasoro algebra, which is the quantum Hamiltonian reduction of the affine KM algebra $g=sl_2$. In that case, it is a straightforward exercise to show that the conjectures agree with the results of Feigin and Fuchs: (1) the embedding diagrams are classified by the double cosets of the reflection subgroups of the affine Weyl group $\hat{a_1}$ (see Table 1) and (2) the multiplicities in the characters are given by the corresponding KL polynomials: $P_{x,y}=Q_{x,y}=1$ for all $x\le y$.

A second check is provided by the $\mathcal{W}_N$ minimal models, which are the quantum Hamiltonian reduction of the affine KM algebra $g=sl_N$ with respect to the principal sl_2 embedding. Consider the dominant weights λ with W_λ isomorphic to W [22]

$$\lambda+\rho=w(\Lambda^+-t\Lambda^-), \tag{4.1}$$

where $t=p/p'$ (p,p' relative prime integers), $\Lambda^+\in P_{++}^p$, $\Lambda^-\in P_{++}^{p'-1}$ and w an arbitrary element of the Weyl group $\overline{W}$ of $\bar g$. The simple roots of $\Delta^{re}_{\lambda,+}$ are given by $\hat\alpha_i=w(\alpha_i)+\Lambda_i^-\delta$. Let $\hat s_i=r_{\hat\alpha_i}$, then

1. W_λ is generated by the simple reflections $\hat s_i$

Feigin-Fuchs	coset
I	trivial
$II_{\pm}$	a_1
II^0	a_1/a_1
$III_{\pm}$	$\hat{a}_1$
$III^0_{\pm}$	$\hat{a}_1/a_1$
$III^{00}_{\pm}$	$a_1\backslash\hat{a}_1/a_1$

Table 1: *classification of embedding patterns of the Virasoro algebra.*

2. W^0_λ is generated by the $\hat{s}_i$ for which $\Lambda^+_i = 0$,

3. W^r_λ is generated by the $\hat{s}_i$ for which $\Lambda^-_i = 0$ and α_i is a simple root of g_r.

The $\mathcal{W}_N$ minimal models arise from dominant weights (4.1) which have trivial W^0_λ and W^r_λ, hence $\Lambda^+ - \rho \in P^{p-N}_{++}$ and $\Lambda^- - \rho \in P^{p'-N}_{++}$. The multiplicities $Q_{e,w}$ for these regular dominant are easily read off from the recusion relation (2.25) for $x = e$, since in that case $Q_{e,ys} = Q_{e,y}$ for all $sy > y$ so it easily follows that $Q_{e,w} = 1$. This reproduces the character formulas of [5, 12] for the $\mathcal{W}_N$ minimal models. Similarly, admissible modules for arbitrary $\mathcal{W}$ algebras can be obtained. As should be clear from above, the only difference will be in the domain of Λ^-.

New testcases for weights λ with $W_\lambda \cong W$ arise when one considers nontrivial subgroups W^0_λ and/or W^r_λ and non-dominant highest weights. In the next section we will do this for the case of the $\mathcal{W}_3$ algebra.

4.2 Classification of $\mathcal{W}_3$ modules

The Zamolodchikov algebra $\mathcal{W}_3$ [1] is the quantum Hamiltonian reduction of the affine KM algebra sl_3 with respect to the principal sl_2 subalgebra [27]. The W_3 weights are (h, w, c), the eigenvalues of the zero modes L_0, W_0 and c respectively. The parametrization of the $\mathcal{W}_3$ weights in terms of the sl_3 weight λ that follows from the BRST construction is

$$h = \frac{1}{2t}|\bar{\lambda} + \bar{\rho}|^2 + \frac{c-2}{24}, \quad w = \frac{1}{27t(t-1)}(\bar{\lambda} + \bar{\rho}, \Lambda_1)(\bar{\lambda} + \bar{\rho}, \Lambda_2)(\bar{\lambda} + \bar{\rho}, \Lambda_1 - \Lambda_2), \tag{4.2}$$

with $c = 50 - 24t - 24/t$ for $t = k + 3$. The character of a $\mathcal{W}_3$ Verma module is given by $\mathrm{ch}\, M^r(\lambda) = q^h \eta(q)^2$ (note that if $t = 1$, the parametrization (4.2) is singular, in that case one replaces $w \to (t-1)w$).

Up to isomorphism, there are two nontrivial parabolic subgroups of the affine Weyl group $\hat{a}_2$ of sl_3, namely $a_1 \simeq Z_2$ and $a_2 \simeq D_3$. This gives rise to 9 inequivalent double cosets (we eliminated the invariance of the $\mathcal{W}_3$ weights under $(t, \bar{\lambda}) \to (1/t, -\bar{\lambda}/t)$, which interchanges W^0_λ and W^r_λ), see table 2. For each of these double cosets, we computed the multiplicities $P^{IJ}_{w,w'}(1)$ (for the first 15 elements) from the KL polynomials $P_{w,w'}(q)$ of sl_3. Together with the associated Hasse diagrams, they are given in tables 5-13. For completeness, we also give the KL polynomials $P_{w,w'}(q)$ and $Q_{w,w'}(q)$ for sl_3, which we computed up till $\ell(w) = 15$,

tables 3 [1] and 4.
To check if the polynomials and Hasse diagrams correspond with multiplicities and embedding diagrams of $\mathcal{W}$ Verma modules, we subsequently calculated (parts of) the irreducible characters and embedding patterns directly on the Verma module.
This goes as follows. Starting from a highest weight vector v_a (eigenvector of L_0, W_0, c with eigenvalues h, w, c, and annihilated by the L_n, W_n for $n > 0$) a basis $M(a)_{h+N}$ of the Verma module at depth N is constructed. The rank of the innerproduct matrix (Shapovalov form) at depth N gives the dimension of the irreducible character at depth N, and the eigenvectors of W_0 in the kernel of L_1, L_2 and W_1 gives the singular vectors.
In practice, even at modest depth these calculations require a lot of computer time: with the Mathematica routines we had available, the singular vectors could generically be determined up to depth 9, and the characters up to depth 6[2] To verify the predictions of the conjectures, we selected a dominant weight with every coset such that the singular vectors in the associated Verma module occur at the lowest possible levels, see table 2. For every coset we computed the characters of the first 15 submodules, and reconstructed the embedding patterns. We found complete agreement with the KL conjecture.

W^r_λ	W^0_λ	t	Λ^+	Λ^-	c	h	w	Table
a_2	a_2	1	(1, 0, 0)	(1, 0, 0)	2	0	0	5
a_2	a'_2	1	(0, 1, 0)	(1, 0, 0)	2	$\frac{1}{3}$	$\frac{2}{27}$	6
a_2	a_1	2	(1, 1, 0)	(1, 0, 0)	-10	$-\frac{1}{3}$	$\frac{1}{27}$	7
a_2	a'_1	2	(0, 1, 1)	(1, 0, 0)	-10	0	0	8
a_2	–	3	(1, 1, 1)	(1, 0, 0)	-30	-1	0	9
a_1	a_1	3/2	(2, 1, 0)	(1, 1, 0)	-2	$-\frac{1}{9}$	$-\frac{1}{81}$	10
a_1	a'_1	3/2	(2, 0, 1)	(1, 1, 0)	-2	$\frac{2}{9}$	$\frac{10}{81}$	11
a_1	–	3/2	(1, 1, 1)	(1, 1, 0)	-2	0	0	12
–	–	4/3	(2, 1, 1)	(1, 1, 1)	0	0	0	13

Table 2: *Classification of $\mathcal{W}_3$ modules with W_λ isomorphic to $\hat{a}_2$.*

4.3 Closed character formulae

The practical upshot of the KL conjectures is that characters of $\mathcal{W}$ algebras can be computed using only the combinatorics of double cosets of affine Weyl groups. In general however, the word problem posed by the recursion relation (2.25) is too complicated to solve in closed form. Only in certain special cases, one does get a closed expression for character formulas, as in the case of the Virasoro algebra ($Q_{x,y} = 1$ for $x \leq y$) and the $\mathcal{W}_N$ minimal models ($Q_{e,y} = 1$). We end this section by discussing two more examples where a closed formula can be obtained: cosets of type $\overline{W}\backslash W$ and of type $\overline{W}\backslash W/\overline{W}$

[1]We thank M. Goresky, who first computed this table for us.

[2]to appreciate the effectiveness of the KL conjecture: applying table 3 to the vacuum module for $c = 0$ the characters are already determined beyond depth 200, where the Verma module has of the order of 10^{20} states.

4.3.1 *The coset* $\overline{W}\backslash W$

Consider a coset of the form $\overline{W}\backslash W$, with W the affine Weyl group and $\overline{W}$ the corresponding finite Weyl group. These cosets correspond to modules on the boundary of the Kac-table (type III^0 of the Virasoro), where the characters are given by finite sums over Verma characters. For the $\mathcal{W}_3$ algebra $W = \hat{a}_2$ and $\overline{W} = a_2$. The KL polynomials $P^{IJ}_{x,y}$ and Hasse diagram of $a_2\backslash\hat{a}_2$ is given in table 9. In that case it can be shown that there are 2 different character formulas, depending only on the length of $\underline{w}$ (the minimal representative of w in the coset).

If the length $\ell(\underline{w})$ is odd, there are precisely 2 adjacent elements of $\underline{w}$ of length $\ell(\underline{w})+1$, of the form $\underline{w}.j$ and $\underline{w}.k$ (where j and k are distinct simple reflections). Let a_2 denote the a_2 generated by j and k. Then the character reads

$$\mathrm{ch}\, L(w.\lambda) = \sum_{x\in a_2} \epsilon_x \mathrm{ch}\, M(w.x.\lambda) \tag{4.3}$$

If the length $\ell(\underline{w})$ is even, there are at most 3 adjacent elements of $\underline{w}$ of length $\ell(\underline{w})+1$, and there is precisely one of the form $\underline{w}.i$ (for i a simple reflection). Then we find

$$\mathrm{ch}\, L(w.\lambda) = \sum_{x\in a_2} \epsilon_x (\mathrm{ch}\, M(w.i.x.i.\lambda) - \mathrm{ch}\, M(w.i.x.\lambda)) \tag{4.4}$$

where again the a_2 is generated by j,k $(i,j,k$ are distinct simple reflections).

It appears that these two formulas are related to the generic decomposition patterns of Weyl modules, studied in [30]. Similarly, there is 1 formula in the case of A_1, 4 different formulas for B_2 and 12 for G_2.
These character formulas apply in particular to the (p,1) topological minimal models. The admissible weights (4.1) in that case are integral, hence $W^r_\lambda = \overline{W}$. The regular integral weights

$$(\lambda+\rho, \alpha_i^\vee) \neq 0 \bmod p \tag{4.5}$$

are on the orbit of the coset $\overline{W}\backslash W$, with dominant weights $\lambda \in P^p_{++}$. For $\mathcal{W}_3$ therefore, provided $p \geq 3$, (4.3) and (4.4) give the character formulas for all the regular weights. We observe that (4.5) is exactly the condition for the physical states in (p,1) topological minimal matter coupled to W-gravity [29]. An interesting open question is whether the non-trivial multiplicities indicate the presence of extra physical states.

The character formulas (4.3) and (4.4) apply in other cases as well, for instance (i) for regular integral weights on the boundary of the (p,p') Kac table, (ii) for weights with $W^0_\lambda = \overline{W}$ and W^r_λ trivial (interchanging left and right multiplication) and (iii) for the other $g = sl_3$ related $\mathcal{W}$ algebras ($W_3^{(2)}$ and sl_3 itself).

4.3.2 *The coset* $\overline{W}\backslash W/\overline{W}$

Consider a coset of the form $\overline{W}\backslash W/\overline{W}$, with W an affine Weyl group and $\overline{W}$ a subgroup isomorphic to the finite Weyl group $\overline{W}$. These cosets correspond to modules in a corner of the Kac-table (type III^{00} of the Virasoro), where again the characters are given by finite sums

over Verma characters, but in addition they are now grouped in $\bar{g}$ multiplets. Specifically, the multiplicities $P^{IJ}_{x,y}(1)$ are related to the dimension of weight spaces in finite dimensional modules of the simple Lie algebra $\bar{g}$ [28]. The correspondence is as follows.
First consider the case where the embedding of the left- and right subgroups is the same and given by $\overline{W}$ (depending on g there may be more ways to embed $\overline{W}$ in W). Since $W = \overline{W} \cdot \overline{Q}$ (semi-direct product) and in $\overline{Q}$ there is a unique dominant element α on each $\overline{W}$ orbit, it follows that there is a one-to-one correspondence between coset elements $w \in \overline{W}\backslash W/\overline{W}$ and dominant roots $\bar{\alpha}_w \in P_+ \cap \overline{Q}$. More generally, if the embeddings are chosen differently, this correspondence is between coset elements $w \in \overline{W}\backslash W/\overline{W}'$ and dominant weights $\bar{\lambda} \in P_+ \cap \overline{Q} + \Lambda_i$ where Λ_i is the fundamental weight that determines the embedding: $\overline{W}'$ is generated by the set of simple reflections s_j for $j \neq i$. Then the result of [28] states that

$$P^{IJ}_{w,w'}(1) = \dim L(\bar{\lambda}_{w'})_{\bar{\lambda}_w} \tag{4.6}$$

where $L(\bar{\lambda})_{\bar{\mu}}$ is the weight space of weight $\bar{\mu}$ in the (finite dimensional) irreducible $\bar{g}$ module of highest weight $\bar{\lambda}$.
This result applies in particular to the $\mathcal{W}_N$ algebras with $c = \mathrm{rank}(\bar{g})$, *i.e.* with $k + h^\vee = 1$. From (4.1) it follows that the dominant weight is determined by a level 1 weight, *i.e.* $\lambda + \rho = \Lambda_i$. $W^r_\lambda = \overline{W}$ and $W^0_\lambda = \overline{W}'$. Using the correspondence described above, to every primitive weight $\mu = w.\lambda$ one associates the dominant weight $\bar{\lambda}_w = \bar{\mu} + \bar{\rho}$. Then the sum $w' \geq w$ can be rewritten as follows

$$\mathrm{ch}\, M^r(\mu) = \sum_{\bar{\mu}' \geq \bar{\mu}} \dim L(\bar{\mu}' + \bar{\rho})_{\bar{\mu}+\bar{\rho}} \mathrm{ch}\, L^r(\mu'). \tag{4.7}$$

Inverting this (using the basis transformation from Verma modules $M(\bar{\mu})$ to singlets $e^{\bar{\mu}}$) gives

$$\mathrm{ch}\, L^r(\mu) = \sum_{x \in \overline{W}} \epsilon_x \mathrm{ch}\, M^r(\mu + \rho - x(\rho)) \tag{4.8}$$

This character formula was first proposed in [31], where it was obtained as a limit of the characters of the $\mathcal{W}_N$ minimal models [5]. The inverse formula (4.7) was obtained for $\mathcal{W}_3$ in [32], using an explicit construction of the singular vectors in the Fock space and comparison of the characters for each side. Again, this character formula applies more generally, in particular to the weights in a corner of the (p, p') Kac table (in that case one replaces $\rho \to p\rho$ everywhere on the RHS of (4.8))

5 Concluding remarks

In this paper we have formulated the KL conjecture for $\mathcal{W}$ algebras associated with arbitrary sl_2 reductions. The result can be described in terms of a double coset $W^r_\lambda\backslash W_\lambda/W^0_\lambda$: the Hasse diagram gives the embedding diagram of the Verma modules, and the KL polynomials give the multiplicities in the characters.

The conjectures also apply to finite $\mathcal{W}$ algebras, which are the Hamiltonian reduction of simple Lie algebras $\bar{g}$ [13]. In that case one simply takes W to be the Weyl group $\overline{W}$ of $\bar{g}$. The character formulas for this class of algebras are given in [14] (for regular integral weights only) and the results agree completely.

We remark that the conjecture is also a useful tool to analyse the structure of the Verma modules in more detail. This is particularly important if one attempts to construct a resolution of the irreducible modules by Verma modules. The physical motivation for doing so is the application to $\mathcal{W}$ gravity/strings: it is much simpler to compute the (string) BRST cohomology on Verma modules than on irreducible modules. The problem is that it is not always possible to find a resolution by Verma modules. For instance, for the $\mathcal{W}_3$ string at $c = 2$ it is found by explicit construction [32] that there is no resolution by Verma modules, but instead one is forced to introduce generalized Verma modules. This is directly linked to the existence of primitive vectors that are pseudo-singular rather than singular (they are not an eigenvector of W_0). For such an analysis it is convenient to have the data presented by the KL conjecture. This way for instance, one can easily show that the character (4.4) of the (p,1) topological minimal models for $\mathcal{W}_3$ cannot be reproduced by a resolution by (generalized) Verma modules. This is due to the occurence of subsingular vectors [14]. It would be very interesting to know what type of modules are needed to build such a resolution, since these modules are going to carry the cohomology of the topological $\mathcal{W}_3$ string. Work on this is in progress.

Appendices

KL polynomials on Coxeter groups

In this appendix, we summarize the definition and some of the properties of KL polynomials $P_{x,y}$ for a Coxeter group W, for details see [7, 17]. The starting point is the Hecke algebra $\mathcal{H}$ with generators T_y (one for each $y \in W$) and defining relations

$$T_x T_y = T_{xy} \quad \text{if} \quad \ell(xy) = \ell(x) + \ell(y) \tag{A.1}$$

$$(T_s + 1)(T_s - q) = 0 \quad \text{if} \quad s \in S \tag{A.2}$$

where S is the set of simple reflections that generate W. The elements T_y are invertible in $\mathcal{H}$, and one can write

$$T_{y^{-1}}^{-1} = \sum_{x \le y} \varepsilon_x \varepsilon_y R_{x,y}(q) q^{-\ell(y)} T_x \tag{A.3}$$

where $\varepsilon_y = (-1)^{\ell(y)}$, and $R_{x,y}(q)$ is a polynomial in q of degree $\ell(y) - \ell(x)$ for $x \le y$, uniquely defined by (A.3). The map $\imath$ defined by

$$\imath(q) = q^{-1}, \qquad \imath(T_y) = T_{y^{-1}}^{-1} \tag{A.4}$$

is an automorphism of $\mathcal{H}$. The KL polynomials are associated with the invariants of $\imath$. For any pair $x \le y$ in W, there is a uniquely defined polynomial $P_{x,y}$ of degree $\le (\ell(y) - \ell(x) - 1)/2$ if $x < y$, and $P_{x,x} = 1$, such that

$$C_y = \sum_{x \le y} \varepsilon_x \varepsilon_y q^{\ell(y)/2 - \ell(x)} P_{x,y}(q^{-1}) T_x \tag{A.5}$$

satisfies

$$\imath(C_y) = C_y \tag{A.6}$$

for all $y \in W$. Equivalently, the P-polynomials satisfy

$$q^{\ell(y)-\ell(x)} P_{x,y}(q^{-1}) = \sum_{x \leq z \leq y} R_{x,z} P_{z,y}(q) \quad \text{for all } x \leq y \tag{A.7}$$

From this, one can extract a recursion relation (expressing the polynomials $P_{x,y}$ in terms of the polynomials $P_{x',y'}$ with $y' < y$). Namely, for $ys > y$ one has [7]

$$P_{x,ys} = q^{1-c} P_{xs,y} + q^c P_{x,y} - q \sum_{\substack{x \leq z < y \\ zs < z}} P_{x,z} \not{P}_{z,y} \tag{A.8}$$

Here $c = 1$ if $xs < x$ and 0 otherwise, and $\not{P}_{z,y}$ is the term in $P_{z,y}$ of (maximal) degree $\frac{1}{2}(\ell(y) - \ell(z) - 1)$. The initial values of the recursion relation are $P_{x,e}(q) = \delta_{x,e}$. This implies in particular that $P_{x,y}(q) = 0$ unless $x \leq y$. From (A.8) it also follows that $P_{x,y}(0) = 1$ if $x \leq y$. In the case crystallographic Coxeter groups (which includes (affine) Weyl groups) the coefficients of $P_{x,y}$ give the dimensions of stalks of cohomology sheaves of the intersection cohomology complexes associated to Schubert varieties [34]. This implies in particular that these coefficients are nonnegative integers.

Similarly, if $ys < y$ it can be shown that $C_y T_s = -C_y$ which implies that

$$P_{x,y} = P_{xs,y} \quad \text{for } x \leq y \text{ and } ys < y \tag{A.9}$$

For finite Coxeter groups (where there is a unique longest element w_0) it easily follows that

$$P_{x,w_0} = 1 \tag{A.10}$$

Inverse KL polynomials on Coxeter groups

The KL polynomials $P_{y,w}$ form an upper triangular matrix with 1's on the main diagonal, which naturally can be inverted. Thus, one can define for each $x \leq y$ in W a polynomial $Q_{x,y}$ such that

$$\sum_{x \leq z \leq y} \varepsilon_x \varepsilon_z P_{x,z} Q_{z,y} = \delta_{x,y} \quad \text{for all } x \leq y \tag{A.11}$$

It is clear that $Q_{x,x}(q) = 1$ and that $Q_{x,y}(q)$ has degree $\leq (\ell(y) - \ell(x) - 1)/2$ for $x < y$, and $\not{Q}_{x,y} = \not{P}_{x,y}$. It also follows that

$$q^{\ell(y)-\ell(x)} Q_{x,y}(q^{-1}) = \sum_{x \leq z \leq y} Q_{x,z} R_{z,y}(q) \quad \text{for all } x \leq y \tag{A.12}$$

The Q-polynomials are associated to invariants of $\imath$. Define elements S_x, D_x of $\mathcal{H}^*$ by

$$\langle S_x, \imath(T_y) \rangle = \langle D_x, C_y \rangle = \delta_{x,y^{-1}} \tag{A.13}$$

and let $\langle \imath(u), h\rangle = \imath(\langle u, \imath(h)\rangle)$. It follows that $\imath(D_x) = D_x$ and

$$D_x = \sum_{x \geq y} q^{\ell(x)/2-\ell(y)} Q_{x,y} S_y \tag{A.14}$$

with $Q_{x,y}$ the inverse polynomials (A.11). From the right $\mathcal{H}$-action on $\mathcal{H}^*$ given by $\langle u \cdot h, h'\rangle = \langle u, hh'\rangle$ one can conclude that $D_x \cdot T_s = qD_x$ if $xs > x$. This implies

$$Q_{x,ys} = Q_{x,y} \quad \text{for } x \leq y \text{ and } xs > x \tag{A.15}$$

From this it easily follows that

$$Q_{e,y} = 1 \quad \text{for all } y \in W \tag{A.16}$$

The analogue of (A.8) for the Q-polynomials is (for $xs < x$)

$$Q_{xs,y} = q^{1-c} Q_{x,ys} + q^c Q_{x,y} - q \sum_{\substack{x<z\leq y \\ zs>z}} \mu_{x,z} Q_{z,y} \tag{A.17}$$

with $c = 1$ for $ys > y$ and $c = 0$ for $ys < y$. In this form (A.17) cannot be used to solve for the Q-polynomials (at least not in the case of infinite Coxeter groups). But, combining (A.17) and (A.15) for $ys > y$, one gets a useful relation (expressing $Q_{z,w}$ in terms of $Q_{z',w'}$ with $w' < w$)

$$Q_{x,ys} = cQ_{xs,y} + (-q)^c Q_{x,y} + cq \sum_{\substack{x<z\leq y \\ zs>z}} \mu_{x,z} Q_{z,y} \tag{A.18}$$

with $c = 1$ for $xs < x$ and $c = 0$ for $xs > x$. In the case of (affine) Weyl groups the coefficients of $Q_{x,y}$ are again nonnegative integers.
In the case of a finite Coxeter group, the KL polynomials P are related to the inverse polynomials Q through

$$Q_{x,y} = P_{w_0 y, w_0 x} \tag{A.19}$$

On the other hand: if the group is not finite, P and Q do not appear to be related in any such way.

KL polynomials on cosets

Similar as for ordinary Coxeter groups W, one can also construct KL polynomials on cosets W/W_I (or $W_I \backslash W$), where W_I is a parabolic subgroup of W [33]. In particular one finds that again there are recursion relation for the associated polynomials $P^I_{x,y}$ and $Q^I_{x,y}$. If the subgroup W_I is finite (which is sufficient for our purposes) the polynomials P^I, Q^I and their inverses $\tilde{Q}^I, \tilde{P}^I$ defined through[3]

$$\sum_{x\leq z\leq y} \tilde{Q}^I_{x,z} P^I_{z,y} = \sum_{x\leq z\leq y} Q^I_{x,z} \tilde{P}^I_{z,y} = \delta_{x,y} \tag{A.20}$$

[3]Note that this notation is slighly different from the notation used in [14].

can be expressed in terms of the KL polynomials of W

$$\begin{array}{ll} P^I_{x,y} = P_{\overline{x},\overline{y}} & Q^I_{x,y} = Q_{\underline{x},\underline{y}} \\ \tilde{P}^I_{x,y} = \sum_{z\in[x]} P_{z,\underline{y}}\varepsilon_z\varepsilon_{\underline{y}} & \tilde{Q}^I_{x,y} = \sum_{z\in[y]} Q_{\overline{x},z}\varepsilon_{\overline{x}}\varepsilon_z \end{array} \tag{A.21}$$

Here $\underline{z}$ is the minimal- and $\overline{z}$ is the maximal representative of the coset $[z]$ of z.

However, the cosets that play a role in this paper are two-sided cosets $W_I\backslash W/W_J$, with respect to parabolic subgroups W_I and W_J. There does not appear to be an abstract set-up for double-sided cosets. In particular, the partial ordering on these cosets is more complicated than in the case of one-sided cosets (for example, the length of adjacent elements may differ by more than 1). So instead of defining KL polynomials through a recursuion relation we take (A.21) as our starting point, *i.e.* we define

$$\begin{array}{ll} P^{IJ}_{w,w'} = P_{\overline{w},\overline{w}'} & Q^{IJ}_{w,w'} = Q_{\underline{w},\underline{w}'} \\ \tilde{P}^{IJ}_{w,w'} = \sum_{x\in[w]} P_{x,\underline{w}'}\varepsilon_x\varepsilon_{\underline{w}'} & \tilde{Q}^{IJ}_{w,w'} = \sum_{x\in[w']} Q_{\overline{w},x}\varepsilon_{\overline{w}}\varepsilon_x \end{array} \tag{A.22}$$

where the polynomials $\tilde{P}^{IJ}, \tilde{Q}^{IJ}$ are again the inverse polynomials

$$\sum_{x\leq z\leq y} \tilde{Q}^{IJ}_{x,z}P^{IJ}_{z,y} = \sum_{x\leq z\leq y} Q^{IJ}_{x,z}\tilde{P}^{IJ}_{z,y} = \delta_{x,y} \tag{A.23}$$

6 Tables

Table 3: *KL polynomials $P_{x,y}$ for the Weyl group $\hat{a}_2$ of the affine KM algebra $g = sl_3$, up to $\ell(y) = 15$. To find $P_{x,y}$ for arbitrary pairs x, y (with $\ell(y) \leq 15$) use that*
1. $P_{x,y} = 0$ unless $x \leq y$,
2. $P_{x,y} = P_{x^{-1},y^{-1}}$,
3. $P_{x,y} = P_{\tau(x),\tau(y)}$ with τ an automorphism of the Dynkin diagram,
4. $P_{x,y} = P_{x',y}$ for $x \leq x'$ and $P_{x',y}(1)$ maximal,
5. if 1-4 does not apply then $P_{x,y} = 1$.
So, given a pair x, y with $x \leq y$, one first fixes i,j,k and an order (i.e. reading from left-to-right or from right-to-left) such that y is in the table. Secondly, one searches for an x' (in the fixed order) in the table such that $x \leq x'$ and $P_{x',y}(1)$ is maximal; then $P_{x,y} = P_{x',y}$. If either of the two steps fail then $P_{x,y} = 1$.

Table 4: *Inverse KL polynomials $Q_{x,y}$ for the Weyl group $\hat{a}_2$ of the affine KM algebra $g = sl_3$, up to $\ell(x) = 14$. To find $Q_{x,y}$ for arbitrary pairs x, y (with $\ell(x) \leq 14$) use the rules of table 3. with x,y interchanged and the ordering reversed.*

$\ell(y)$	y	x	$P_{x,y}-1$	$\ell(y)$	y	x	$P_{x,y}-1$	$\ell(y)$	y	x	$P_{x,y}-1$
4	ijki	i	q			ijk	$3q^2+3q$	14	ijikijikijikij	ijikijikijk	q
5	ijkij	ij	q		ijkijikijki	ijikijki	q			ijikijikiji	q
	ijkji	i	q			ikijkij	q			ijikijikij	$2q$
6	ijikij	iji	q			ijkijki	$2q$			ijikijk	q^2+2q
	ijkijk	iki	q			ijiki	q^2+2q			ijikiji	q^2+2q
		ijk	q		ijkijkijkij	ijkijkij	q			ijikij	$2q^2+2q$
7	ijikiji	iji	q	12	ijikijikijik	ijikijiki	q			iji	q^3+2q^2+2q
	ijkijik	ijik	q			ijikijik	$2q$		ijikijikijkijk	ijikijkijki	q
		iki	q			ijiki	q^2+2q			ijikijikijk	q
		ijk	$2q$			ijik	$2q^2+2q$			ijikijkijk	$2q$
	ijkijki	ijki	q		ijikijikijki	ijikijkij	q			ijikijk	q^2+2q
8	ijikijik	ijiki	q			ijikijiki	q		ijikijkijkijki	ijikijkijki	q
		ijik	$2q$			ijikijki	$2q$		ijkijkijkijkij	ikijkijkijk	q
	ijikijki	ijiki	q			ijiki	q^2+2q			ijkijkijkij	q
	ijkijkij	ikijk	q		ijikijkijkij	ijikijkij	q	15	ijikijikijikiji	ijikijikiji	q
		ijkij	q		ijkijkijkijk	ikijkijki	q			ijikiji	q^2+q
9	ijikijiki	ijiki	q			ijkijkijk	q			iji	q^3+q^2+q
	ijkijikij	ijikij	q	13	ijikijikijiki	ijikijiki	q		ijikijikijikijk	ijikijikijk	q
		ikijk	q			ijiki	q^2+q			ijikijk	q^2+q
		ijkij	$2q$		ijikijikijkij	ijikijkij	q		ijikijikijkijki	ijikijkijki	q
		iji	q^2+2q		ijkijikijikij	ijkijikijk	q		ijkijikijikijik	ijkijikijiki	q
	ijkijikji	ijikji	q			ijikijikij	q			ijikijikijik	q
		ijkji	$3q$			ikijikijk	$2q$			ikijikijiki	$2q$
		ijiki	$2q$			ijkijikij	$3q$			ijkijikijik	$3q$
		iji	q^2+3q			ijikijkij	$2q$			ijikijikijk	$2q$
		i	$2q^2+3q$			ijikijk	q^2+3q			ikijkijki	q^2+2q
	ijkijkijk	ijkijk	q			ijikiji	q^2+2q			ijikijikj	q^2+2q
10	ijikijikij	ijikijk	q			ijikij	$2q^2+3q$			ijikijiki	q^2+3q
		ijikiji	q			ikijk	$2q^2+3q$			ijkijiki	$2q^2+3q$
		ijikij	$2q$			ijkij	$3q^2+3q$			ijikijik	$2q^2+3q$
		iji	q^2+2q			iji	q^3+3q^2+3q			ikijiki	$3q^2+3q$
	ijikijkijk	ijikijk	q		ijkijikijikji	ijikijikji	q			ijkijik	$4q^2+3q$
	ijkijkijki	ikijkij	q			ijkijikji	$3q$			ijikijk	$3q^2+3q$
		ijkijki	q			ijikijiki	$2q$			ijikj	q^3+3q^2+3q
11	ijikijikiji	ijikiji	q			ijikjik	q^2+2q			ijiki	q^3+4q^2+3q
		iji	q^2+q			ijikiji	q^2+3q			ijik	$2q^3+4q^2+3q$
	ijikijikijk	ijikijk	q			ijikji	$2q^2+3q$			iki	$2q^3+4q^2+3q$
	ijkijikijik	ijkijiki	q			ijkji	$4q^2+3q$			ijk	$3q^3+4q^2+3q$
		ijikijik	q			ijiki	$3q^2+3q$		ijkijikijikijki	ijkijikijkij	q
		ikijiki	$2q$			iji	q^3+4q^2+3q			ijikijikijki	q
		ijkijik	$3q$			i	$2q^3+4q^2+3q$			ikijikijkij	$2q$
		ijikijk	$2q$		ijkijikijkijk	ijikijkijk	q			ijkijikijki	$3q$
		ijikj	q^2+2q			ikijkijki	q			ijikijkijki	$2q$
		ijiki	q^2+3q			ijkijkijk	$2q$			ijikijkij	q^2+3q
		ijik	$2q^2+3q$			ijikijk	q^2+2q			ijikijiki	q^2+2q
		iki	$2q^2+3q$		ijkijkijkijki	ijkijkijki	q			ijikijki	$2q^2+3q$

Table 3: *KL polynomials for affine sl_3. For explanation see section 6.*

$\ell(x)$	x	y	$Q_{x,y}-1$
1	i	ijki	q
		ijkijki	q^2+q
		ijkijkijki	q^3+q^2+q
		ijkijkijkijki	$q^4+q^3+q^2+q$
2	ij	ijkij	q
		ikjikj	q
		ijkijkij	q^2+q
		ikjikjikj	q^2+q
		ijkijkijkij	q^3+q^2+q
		ikjikjikjikj	q^3+q^2+q
		ijkijkijkijkij	$q^4+q^3+q^2+q$
3	iji	ijikij	q
		ikjikj	q
		ijikiji	$3q$
		ijkijki	$2q$
		ijkjikj	$2q$
		ijkijkij	$3q$
		ijkijikij	q^2+3q
		ijkijikji	q^2+3q
		ikjikjikj	q^2+3q
		ijikijikij	$2q^2+3q$
		ijkijkijki	$2q^2+3q$
		ijikijikiji	$4q^2+3q$
		ijkijikijki	$3q^2+3q$
		ijkijikjikj	$3q^2+3q$
		ijkijkijkij	$4q^2+3q$
		ikjikjikjikj	q^3+3q^2+3q
		ijkijikijikij	q^3+4q^2+3q
		ijkijikijikji	q^3+4q^2+3q
		ijkijkijkijki	$2q^3+4q^2+3q$
		ijkjikjikjikj	q^3+4q^2+3q
		ijikijikijikij	$2q^3+4q^2+3q$
		ijkijkijkijkij	$4q^3+4q^2+3q$
	ijk	ijkijk	q
		ikjikjik	q
		ijkijkijk	q^2+q
		ikjikjikjik	q^2+q
		ijkijkijkijk	q^3+q^2+q
		ikjikjikjikjik	q^3+q^2+q
4	ijik	ijkijik	q
		ijikijik	$2q$
		ikjikjik	q
		ijkijkijk	$2q$
		ijkijikijik	q^2+2q
		ikjikjikjik	q^2+2q
		ijikijikijik	$2q^2+2q$
		ijkijkijkijk	$2q^2+2q$
		ikjikjikjikjik	q^3+2q^2+2q
	ijki	ijkijki	q
		ijkijkijki	q^2+q
		ikjikjikji	q
		ijkijkijkijki	q^3+q^2+q
		ikjikjikjikji	q^2+q
5	ijiki	ijikijik	q
		ijikijki	q
		jkijkijk	q
		ijikijiki	$3q$
		ijkijikji	$2q$
		ijkijkijk	$2q$

$\ell(x)$	x	y	$Q_{x,y}-1$
		jikijkijk	$2q$
		ijkijkijki	$3q$
		ikjikjikji	q
		ijkijikijik	q^2+3q
		ijkijikijki	q^2+3q
		ikjikjikjik	$2q$
		jikijikijik	q^2+2q
		jkijkijkijk	q^2+3q
		ijikijikijik	$2q^2+3q$
		ijikijikijki	$2q^2+3q$
		ijkijkijkijk	$2q^2+3q$
		jikjikjikjik	$3q$
		ijikijikijiki	$4q^2+3q$
		ijkijikijikji	$3q^2+3q$
		ijkijikijkijk	$3q^2+3q$
		ijkijkijkijki	$4q^2+3q$
		ikjikjikjikji	q^2+3q
		jikijikijkijk	$3q^2+3q$
		jikijikjikjik	q^2+3q
		ikjikjikjikjik	$2q^2+3q$
		jkijkijkijkijk	q^3+3q^2+3q
	ijkij	ijkijkij	q
		ijkijkijkij	q^2+q
		ikjikjikjikj	q
		ijkijkijkijkij	q^3+q^2+q
	ijkji	ijkijikji	q
		ijkijkijki	q
		ijkijikijikji	q^2+q
		ijkijkijkijki	q^2+q
6	ijikij	ijkijikij	q
		jikijikij	q
		ijikijikij	$2q$
		jkijkijkij	q
		ijkijkijkij	$2q$
		ikjikjikjikj	q
		ijkijikijikij	q^2+2q
		jikijikijikij	q^2+2q
		jikjikjikjikj	$2q$
		jkijkijkijkij	q^2+2q
		ijikijikijikij	$2q^2+2q$
		ijkijkijkijkij	$2q^2+2q$
	ijkijk	ijkijkijk	q
		ijkijkijkijk	q^2+q
		ikjikjikjikjik	q
7	ijikiji	ijikijikij	q
		ijikijikiji	$3q$
		ijkijikjikj	$2q$
		ikjikjikjikj	q
		ijkijikijikij	q^2+2q
		ijkijikijikji	q^2+3q
		ijkijkijkijki	$2q$
		ijikijikijikij	$2q^2+3q$
		ijkijkijkijkij	$3q$
	ijikijk	ijikijikij	q
		ijikijkijk	q
		jkijkijkij	q
		ijikijikijk	$3q$
		ijkijikijik	$2q$
		ijkijkijkij	$2q$

$\ell(x)$	x	y	$Q_{x,y}-1$
		jikijkijkij	$2q$
		ijkijkijkijk	$3q$
		ijkijikijikij	q^2+3q
		ijkijikijkijk	q^2+3q
		jikijikijikij	q^2+2q
		jkijkijkijkij	q^2+3q
		ijikijikijikij	$2q^2+3q$
		ijikijikijkijk	$2q^2+3q$
		ijkijkijkijkij	$2q^2+3q$
		ikjikjikjikjik	q
	ijkijik	ijkijikijik	q
		ijkijkijkijk	q
		ikjikjikjikjik	q
	ijkijki	ijkijkijki	q
		ijkijkijkijki	q^2+q
8	ijikijik	ijkijikijik	q
		ijikijikijik	$2q$
		ikjikjikjikjik	q
	ijikijki	ijkijikijki	q
		jikijikijki	q
		ijikijikijki	$2q$
		jkijkijkijki	q
		ijkijkijkijki	$2q$
	ijkijkij	ijkijkijkij	q
		ijkijkijkijkij	q^2+q
9	ijikijiki	ijikijikijik	q
		ijikijikijki	q
		ijikijikijiki	$3q$
		ijkijikijikji	$2q$
		jikijikijkijk	$2q$
		jkijkijkijkijk	q
	ijikijkij	ijikijikijki	q
		ijikijkijkij	q
		jkijkijkijki	q
		ijikijikijkij	$3q$
		ijkijikijikij	$2q$
		ijkijkijkijki	$2q$
		jikijkijkijki	$2q$
		ijkijkijkijkij	$3q$
	ijkijikij	ijkijikijikij	q
		ijkijkijkijkij	q
	ijkijikji	ijkijikijikji	q
	ijkijkijk	ijkijkijkijk	q
10	ijikijikij	ijkijikijikij	q
		jikijikijikij	q
		ijikijikijikij	$2q$
	ijikijkijk	ijkijikijkijk	q
		jikijikijkijk	q
		ijikijikijkijk	$2q$
		jkijkijkijkijk	q
	ijkijkijki	ijkijkijkijki	q
11	ijikijikiji	ijikijikijikij	q
	ijikijikijk	ijikijikijikij	q
		ijikijikijkijk	q
	ijikijkijki	ijikijikijkijk	q
		ijikijkijkijki	q
		jkijkijkijkijk	q
	ijkijkijkij	ijkijkijkijkij	q

Table 4: *Inverse KL polynomials for affine sl_3. For explanation see section 6.*

P^{IJ}	$\{2,3\},\{2,3\}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	(h,w)
1	e	1	2	1	1	3	2	2	4	1	1	3	3	5	2	2	$(0,0)$
2	1	·	1	1	1	2	2	2	3	1	1	3	3	4	2	2	$(1,0)$
3	1231	·	·	1	·	1	2	1	2	1	1	3	2	3	2	2	$(3,-2)$
4	1321	·	·	·	1	1	1	2	2	1	1	2	3	3	2	2	$(3,2)$
5	12321	·	·	·	·	1	1	1	2	1	1	2	2	3	2	1	$(4,0)$
6	1231231	·	·	·	·	·	1	·	1	1	·	2	1	2	2	1	$(7,-6)$
7	1321321	·	·	·	·	·	·	1	1	·	1	1	2	2	1	2	$(7,6)$
8	123121321	·	·	·	·	·	·	·	1	·	·	1	1	2	1	1	$(9,0)$
9	1231231231	·	·	·	·	·	·	·	·	1	·	1	·	1	2	·	$(12,-16)$
10	1321321321	·	·	·	·	·	·	·	·	·	1	·	1	1	·	2	$(12,16)$
11	12312131231	·	·	·	·	·	·	·	·	·	·	1	·	1	1	·	$(13,-12)$
12	12321321321	·	·	·	·	·	·	·	·	·	·	·	1	1	·	1	$(13,12)$
13	1231213121321	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	$(16,0)$
14	1231231231231	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	$(19,-30)$
15	1321321321321	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	$(19,30)$

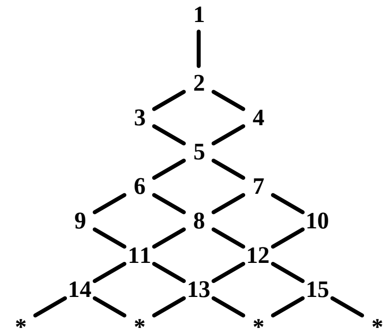

Table 5: *multiplicities* $P^{IJ}_{x,y}(1)$ *and Hasse diagram for the coset* $a_2\backslash\hat{a}_2/a_2$.

P^{IJ}	$\{2,3\},\{1,3\}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	(h,w)
1	e	1	1	2	2	1	3	1	3	2	4	2	1	4	3	1	$(\frac{1}{3},\frac{2}{27})$
2	12	·	1	1	2	1	2	1	3	2	3	2	1	4	3	1	$(\frac{4}{3},-\frac{16}{27})$
3	132	·	·	1	1	1	2	1	2	2	3	2	1	3	3	1	$(\frac{7}{3},\frac{20}{27})$
4	12312	·	·	·	1	·	1	1	2	1	2	2	1	3	2	1	$(\frac{13}{3},-\frac{70}{27})$
5	132132	·	·	·	·	1	1	·	1	2	2	1	1	2	3	1	$(\frac{16}{3},\frac{128}{27})$
6	1232132	·	·	·	·	·	1	·	1	1	2	1	1	2	2	1	$(\frac{19}{3},\frac{56}{27})$
7	12312312	·	·	·	·	·	·	1	1	·	1	2	·	2	1	1	$(\frac{25}{3},-\frac{250}{27})$
8	123121312	·	·	·	·	·	·	·	1	·	1	1	·	2	1	1	$(\frac{28}{3},-\frac{160}{27})$
9	132132132	·	·	·	·	·	·	·	·	1	1	·	1	1	2	·	$(\frac{31}{3},\frac{308}{27})$
10	12312132132	·	·	·	·	·	·	·	·	·	1	·	·	1	1	·	$(\frac{37}{3},\frac{110}{27})$
11	12312312312	·	·	·	·	·	·	·	·	·	·	1	·	1	·	1	$(\frac{43}{3},-\frac{520}{27})$
12	132132132132	·	·	·	·	·	·	·	·	·	·	·	1	·	1	·	$(\frac{49}{3},\frac{686}{27})$
13	1231213121312	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	$(\frac{49}{3},-\frac{286}{27})$
14	1232132132132	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	$(\frac{52}{3},\frac{52}{27})$
15	12312312312312	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	$(\frac{64}{3},\frac{64}{27})$

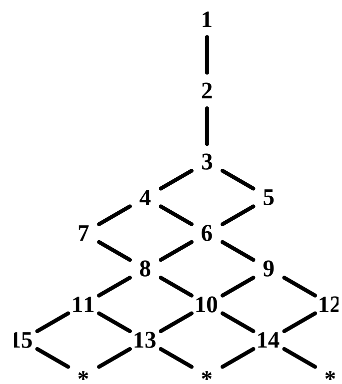

Table 6: *multiplicities* $P^{IJ}_{x,y}(1)$ *and Hasse diagram for the coset* $a_2\backslash\hat{a}_2/a_2'$

P^{IJ}	$\{2,3\},\{3\}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	(h,w)
1	e	1	1	2	2	1	2	1	2	4	1	2	3	1	2	4	$(-\frac{1}{3},\frac{1}{27})$
2	1	·	1	1	1	1	2	1	2	3	1	2	2	1	2	4	$(\frac{2}{3},\frac{10}{27})$
3	12	·	·	1	1	1	1	1	1	3	1	2	2	1	2	3	$(\frac{5}{3},-\frac{35}{27})$
4	132	·	·	·	1	·	1	1	1	2	1	1	2	1	2	3	$(\frac{8}{3},\frac{28}{27})$
5	1231	·	·	·	·	1	·	1	1	2	·	2	1	1	2	2	$(\frac{11}{3},-\frac{125}{27})$
6	1321	·	·	·	·	·	1	·	1	1	1	1	1	1	1	3	$(\frac{14}{3},\frac{154}{27})$
7	12312	·	·	·	·	·	·	1	·	1	·	1	1	·	2	1	$(\frac{20}{3},-\frac{260}{27})$
8	12321	·	·	·	·	·	·	·	1	1	·	1	1	1	1	2	$(\frac{17}{3},\frac{55}{27})$
9	123121	·	·	·	·	·	·	·	·	1	·	1	1	·	1	1	$(\frac{23}{3},-\frac{143}{27})$
10	132132	·	·	·	·	·	·	·	·	·	1	·	1	1	·	2	$(\frac{23}{3},\frac{343}{27})$
11	1231231	·	·	·	·	·	·	·	·	·	·	1	·	·	1	·	$(\frac{32}{3},-\frac{440}{27})$
12	1232132	·	·	·	·	·	·	·	·	·	·	·	1	·	·	1	$(\frac{29}{3},\frac{91}{27})$
13	1321321	·	·	·	·	·	·	·	·	·	·	·	·	1	·	1	$(\frac{35}{3},\frac{595}{27})$
14	12312312	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	$(\frac{44}{3},-\frac{836}{27})$
15	12321321	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	$(\frac{38}{3},\frac{442}{27})$

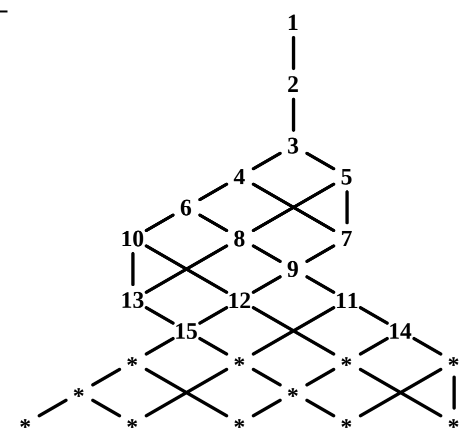

Table 7: *multiplicities* $P^{IJ}_{x,y}(1)$ *and Hasse diagram for the coset* $a_2\backslash\hat{a}_2/a_1$.

P^{IJ}	$\{2,3\},\{1\}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	(h,w)
1	e	1	1	1	1	1	3	2	2	2	2	2	2	5	1	1	$(0,0)$
2	12	·	1	·	1	1	2	2	1	2	2	2	2	4	1	1	$(1,-1)$
3	13	·	·	1	1	1	2	1	2	2	2	2	2	4	1	1	$(1,1)$
4	123	·	·	·	1	·	1	1	1	2	1	2	1	3	1	1	$(3,-3)$
5	132	·	·	·	·	1	1	1	1	1	2	1	2	3	1	1	$(3,3)$
6	1232	·	·	·	·	·	1	1	1	1	1	1	1	3	1	1	$(4,0)$
7	12312	·	·	·	·	·	·	1	·	1	·	1	1	2	1	·	$(6,-6)$
8	13213	·	·	·	·	·	·	·	1	·	1	1	1	2	·	1	$(6,6)$
9	123123	·	·	·	·	·	·	·	·	1	·	1	·	1	1	·	$(9,-15)$
10	132132	·	·	·	·	·	·	·	·	·	1	·	1	1	·	1	$(9,15)$
11	1231213	·	·	·	·	·	·	·	·	·	·	1	·	1	·	·	$(10,-10)$
12	1232132	·	·	·	·	·	·	·	·	·	·	·	1	1	·	·	$(10,10)$
13	12312132	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	$(12,0)$
14	12312312	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	$(13,-27)$
15	13213213	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	$(13,27)$

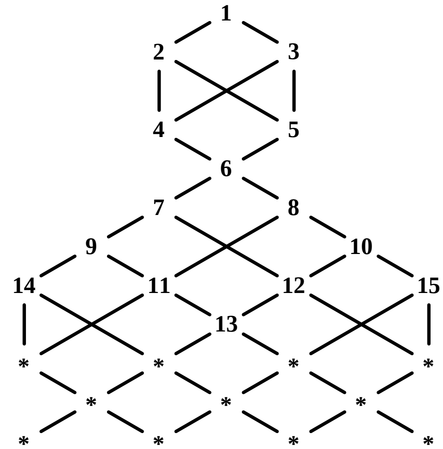

Table 8: *multiplicities* $P^{IJ}_{x,y}(1)$ *and Hasse diagram for the coset* $a_2'\backslash\hat{a}_2/a_1$.

P^{IJ}	$\{2,3\},\emptyset$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	(h,w)
1	e	1	1	1	1	2	2	1	2	1	2	3	2	2	1	2	$(-1,0)$
2	1	·	1	1	1	1	1	1	1	1	2	3	2	2	1	2	$(0,0)$
3	12	·	·	1	·	1	1	1	1	1	2	2	1	2	1	1	$(1,-1)$
4	13	·	·	·	1	1	1	1	1	1	1	2	2	1	1	2	$(1,1)$
5	123	·	·	·	·	1	·	1	1	·	1	1	1	1	1	1	$(3,-2)$
6	132	·	·	·	·	·	1	·	1	1	1	1	1	1	1	1	$(3,2)$
7	1231	·	·	·	·	·	·	1	·	·	1	1	·	1	1	1	$(5,-5)$
8	1232	·	·	·	·	·	·	·	1	·	1	1	1	1	1	1	$(4,0)$
9	1321	·	·	·	·	·	·	·	·	1	·	1	1	1	·	1	$(5,5)$
10	12312	·	·	·	·	·	·	·	·	·	1	·	·	1	1	·	$(8,-8)$
11	12321	·	·	·	·	·	·	·	·	·	·	1	·	1	·	1	$(7,0)$
12	13213	·	·	·	·	·	·	·	·	·	·	·	1	·	·	1	$(8,8)$
13	123121	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	$(9,-5)$
14	123123	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	$(11,-14)$
15	123213	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	$(9,5)$

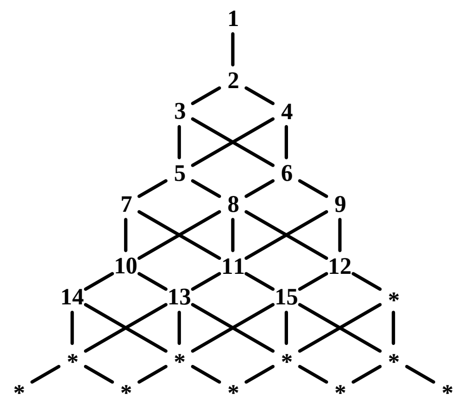

Table 9: *multiplicities* $P^{IJ}_{x,y}(1)$ *and Hasse diagram for the coset* $a_2\backslash\hat{a}_2$

P^{IJ}	$\{3\},\{3\}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	(h,w)
1	e	1	1	1	2	2	2	1	1	2	2	2	2	2	2	2	$(-\frac{1}{9},-\frac{1}{81})$
2	1	·	1	·	1	1	1	1	1	2	2	1	1	2	2	1	$(\frac{17}{9},-\frac{55}{81})$
3	2	·	·	1	1	1	1	1	1	1	1	2	2	2	1	2	$(\frac{8}{9},-\frac{28}{81})$
4	12	·	·	·	1	·	1	1	·	1	1	1	1	2	1	1	$(\frac{35}{9},-\frac{595}{81})$
5	21	·	·	·	·	1	1	·	1	1	1	1	1	1	1	1	$(\frac{44}{9},\frac{836}{81})$
6	121	·	·	·	·	·	1	·	·	1	1	1	1	1	1	1	$(\frac{53}{9},\frac{323}{81})$
7	132	·	·	·	·	·	·	1	·	·	1	1	·	1	1	1	$(\frac{44}{9},-\frac{136}{81})$
8	231	·	·	·	·	·	·	·	1	1	·	·	1	1	1	1	$(\frac{62}{9},-\frac{190}{81})$
9	1231	·	·	·	·	·	·	·	·	1	·	·	·	1	1	·	$(\frac{107}{9},-\frac{2755}{81})$
10	1321	·	·	·	·	·	·	·	·	·	1	·	·	·	1	·	$(\frac{98}{9},\frac{2618}{81})$
11	2132	·	·	·	·	·	·	·	·	·	·	1	·	·	·	1	$(\frac{80}{9},\frac{1700}{81})$
12	2312	·	·	·	·	·	·	·	·	·	·	·	1	1	·	1	$(\frac{89}{9},-\frac{1729}{81})$
13	12312	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	$(\frac{152}{9},-\frac{5320}{81})$
14	12321	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	$(\frac{134}{9},-\frac{406}{81})$
15	21312	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	$(\frac{107}{9},-\frac{325}{81})$

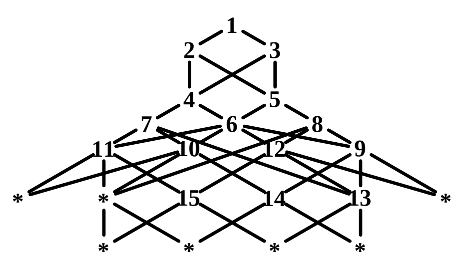

Table 10: *multiplicities* $P^{IJ}_{x,y}(1)$ *and Hasse diagram for the coset* $a_1\backslash\hat{a}_2/a_1$

P^{IJ}	$\{3\},\{2\}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	(h,w)
1	e	1	1	1	1	1	1	2	1	2	2	1	2	3	1	1	$(\frac{2}{9},\frac{10}{81})$
2	1	·	1	1	1	·	1	1	1	1	2	1	1	3	1	1	$(\frac{20}{9},-\frac{260}{81})$
3	13	·	·	1	·	·	1	1	·	1	1	1	1	2	1	1	$(\frac{29}{9},\frac{91}{81})$
4	21	·	·	·	1	·	·	1	1	1	1	1	1	2	1	1	$(\frac{38}{9},\frac{442}{81})$
5	23	·	·	·	·	1	1	1	1	2	2	·	2	2	1	1	$(\frac{11}{9},-\frac{125}{81})$
6	123	·	·	·	·	·	1	·	·	1	1	·	·	1	1	1	$(\frac{47}{9},-\frac{665}{81})$
7	213	·	·	·	·	·	·	1	·	1	·	·	1	1	·	1	$(\frac{65}{9},\frac{1495}{81})$
8	231	·	·	·	·	·	·	·	1	·	1	·	1	1	1	·	$(\frac{65}{9},-\frac{935}{81})$
9	1213	·	·	·	·	·	·	·	·	1	·	·	·	1	·	1	$(\frac{74}{9},\frac{874}{81})$
10	1231	·	·	·	·	·	·	·	·	·	1	·	·	1	1	·	$(\frac{119}{9},-\frac{3689}{81})$
11	1321	·	·	·	·	·	·	·	·	·	·	1	·	·	1	1	$(\frac{83}{9},\frac{2197}{81})$
12	2131	·	·	·	·	·	·	·	·	·	·	·	1	1	·	·	$(\frac{83}{9},\frac{253}{81})$
13	12131	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	$(\frac{128}{9},-\frac{2852}{81})$
14	12321	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	$(\frac{137}{9},-\frac{2015}{81})$
15	13213	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	$(\frac{128}{9},\frac{3952}{81})$

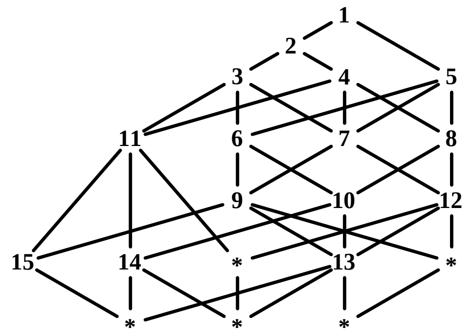

Table 11: *multiplicities* $P^{IJ}_{x,y}(1)$ *and Hasse diagram for the coset* $a_1\backslash\hat{a}_2/a_1'$

P^{IJ}	$\{3\},\emptyset$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	(h,w)
1	e	1	1	1	1	1	1	1	1	2	1	2	1	2	2	1	$(0,0)$
2	1	·	1	·	1	1	1	·	1	1	1	1	1	1	2	1	$(1,-1)$
3	2	·	·	1	1	·	1	1	1	1	1	1	1	1	1	1	$(1,1)$
4	12	·	·	·	1	·	·	·	1	1	1	·	·	1	1	1	$(3,-5)$
5	13	·	·	·	·	1	·	·	·	1	1	1	·	1	1	1	$(2,2)$
6	21	·	·	·	·	·	1	·	1	·	·	1	1	1	1	·	$(3,5)$
7	23	·	·	·	·	·	·	1	·	1	·	1	1	1	1	1	$(2,-2)$
8	121	·	·	·	·	·	·	·	1	·	·	·	·	1	1	·	$(4,0)$
9	123	·	·	·	·	·	·	·	·	1	·	·	·	1	1	1	$(6,-14)$
10	132	·	·	·	·	·	·	·	·	·	1	·	·	·	·	1	$(5,5)$
11	213	·	·	·	·	·	·	·	·	·	·	1	·	1	·	·	$(6,14)$
12	231	·	·	·	·	·	·	·	·	·	·	·	1	·	1	·	$(5,-5)$
13	1213	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	$(8,0)$
14	1231	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	$(10,-30)$
15	1232	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	$(7,-7)$

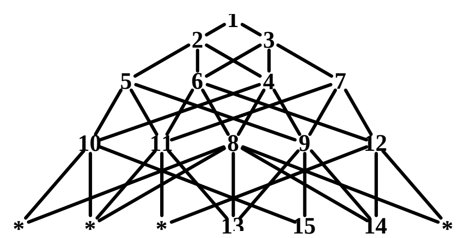

Table 12: *multiplicities* $P^{IJ}_{x,y}(1)$ *and Hasse diagram for the coset* $a_1\backslash\hat{a}_2$

P^{IJ}	$\emptyset,\emptyset$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	(h,w)
1	e	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	$(0,0)$
2	1	·	1	·	·	1	1	1	·	1	·	1	1	1	1	1	$(2,0)$
3	2	·	·	1	·	1	·	1	1	·	1	1	1	·	1	1	$(1,-1)$
4	3	·	·	·	1	·	1	·	1	1	1	·	1	1	1	1	$(1,1)$
5	12	·	·	·	·	1	·	·	·	·	·	1	1	·	1	·	$(4,-10)$
6	13	·	·	·	·	·	1	·	·	·	·	·	1	1	1	1	$(4,10)$
7	21	·	·	·	·	·	·	1	·	·	·	1	·	·	·	1	$(5,15)$
8	23	·	·	·	·	·	·	·	1	·	·	·	1	·	·	1	$(3,-7)$
9	31	·	·	·	·	·	·	·	·	1	·	·	·	1	·	·	$(5,-15)$
10	32	·	·	·	·	·	·	·	·	·	1	·	·	·	1	·	$(3,7)$
11	121	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	$(6,6)$
12	123	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	$(8,-22)$
13	131	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	$(6,-6)$
14	132	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	$(8,22)$
15	213	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	$(9,35)$

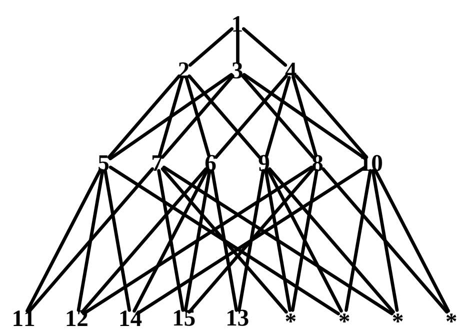

Table 13: *multiplicities* $P^{IJ}_{x,y}(1)$ *and Hasse diagram for the coset* $\hat{a}_2$

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