Examples of q-regularization.

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Abstract. An introduction to Hopf algebras as a tool for the regularization of relevant quantities in quantum field theory is given. We deform algebraic spaces by introducing q as a regulator of a non-commutative and non-cocommutative Hopf algebra. Relevant quantities are finite provided $q \neq 1$ and diverge in the limit $q \to 1$. We discuss q-regularization on different q-deformed spaces for $\lambda\phi^4$ theory as example to illustrate the idea .

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1 Introduction

Thanks to the work done in expressing vector bundles, forms, integration, etc., on locally compact topological spaces X, entirely in terms of the algebra $C(X)$ of complex continuous functions on X vanishing at infinity which forms a commutative C^* algebra; a generalization of ordinary geometry can be introduced. Namely, when expressed in terms of a C^* algebra the above cited notions make sense even when the C^* algebra is not commutative, therefore not of the form $C(X)$ [1]. The simplest non-commutative geometries that have been studied are non-commutative and non-cocommutative Hopf algebras, corresponding to both quantization and curvature.

Meanwhile in classical mechanics states are points of a manifold M and observables are functions on M; in the quantum case, states are one-dimensional subspaces of a Hilbert space H and observables are operators in H. Observables, in both classical and quantum mechanics, form an associative algebra, which is commutative in the classical case and non-commutative in the quantum case. So, we can think of quantization as a procedure that replaces the classical algebra of observables by a non-commutative quantum algebra of observables. The non-commutative Heisenberg algebra, i.e. the algebra that comes up because momentum and space are not simultaneously measurable (so called Heisenberg's uncertainty principle), is the best example to illustrate this idea. Generally speaking, it is expected that even using non commutative geometry, one might nevertheless extend our regular notions of symmetry to the quantum world. If we consider the space of states endowed by a group structure, the functions on this are observables. To quantize such a system one has to construct a non-commutative associative algebra of functions on a locally compact topological group space; i.e. a quantum group [2].

Thinking about quantization of the space-time metric itself, where we cannot use path integration techniques to express quantization in terms of classical fields; we claim the assumption of a smooth manifold structure for space-time to be meaningless in extremely small scales from the experimental viewpoint. The problem is that the finer the accuracy in the observation we ask for, the heavier the test particle we need; eventually the space-time curvature due to both the test particle and the space-time itself can be of the same magnitud. In this context, by relaxing the assumption of smoothness of the space-time manifold and introducing non-commutative algebraic geometry, we propose a scheme called q-regularization, so we can regulate relevant quan-

tities in field theory before renormalising. q (being $q^2 \neq -1$) parametrizes the deformation to the non-commutative and non-cocommutative framework in which relevant quantities in quantum field theories are finite for $q \neq 1$, and reduce to the unregulated, divergent, physical theory as $q \to 1$. Namely, as well as in dimensional regularization we interpolate consistently to dimension $4-\epsilon$ where the relevant quantities are finite (these would be infinite at dimension four); in q-regularization we extend relevant quantities in quantum field theory to a non-commutative and non-cocommutaive Hopf algebra or quantum group (by introducing the parameter q) where the relevant quantities are finite (these would be infinite at $q = 1$; i.e. in $C(X)$, the commutative limit).

We present two examples, the first one is constructed in a four dimensional representation of a particular non-commutative space previously reported [3]. The second example is proposed having in mind q-spinors (two dimensional objects with the generators of $A_q^{2/0}$, Manin's quantum plane [4], as entries) constructed by the projective representation of the Heisenberg algebra, they are braided in a very specific way to obtain a q-deformed space.

Second example is intended as a first step to approach q-regularization in q-Minkowski space-time. We work out this example in a q-deformed space which can be related with both, first example's q-mutator algebra and previously reported [5][6] braided two copies of Manin's quantum planes. Since we do not impose reality conditions, among others, we are not working in anyway in q-Minkowski space-time.

For the second example we want to learn more about the symmetries of our measure, we study a projection in the q-deformed space used and its relation to the $SU_q(2)$ measure. Moreover, we analyze the null directions of the corresponding Hopf algebra that lead to a q-deformed Galilei group.

This paper is organized as follows; in Section 2 we construct the Manin quantum plane out of the non-commutative Heisenberg algebra and introduce the q-spinors as a way to link q-regularization scheme with physically meaningful concepts. In section 3, we present two examples of q-regularization on q-deformed Euclidean spaces for $\lambda\phi^4$ theory. Our scheme can only be carried out in a very particular basis for functions defined on the q-deformed spaces chosen such that we end with a Haar weight that reduces to an ordinary integration. Further work should be done to generalize this. Finally, in order to learn about desired properties of symmetry in this q-regularization we study the zero time projection of the measure we have just introduced in second

example in terms of the $SU_q(2)$ measure and the null directions of the Hopf algebra that lead to a q-deformed Galilei group. The quantum Galilei group has been found as symmetry in condensed matter [7].

2 From Heisenberg algebra to q-spinors.

The goal of this section is to link non-commutative Heisenberg algebra with two co-cycles and q-spinors as defined by Manin [4]. Let us start by the fundamental Heisenberg commutator algebra generated by translations on phase space $(\mathbf{r}, \mathbf{p})$;

$$\begin{aligned} \left[r^i, p^j\right] &= i\hbar\delta^{ij} \\ \left[r^i, r^j\right] &= \left[p^i, p^j\right] = 0. \end{aligned} \tag{1}$$

We propose the following translation operator on phase space;

$$U(\mathbf{a}, \mathbf{b}) = e^{i(\mathbf{a}\cdot\mathbf{p} - \mathbf{b}\cdot\mathbf{r})/\hbar} \text{ where } \mathbf{a} \text{ and } \mathbf{b} \in R^n \ . \tag{2}$$

In a ray or projective representation, eq(2) obeys the composition law [8].

$$U(\mathbf{a}_2, \mathbf{b}_2) \cdot U(\mathbf{a}_1, \mathbf{b}_1) = e^{[2\pi i\alpha_2(\mathbf{r};(\mathbf{a}_1,\mathbf{b}_1),(\mathbf{a}_2,\mathbf{b}_2))]} \cdot U(\mathbf{a}_1 + \mathbf{a}_2, \mathbf{b}_1 + \mathbf{b}_2), \tag{3}$$

where $\mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_2, \mathbf{b}_2 \in R^n$ and, for a free particle in quantum mechanics, the two co-cycle α_2 for translations in the phase space is given by

$$2\pi\alpha_2\left(\mathbf{r}; (\mathbf{a}_1, \mathbf{b}_1), (\mathbf{a}_2, \mathbf{b}_2)\right) = \frac{1}{2\hbar}(\mathbf{a}_1 \cdot \mathbf{b}_2 - \mathbf{a}_2 \cdot \mathbf{b}_1). \tag{4}$$

Let us now consider the following infinitesimal Galilei transformation [8]

$$\begin{aligned} \mathbf{r}' = \mathbf{r} + \mathbf{a}_1 = \mathbf{r} + \hbar\mathbf{u} \qquad \mathbf{r}'' = \mathbf{r} + \mathbf{a}_2 = \mathbf{r} \\ \mathbf{p}' = \mathbf{p} + \mathbf{b}_1 = \mathbf{p} \qquad \mathbf{p}'' = \mathbf{p} + \mathbf{b}_2 = \mathbf{p} + \hbar\mathbf{u}, \end{aligned} \tag{5}$$

where $\mathbf{u}$ is a unit vector in R^n.

If we define

$$q = e^{-i\hbar}, \tag{6}$$

impose eq(5) as symmetry in eq(4) and substitute the result in eq(3), it is straightforward to prove that

$$U(\hbar\mathbf{u},0)\mathrm{U}(0,\hbar\mathbf{u}) = \mathrm{qU}(0,\hbar\mathbf{u})\mathrm{U}(\hbar\mathbf{u},0) \tag{7}$$

is a realization of $A_q^{(2/0)}$; i.e. this fulfills the non-commutative algebra of the Manin's quantum plane [4].

Like other authors [5][6], we call the following two dimensional object a q-spinor (more properly Weyl q-spinor).

$$Z^\rho = \begin{bmatrix} Z^1 \\ Z^2 \end{bmatrix} = \begin{bmatrix} U(\hbar\mathbf{u},0) \\ U(0,\hbar\mathbf{u}) \end{bmatrix}, \text{ i.e. } \rho = 1,2\ . \tag{8}$$

In example 2 we use an approach [5] [6] in which the q-deformed space can be related with the tensor product representation of two q-spinor spaces called $(Z^i, \tilde{Z}^i)$. A pair of q-spinors ($i = 1,2$) is introduced in each space. Hereafter greek indices are for spinor suffix and roman ones for different spinors. Besides it is required the following braiding

$$Z^i \tilde{Z}^j = \hat{R}^{ij}_{j'i'} \tilde{Z}^{j'} Z^{i'} \tag{9}$$

where $\hat{R}^{ij}_{j'i'}$ is the Yang-Baxter matrix for $SL_q(2,C)$.

3 Examples on q-regularization

In this section we present two examples of q-regularization for $\lambda\phi^4$ theory on two apparently different q-deformed spaces, both Euclidean. The first case involves a four dimensional version of a Hopf algebra previously reported [3]; we propose to extend momenta internal to Feynman loops to a non-commutative structure. Second example involves a braided 4-dimensional representation of Manin's quantum plane (so called q-spinors) where some particular transformations on the generators of this Hopf algebra relates to the one used in example 1. Actually, example 1 is posed in order to better explain example 2 which is considered as a preliminary step for formulating q-regularization on q-Minkowski space-time.

EXAMPLE 1. From reference 3, let us consider the Hopf algebra L generated by (l_1, l_2, l_3, l_4) and

$$[l_k, l_j] = i l_j Q', \text{ for k=2,4 and j=1,3} \tag{10}$$

where $Q' = \sqrt{1-q}$. Define on this, the antipode map as

$$S(l_k) = -l_k \qquad S(l_j) = -q^{-i} l_j q^{-l_k/Q'}, \tag{11}$$

the coproduct map is given by

$$\triangle l_k = l_k \otimes 1 + 1 \otimes l_k \qquad \triangle\, l_j = l_j \otimes 1 + q^{\frac{l_k}{Q'}} \otimes l_j \text{ and} \tag{12}$$

the counit

$$\epsilon(l_k) = \epsilon(l_j) = 0. \tag{13}$$

Additionally, L can become a C^* algebra if we define

$$l_k^* = l_k \qquad l_j^* = l_j^* q^{i/2} \tag{14}$$

For every finite-dimensional Hopf algebra there is an invariant integration, the Haar weight $\int$, unique up to normalization.

A basis $B^{a_1,\dots a_4} = e^{ia_1 l_1} \dots e^{ia_4 l_4}$ where $a_n \in C$, being C the complex is chosen, then the dual basis $D_{a_1',\dots a_4'}$ is given via

$$B^{a_1 \dots a_4} D_{a_1' \dots a_4'} = \delta(a_1' - a_1) \dots \delta(a_4' - a_4), \quad a_n' \in C \tag{15}$$

where the Dirac delta functions δ have been defined with respect to the usual Lebesgue integration, then it is straightforward, by analogy with the case of finite dimensional Hopf algebras [9], to prove [10]

$$\int \int f = [2\pi\delta(0)]^k \int \prod_j da_j' f'(0, a_j'(1 - q^{-i})) \tag{16}$$

for all j and f suitable of being written on the basis $B^{a_1 \dots a_4}$. In case $q \neq 1$ and assuming proper analycity and decay of f' (the Fourier transform of the Wick ordered function f), eq(16) might be finite for suitable f. If $q = 1$ eq(16) certainly diverges.

We propose, from $\lambda\phi^4$ theory, to q-regularize the vertex corrections with contributions given by

$$\Gamma(s) = \frac{(-i\lambda)^2}{2} \int \int \frac{d^4 l}{(2\pi)^4} \frac{i}{(l-p)^2 - \mu_0^2 - i\epsilon} \frac{i}{l^2 - \mu_0^2 + i\epsilon} \tag{17}$$

where s is any of the Mandelstam variables. These corrections diverge logarithmically.

Let us extend the internal momentum in the Feynman loop in $\Gamma(s)$ to the non-commutative algebraic framework by considering instead of the standard Lebesgue integration, the Haar weight above defined on the basis $B^{a_1 \cdots a_4}$, thus;

$$\Gamma_q(s) = \tag{18}$$

$$\frac{\lambda_0^2 \delta(0)}{2(2\pi)^3} \int \frac{d^j l'_j}{\left[p_k^2 + \left(l'_j(1-q^{-i}) - p_j\right)^2 - \mu_0^2 + i\epsilon\right]\left[\left(l'_j(1-q^{-i})\right)^2 - \mu_0^2 + i\epsilon\right]}$$

where l'_j are the odd components of the dual internal momentum that was extended to non-commutative geometry and $p_k(p_l)$ are the even (odd) components of the external momentum in standard Euclidean commutative four dimensional space-time. Unless $q = 1$, eq(18) is finite, thus we have a regularization scheme. An additional attempt of q-renormalization has recently been presented [10]. Since we extend to the non-commutative framework only the internal momenta degrees of freedom, the lack of locality, consequence of this extention has not experimental consequences in this case. [11].

EXAMPLE 2. Let us consider the Hopf algebra H generated by 1 and $(a, \bar{a}, b, \bar{b})$ such that;

$$\left[b, \bar{b}\right] = 0, \qquad [a, \bar{a}] = 2(q^{-1} - q) q^{\frac{1}{2Q'}(b + 3\bar{b})} \tag{19}$$

$$\left[\bar{b}, a\right] = [b, a] = 2Q'\bar{a}, \qquad \left[\bar{b}, \bar{a}\right] = [b, \bar{a}] = 2Q'a.$$

The coproduct map $\triangle$ in this Hopf algebra is

$$\triangle a = a \otimes 1 + q^{\frac{b}{Q'}} \otimes a \qquad \triangle b = b \otimes 1 + 1 \otimes b \tag{20}$$

$$\triangle \bar{a} = \bar{a} \otimes 1 + q^{\frac{\bar{b}}{Q'}} \otimes \bar{a} \qquad \triangle \bar{b} = \bar{b} \otimes 1 + 1 \otimes \bar{b},$$

the antipode map S is

$$\begin{aligned} S(a) &= \frac{1}{2}\left\{-(q^{-2} + q^2) a q^{-\frac{b}{Q'}} + (q^2 - q^{-2}) \bar{a} q^{-\frac{b}{Q'}}\right\} \qquad (21) \\ S(\bar{a}) &= \frac{1}{2}\left\{(q^2 - q^{-2}) a q^{-\frac{\bar{b}}{Q'}} - (q^{-2} + q^2) \bar{a} q^{-\frac{\bar{b}}{Q'}}\right\} \\ S(b) = -b & \qquad S(\bar{b}) = -\bar{b}, \end{aligned}$$

and finally the counit map ϵ is

$$\epsilon(a) = \epsilon(\bar{a}) = \epsilon(b) = \epsilon(\bar{b}) = 0 \tag{22}$$

Furthermore, we can make this into a $*$-algebra via

$$b^* = b, \quad \bar{b}^* = \bar{b} \quad a^* = aq^{i/2}, \quad \bar{a}^* = \bar{a}q^{i/2}$$

iff q is a primitive root of unity such that $q^4 = 1$.

We would like to relate H with

$$X^{ij} = \tilde{Z}^i Z^j \ \in A_q^{2/0} \otimes A_q^{2/0} \ \ i,j = 1,2; \tag{23}$$

where $\tilde{Z}^i$ and Z^j where introduced in Section 2 (eq(8) and eq(9)). It is straightforward to prove that $A_q^{2/0} \otimes A_q^{2/0}$ is isomorphic to the real algebra generated by 1 and (A,$\bar{A}$,B,$\bar{B}$), where

$$A = X + Y, \quad \bar{A} = X - Y, \quad B = Z + T, \quad \bar{B} = Z - T \tag{24}$$

and

$$X = q^{-1/2}X_{11}, \quad Y = q^{-1/2}X_{12}, \quad Z = \frac{q^{-1}X_{21} - qX_{22}}{\sqrt{q+q^{-1}}}, \quad T = \frac{X_{21} + X_{22}}{q\sqrt{q+q^{-1}}}. \tag{25}$$

To relate H with $A_q^{2/0} \otimes A_q^{2/0}$ let us rewrite the $(A, \bar{A}, B, \bar{B})$ generators, for $q \neq 1$, as follows

$$\begin{aligned} A = a, \quad \bar{A} = \bar{a}, \\ B = q^{\frac{b}{Q'}}, \quad \bar{B} = q^{\frac{\bar{b}}{Q'}}. \end{aligned} \tag{26}$$

On the other hand, it is straightforward to prove that in H, $((a+\bar{a})$, $(a-\bar{a})$, b, $\bar{b})$ corresponds to the algebra L with generators (l_1, l_2, l_3, l_4) defined in example 1. If $q \to 1$, the algebra becomes the commutative algebra of functions on the space generated by $(a, \bar{a}, b, \bar{b})$ and the unit.

Like in example 1, we proceed defining the Haar measure $\int\int$ as a map $H \to C$, such that

$$\int\int f = f_{(1)} \int\int f_{(2)} \quad \forall f \in H \tag{27}$$

Here we have expressed the action of $\triangle$ on f as $\triangle f = f_{(1)} \otimes f_{(2)}$. We remark that it is well known in the theory of Hopf algebras [12] that eq(27) is the dual formulation of the usual left invariance.

By analogy with the case of finite dimensional Hopf algebras [9], we use the following formal expression for eq(27)

$$\int \int f = Tr_H L_f S^2 \tag{28}$$

where L_f stands for f acting by left multiplication on H.

From eq(21) follows

$$S^2(a-\bar{a}) = w^{-1}(a-\bar{a}) \quad ; \quad S^2(a+\bar{a}) = w^{-1}(a+\bar{a}) \quad ; \quad S^2 b = b \quad ; \quad S^2\bar{b} = \bar{b} \tag{29}$$

where $w^{-1} = f(q)$ and $lim_{q\to 1} w^{-1} = 1$. This shall be used below.

To compute $\int \int$ we propose the following basis in H:

$$F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6} = (F^{\lambda_1\lambda_2}, F^{\lambda_3\lambda_4}, F^{\lambda_5\lambda_6}) = \tag{30}$$

$$\left(e^{i\lambda_1\bar{b}}e^{i\lambda_2\frac{(a-\bar{a})}{2}}, e^{i\lambda_3\bar{b}}e^{i\lambda_4\frac{(a+\bar{a})}{2}}, e^{i\lambda_5\bar{b}}e^{i\lambda_6 b}\right)$$

where

$$F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6} \in H \text{ and } (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6) \in R.$$

We associate to $F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6}$ a dual basis $F_{\lambda_1'\lambda_2',\lambda_3'\lambda_4',\lambda_5'\lambda_6'} \in (A_q^{2/0} \otimes A_q^{2/0})^!$ where $(A_q^{2/0} \otimes A_q^{2/0})^!$ is the dual Hopf algebra of $A_q^{2/0} \otimes A_q^{2/0}$, such that

$$F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6} F_{\lambda_1'\lambda_2',\lambda_3'\lambda_4',\lambda_5'\lambda_6'} = (\delta(\lambda_1'-\lambda_1)\delta(\lambda_2'-\lambda_2), \tag{31}$$

$$\delta(\lambda_3'-\lambda_3)\delta(\lambda_4'-\lambda_4), \delta(\lambda_5'-\lambda_5)\delta(\lambda_6'-\lambda_6))$$

In basis eq(30) we have introduced six parameters λ_i, one for each generator involved. They are dual variables to the non-commutative parameter.

Theorem 1. The Haar weight $\int \int F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6}$ defined in eq(28), for a basis $F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6}$ chosen as in eq(30), reduces to an ordinary integration.

Proof. From eq(19) we know that

$$\left[\bar{b}, (a-\bar{a})\right] = -2Q'(a-\bar{a})$$

$$\left[\bar{b},(a+\bar{a})\right] = 2Q'(a+\bar{a})$$

$$\left[\bar{b},b\right] = 0$$

Note that $\frac{(a+\bar{a})}{2} = X$, $\frac{(a-\bar{a})}{2} = Y$ in eq(25). Substituting the basis given by eq(30) in eq(28) and using the Glaube formula for operators we obtain the ordinary integral

$$\int\int F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6} = \left(\int_{-\infty}^{\infty} d\lambda_1' d\lambda_2' \delta(\lambda_1' - (\lambda_1 + \lambda_1'))\delta(\lambda_2' - (\lambda_2 e^{-2i\lambda_1' Q'} + w^{-1}\lambda_2')), \right. \tag{32}$$

$$\int_{-\infty}^{\infty} d\lambda_3' d\lambda_4' \delta(\lambda_3' - (\lambda_3 + \lambda_3'))\delta(\lambda_4' - (\lambda_4 e^{2i\lambda_3' Q'} + w^{-1}\lambda_4')),$$

$$\left. \int_{-\infty}^{\infty} d\lambda_5' d\lambda_6' \delta(\lambda_5' - (\lambda_5 + \lambda_5'))\delta(\lambda_6' - (\lambda_6 + \lambda_6'))\right)$$

Q.E.D.

The basis $F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6}$ in eq(30) admits an expression in terms of the q-spinor defined in eq(8) out of the projective representation for the Heisenberg algebra. Furthermore, this basis can be rewritten in terms of q-Majorana spinors built using q-Weyl spinors in analogy with the commutative algebraic formulation. As a result of this we can show how does not matter if we think in terms of integrating out non-commutative light cone coordinates, Weyl q-spinors or Majorana q-spinors degrees of freedom; the result is exactly the same. Furthermore, the Haar measure $\int\int$ defined on H can be written in terms of ordinary integration.

Theorem 2. For a suitable $f \in H$ that can be expressed on the basis $F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6}$ given in eq(30) (or any of their different q-spinor representations), $\int\int f$ as defined in eq(28) contains a component that can be q-regularized, i.e. is finite provided $q \neq 1$, but infinite in the limit $q = 1$.

Proof.

It is straightforward to show that for any function f defined on H with basis $F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6}$ the following transformation holds

$$f =: f' := \int_{-\infty}^{\infty} d\lambda_1 d\lambda_2 \tilde{f}(\lambda_1,\lambda_2)F^{\lambda_1\lambda_2} + \tag{33}$$

$$\int_{-\infty}^{\infty} d\lambda_3 d\lambda_4 \tilde{f}(\lambda_3,\lambda_4)F^{\lambda_3\lambda_4} + \int_{-\infty}^{\infty} d\lambda_5 d\lambda_6 \tilde{f}(\lambda_5,\lambda_6)F^{\lambda_5\lambda_6}$$

where we express f as a normal ordered form of f', in terms of the generators. Namely putting $\bar{b}$ to the left of a, $\bar{a}$ and b in the light cone coordinate approach; $\{\sigma^3, \sigma^0\}$ to the left of $\{\sigma^+, \sigma^-\}$ in the Weyl q-spinor formulation and finally $\{\gamma^3, \gamma^0\}$ to the left of $\{\gamma^1, \gamma^2\}$ in the Majorana q-spinor basis. Here $(\sigma^0, \sigma^3, \sigma^+, \sigma^-)$ and $(\gamma^0, \gamma^1, \gamma^2, \gamma^3)$ are q-deformed Pauli and Dirac matrices [6]. Additionally $\tilde{f}$ is the Fourier transform of f', i.e.

$$\tilde{f}(\lambda_i, \lambda_j) = (2\pi)^{-2} \int_{-\infty}^{\infty} d\mu_i d\mu_j f'(\mu_i \mu_j) e^{-i\mu_i \lambda_i} e^{-i\mu_j \lambda_j} \tag{34}$$

$$i, j = (1,2), (3,4), (5,6).$$

Then carrying on integration on λ_1, λ_3, λ_5, λ_6 we obtain

$$\begin{aligned} \int\int f = \int_{-\infty}^{\infty} d\lambda_1' d\lambda_2' d\lambda_2 \tilde{f}(0, \lambda_2) \delta(\lambda_2'(1 - w^{-1}) - \lambda_2 e^{-2i\lambda_1' Q'}) + \\ \int_{-\infty}^{\infty} d\lambda_3' d\lambda_4' d\lambda_4 \tilde{f}(0, \lambda_4) \delta(\lambda_4'(1 - w^{-1}) - \lambda_4 e^{-2i\lambda_3' Q'}) + \\ \int_{-\infty}^{\infty} d\lambda_5' d\lambda_6' \tilde{f}(0,0), \end{aligned} \tag{35}$$

that after changing the order of integration and integrating on λ_2 and λ_4 becomes

$$\begin{aligned} \int\int f = \int_{-\infty}^{\infty} d\lambda_1' d\lambda_2' e^{\lambda_1' Q'} \tilde{f}(0, \lambda_2'(1 - w^{-1}) e^{2i\lambda_1' Q'}) + \\ \int_{-\infty}^{\infty} d\lambda_3' d\lambda_4' e^{-\lambda_3' Q'} \tilde{f}(0, \lambda_4'(1 - w^{-1}) e^{-2i\lambda_3' Q'}) + \\ \int_{-\infty}^{\infty} d\lambda_5' d\lambda_6' \tilde{f}(0,0) \end{aligned} \tag{36}$$

The last term in eq(36) corresponds to the ordinary divergent term that appears in the standard commutative algebraic formulation of quantum field theory; there is no way we can recover a finite term out of this in the limit $q \to 1$. Checking the non-commutative Hopf algebra generated by X^{ij} we find the reason why this happens to be so; T is central with respect to (X, Y, Z),

so this part of the Haar measure is not really defined on a non-commutative algebraic variety. Therefore we can extract out of $\int\int f$ a q-regularizable part

$$\int\int f - \int_{-\infty}^{\infty} d\lambda_5' d\lambda_6' \tilde{f}(0,0) = \tag{37}$$

$$(2\pi\delta(0)) \left\{ \int_{-\infty}^{\infty} d\lambda_2' \tilde{f}(0, \lambda_2'(1-w^{-1})) + \int_{-\infty}^{\infty} d\lambda_4' \tilde{f}(0, \lambda_4'(1-w^{-1}) \right\}.$$

But $lim_{q\to 1} w^{-1} = 1$ thus, as $q \to 1$, $\int\int f - \int_{-\infty}^{\infty} d\lambda_5' d\lambda_6' \tilde{f}(0,0)$ diverges, by contrast at $q \neq 1$ and assuming suitable analicity and decay of $\tilde{f}$ to allow contour integration, eq(37) can be made finite for suitable f; moreover, this is proportional to $(1-w^{-1})^{-1}$.

In the limit $q \to 1$, the transformation described in eq(26) is non sense, because in this limit the map $H \to L$ is singular. We remark that this does not mean that the q-regularization scheme performed on the algebra H and described up to here is lacking sense in case $q = 1$ but only the map that relates this with $A_q^{2/0} \otimes A_q^{2/0}$. We are interested in this map because might be of some help in the future construction of q-regularization on q-Minkowski space-time. Further work should be done in this direction.

Q.E.D.

For obvious reasons, the vertex correction for $\lambda\phi^4$ theory described in example 1 is suitable of being q-regularized on H as well as in example 2 was on L. Further work should be done to generalize these examples to more interesting cases. Since this scheme is strongly basis dependant, a complete analysis of the class of functions suitable of being q-regularized on physically interesting basis is needed. Note that q-regularization may be considered equivalent to dimensional regularization in a similar sense to the McKane and Parisi-Sourlas case [13].

4 Comments and Remarks.

In the paper wherein Woronowicz [14] proves the existence and uniqueness of the Haar measure, i.e. the unique state invariant under left (and simultaneusly right) shifts, for any compact quantum group, he proposes the following q-integration on $SU_q(2)$

$$\int_{q^0}^{1} \mathbf{f} = (1-q) \sum_{k=0}^{\infty} q^k \mathbf{f}(q^k) \quad \text{for any } \mathbf{f} \in SU_q(2), \tag{38}$$

On the other hand, let us set $T = 0$ in eq(25) and from eq(7), we get

$$U_2(\hbar\mathbf{u}, 0) = q^{\frac{1}{2}}\tilde{u}_1(0, \hbar\mathbf{u}) \text{ and } U_2(0, \hbar\mathbf{u}) = -q^{-\frac{1}{2}}\tilde{u}_1(\hbar\mathbf{u}, 0), \qquad (39)$$

this is equivalent to set $\tilde{Z}^\rho = \epsilon^{\rho\sigma}\bar{Z}_\sigma$ in eq(9). Thus

$$A = X + Y, \quad \bar{A} = X - Y \quad B = \bar{B} = Z.$$

We call $X^1 = \frac{1}{2}(A+\bar{A})$,$X^2 = \frac{1}{2}(A-\bar{A})$ and $X^3 = B$; then in terms of this (X^1, X^2, X^3) 3-dimensional vector representation we propose the following basis to be used;

$$F^{\lambda_1\lambda_2,\lambda_3\lambda_4} = \left(e^{i\lambda_1 x^3}e^{i\lambda_2 x^1}, e^{i\lambda_3 x^3}e^{i\lambda_4 x^2}\right) \qquad (40)$$

where $X^1 = x^1$, $X^2 = x^2$, and $X^3 = q^{\frac{x^3}{Q'}}$, as was done in eq(26).

From the work done on the category of representations of a Hopf algebra we can write the action of any function f of the Hopf algebra $SU_q(2)$ on its vector representation V, through the corresponding basis;

$$\mathbf{f}.e^j_m = \sum_{i=+,-,0} f^i \mathbf{e_i}.e^j_m \in V \qquad \forall \mathbf{f} \in SU_q(2) \qquad (41)$$

where $\mathbf{e}_+ = X^+$,$\mathbf{e}_- = X^-$, $\mathbf{e_0} = H$ is the $SU_q(2)$ basis and $f^i \in C$.

From eq(41) is clear that the Woronowicz's map $\int \mathbf{f} \to C$ for the $SU_q(2)$ Haar measure induces a $\int\int f \to C$ map for the vector representation of the Hopf algebra $SU_q(2)$, inducing another for Λ which is written in terms of $SU_q(2)$. We define the Λ matrix as

$$\Lambda^{(ij)}_{(i'j')} \equiv \tilde{M}^i_{i'}M^j_{j'} \quad ; \quad M \in SL_q(2, C),\ \tilde{M} \in \tilde{SL}_q(2, C)\ . \qquad (42)$$

Thus, the similarity of eq(34) and eq(38) can be understood in these terms.

From these two facts and the obvious similarity of the q-integration carried out in eq(34) and the one depicted in eq(38), we think that the q-Time zero-projection in the q-deformed space defined for the second example corresponding to the $\tilde{M}^\dagger = M^{-1}$ identification, reduces $\int\int f - \int_{-\infty}^{\infty} d\lambda'_5 d\lambda'_6 \tilde{f}(0,0)$ in Theorem 2 to the Haar weight on the vector representation of Λ written in terms of $SU_q(2)$.

Finally, we show how null directions in Λ can lead us to obtain the quantum mechanical Galilei group. By imposing the following null bi-ideals

$$u_2^1 = 0, \quad u_1^2 = 0,$$

$$u_1^1 u_2^2 = u_2^2 u_1^1 = 1,$$

being $u_j^i \in M$ (the same is asked for $\tilde{u}_j^i$ being $\tilde{u}_j^i \in \tilde{M}$), we obtain a direct product representation of the quantum Galilei group.

We can see this from the viewpoint of cohomological formalism. We construct the quantum mechanical Galilei group, choosing the following Galilei transformations

$$\mathbf{r}' = \mathbf{r} + \mathbf{v}t \qquad \mathbf{p}' = \mathbf{p} + m\mathbf{v} \tag{43}$$

where m is the particle mass.

Then eq(2) transforms to

$$U(\mathbf{v}) = e^{i\frac{\mathbf{v}\cdot(\mathbf{p}t - m\mathbf{r})}{\hbar}}, \tag{44}$$

and its action on a wave function $\Psi(\mathbf{r})$ introduces a phase (one-cocycle) α_1, i.e.

$$U(\mathbf{v}) \cdot \Psi(\mathbf{r}) = e^{2i\pi\alpha_1(\mathbf{r};\mathbf{v})} \cdot \Psi(\mathbf{r} + \mathbf{v}t). \tag{45}$$

We shall consider this one-cocycle as trivial, so

$$\alpha_1(\mathbf{r};\mathbf{v}) = \delta\alpha_0 = \alpha_0(\mathbf{r}') - \alpha_0(\mathbf{r}) \tag{46}$$

where α_0 is a function, called 0-cocycle, which depends only on $\mathbf{r}$.

Therefore, the group law of the quantum mechanical Galilei group for translations on phase space (or $U(1)$ extended Galilei group) is expressed such that

$$e^{2i\pi\alpha_0(\mathbf{r}')} = e^{2i\pi(\alpha_0(\mathbf{r})+\phi+\alpha_1(\mathbf{r};\mathbf{v}))} \tag{47}$$

where

$$2\pi\alpha_1 = \frac{1}{\hbar}(m\mathbf{v}\cdot\mathbf{r} + \frac{1}{2}mv^2t) \tag{48}$$

and ϕ is the central parameter of the quantum mechanical Galilei group.

On the other hand, we require $M = \begin{pmatrix} u_1^1 & u_2^1 \\ u_1^2 & u_2^2 \end{pmatrix} \in SL_q(2)$ and $M = \begin{pmatrix} \tilde{u}_1^1 & \tilde{u}_2^1 \\ \tilde{u}_1^2 & \tilde{u}_2^2 \end{pmatrix} \in \tilde{S}L_q(2)$ to belong to

the quantum mechanical Galilei group; i.e. M (equivalently $\tilde{M}$) must fulfill eq(5). It is straightforward to prove that, in this case, the following null bi-ideals have to be imposed on M (equivalently on $\tilde{M}$)

$$\begin{aligned} u_2^1 = 0 \qquad & u_1^2 = 0 \\ u_1^1 u_2^2 \;\; = \;\; & u_2^2 u_1^1 = 1, \end{aligned} \tag{49}$$

so to end up with a group that has only one generator as should be.

Besides, we can prove that the null bi-ideals once imposed on $M \in SL_q(2)$ (thereby defining the quantum mechanical Galilei group) produce the following pairing

$$\begin{aligned} < u_2^1, t_m^{\dagger k} >= R_{2m}^{1k} = 0 \quad k,m,s = 1,2 \\ < u_1^2, t_m^{\dagger k} >= R_{1m}^{2k} = 0 \\ < u_1^1 u_2^2, t_s^{\dagger k} >= R_{1m}^{1k} R_{2s}^{2m} = 1 \\ < u_2^2 u_1^1, t_s^{\dagger k} >= R_{2m}^{2k} R_{1s}^{1m} = 1 \end{aligned} \tag{50}$$

where $t_m^{\dagger k}$ $(k,m = 1,2)$ are generators of the dual Hopf algebra for $SL_q(2)$ and R_{kl}^{ij} $(i,j,k,l = 1,2)$ entries of the Yang-Baxter matrix R_G associated with the quantum mechanical Galilei group. This does not determine R_G but restricts the solution to block diagonal matrices.

Finally, if we impose the null-directions given by eq(50) in Λ we obtain the following representation of the quantum Galilei group

$$\Lambda = \begin{pmatrix} (\tilde{u}_1^1)^{-1} u_1^1 & 0 & 0 & 0 \\ 0 & \frac{\tilde{u}_1^1 u_1^1 + q^2 (\tilde{u}_1^1 u_1^1)^{-1}}{1+q^2} & 0 & \frac{q^2 (\tilde{u}_1^1 u_1^1 - (\tilde{u}_1^1 u_1^1)^{-1})}{1+q^2} \\ 0 & 0 & \tilde{u}_1^1 (u_1^1)^{-1} & 0 \\ 0 & \frac{\tilde{u}_1^1 u_1^1 - (\tilde{u}_1^1 u_1^1)^{-1}}{1+q^2} & 0 & \frac{q^2 (\tilde{u}_1^1 u_1^1 + (\tilde{u}_1^1 u_1^1)^{-1})}{1+q^2} \end{pmatrix} \tag{51}$$

Summarizing, in this paper we have introduced the concept of q-regularization and used the projective representation of the non-commutative Heisenberg algebra to construct the Manin quantum plane, thereby defining q-spinors. Using this as a building block we present a q-regularization in terms of a four

dimensional representation of a particular two dimensional non-commutative space at first. Besides, regularization on a q-deformed space that can be maped into a particular braided product of Manin's quantum plane and it is related to the first q-space is studied too. We show how to extract, from relevant quantities, finite components (provided $q \neq 1$) that can become infinite at $q = 1$. To compute the Haar weight, we propose particular basis projected from q-deformed spaces, so the functions to be q-regularized are to be considered on this frame of reference. An example for $\lambda\phi^4$ field theory is presented. Additional work must be done to generalize our scheme to any arbitrary function on q-Minkowski space-time basis.

Finally, in order to learn about the general scheme and its symmetries, we study the T=0 Haar measure in terms of the $SU_q(2)$ measure and the null directions in the Hopf algebra that lead to a quantum mechanical Galilei group.

Although in this paper we can q-regularize only a class of suitable functions (restricted by the particular basis chosen), we think that the full prescription, derived from physical considerations, might apply to make relevant quantities in field theory finite at $q \neq 1$.

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