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General properties of the decay amplitudes for massless particles.

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Abstract

We derive the kinematical constraints which characterize the decay of any massless particle in flat spacetime. We show that in perturbation theory the decay probabilities of photons and Yang-Mills bosons vanish to all orders; the decay probability of the graviton vanishes to one-loop order for graviton loops and to all orders for matter loops. A decay of the graviton might occur in the presence of a short-scale cosmological constant.

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The massless particle which we best know, the photon, is certainly stable for very long times. As for the neutrino, admitted it is really massless, the experimental evidence is less strong, but it is generally regarded as stable too.

Nevertheless, kinematics allows in principle the decay of a massless particle, provided the products are massless and their momenta have the same direction and versus of the initial momentum [1]. This means that the Mandelstam variables of the process vanish, so that its amplitude, regarded as a function of Mandelstam variables, must be computed in this particular "infrared" limit [§]. Moreover, even if the limit of the amplitude is not zero, the phase space for the products reduces to a line in momentum space and then its volume tends to vanish.

In the case of QED it is possible to show in a general way through the Ward identities that the decay amplitude for $\gamma \to \gamma_1 + ... + \gamma_n$ (n odd) can be factorized into a scalar part which is finite and a tensor part which vanishes when all the external momenta are aligned. An analogous reasoning holds for the neutrino. In both cases, it is crucial that the loop amplitudes contain in the denominator the masses of the fermions or of the vector bosons, respectively.

Another example of massless particle is the graviton. Here we do not have any experimental evidence yet. It has been hypotesized [2] that the non-linearity of Einstein equations could lead to a "frequency degeneration" in gravitational waves, a phenomenon which from the quantum point of view would correspond to a decay of the graviton into more gravitons of smaller energy. We were able however to prove through a generalization of the procedure applied to QED that the amplitude of this process vanishes in the perturbation theory on a flat background. The negative mass dimensionality of the Newton constant plays in this case a role analogous to the fermion masses in QED. At the non perturbative level, the hypotesized existence of a cosmological constant could change the situation (see below).

The case of the gluon, although physically quite academic due to the confinement, is particularly interesting because the amplitude of the decay $g \to g_1 + ... + g_n$ (n odd) is finite for $n = 3$ and divergent for $n \geq 5$. (The Ward identities still allow a factorization of this amplitude, but the scalar part contains poles.) Nevertheless, the total decay probability is zero because the phase space for the products is suppressed strongly enough to compensate for the divergence in the amplitude. We thus have here a typical example of cancellation of infrared divergences in the computation of a physical quantity.

[§] In the four-particle amplitude we mean by Mandelstam variables the usual ones, s, t, u; for amplitudes with more external massless particles, they are taken to be all the possible scalar products between the external four-momenta.

A general power counting argument suggests us in which conditions a real decay of a massless particle could be possible: the lagrangian should contain a self-coupling without derivatives and with a coupling of positive mass dimension. This is precisely what happens in quantum gravity in the presence of a cosmological constant, and in fact it has been suggested that in this theory strong infrared effects could become relevant [3]. But one must remind that the cosmological constant also multiplies in the lagrangian a term which is quadratic in the field and thus generates an effective mass for the graviton (if $\Lambda < 0$) or an unstable theory (if $\Lambda > 0$) [4]. A possible way to elude this problem is to admit, like in lattice theory, that the effective cosmological constant vanishes on large scales but not on small scales and is negative in sign [5]. This approach is however out of the scopes of our paper.

The structure of the article is the following. In Section 1 we prove a list of general kinematical properties which characterize the decay of any massless particle. These properties are only due to Lorentz invariance and to the conservation of the total four-momentum. We then introduce an infrared regularization which allows the computation of the decay amplitudes in the limit of vanishing Mandelstam variables. In Section 2 we employ Property 7 (factorization of the partial decay probability) to give a dimensional estimate of the decay probability of the photon, the neutrino, the gluon and the graviton. After recalling in Section 3 how the exact proper vertices are connected to the complete perturbative expression for the decay amplitude, in Section 4 we use the Ward identities for QED, Yang-Mills theory (YM) and Einstein quantum gravity (QG) to find the most general form and momenta-dependence of these vertices. In Section 5 we present our conclusions and a few brief speculations about the possible role of a non-vanishing cosmological constant in the decay of the graviton.

1 General kynematical properties.

In this Section we derive the most general properties of the decay of a massless particle. They are due only to the Lorentz invariance of the process and to the conservation of the total four-momentum.

Property 1. – A massless particle can only decay into massless particles. – In fact, through a suitable Lorentz boost we can make the energy of the initial state arbitrarily small. If, *per absurdum*, in the final state massive particles were present, the energy of this state would be in any reference frame equal or bigger than the sum of the masses.

Property 2. – Let us suppose that the impulse $\vec{p}^{\,0}$ of the initial particle is oriented in a certain direction and versus, for instance let its four-momentum have the form

$$p^0 = (E^0,\, 0,\, 0,\, E^0) \tag{1}$$

Then also the impulses $\vec{p}^{\,1}...\vec{p}^{\,n}$ of the n product particles are oriented in the same direction and versus; in our example we shall have (Fig. 1)

$$p^i = (E^i,\, 0,\, 0,\, E^i); \qquad i = 1, ..., n; \qquad \sum_{i=1}^{n} E^i = E^0 \tag{2}$$

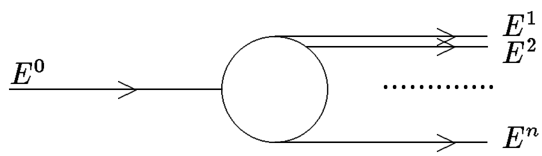

Figure 1: Collinearity property (Property 2).

– Also this property depends on the fact that through a suitable Lorentz boost along z we can make the energy of the initial state arbitrarily small; while if by absurd in the final state some transversal momenta were present, their contribution to the energy would not be affected by the boost. More explicitly, let us consider the $(n+1)$ four-momenta $p^i = (E^i,\, \vec{p}^{\,i}),\ i = 0, 1, ..., n$ in Minkowski space and the following expression

$$G(p^i) = \delta^4\left(\sum_{i=0}^{n} p^i\right) \prod_{i=0}^{n} \delta\left[(p^i)^2\right] \theta(E^0) \prod_{i=1}^{n} \theta(-E^i). \tag{3}$$

The first factor in G expresses the overall four-momentum conservation; the factor $\prod_{i=0}^{n} \delta\left[(p^i)^2\right]$ contrains all the four-momenta to be on-shell; finally, the θ-functions specify that the particle 0 is in-going and the particles $1, ..., n$ are out-going. We shall show that G has support concentrated in a region where all $\vec{p}^{\,i}$ are parallel to each other, more precisely in the region where there exist $\lambda_i < 0$, $i = 1, ...n$ such that

$$p^i = \lambda_i p^0 \tag{4}$$

Proof – We have, from $(p^0)^2 = 0$ and $\sum_{i=0}^{n} p^i = 0$ that

$$\left(\sum_{i=1}^{n} E^i\right)^2 - \left(\sum_{i=1}^{n} \vec{p}^{\,i}\right)^2 = \left(\sum_{i=1}^{n} p^i\right)^2 = 0. \tag{5}$$

Note that the 3-vector $\sum_{i=1}^{n} \vec{p}^{\,i}$ has a length $\ell \leq \sum_{i=1}^{n} |\vec{p}^{\,i}|$, and the equality holds only if $\vec{p}^{\,i} = \lambda_i \vec{p}$ for some $\vec{p}$; on the other hand,

$$\left(p^i\right)^2 = 0, \quad i = 1, ..., n \Rightarrow E^i = |\vec{p}^{\,i}|, \tag{6}$$

therefore (5) reads

$$\left(\sum_{i=1}^{n} |\vec{p}^{\,i}|\right)^2 - \left|\sum_{i=1}^{n} \vec{p}^{\,i}\right|^2 = 0 \tag{7}$$

which can be satisfied only if there exist $\vec{p}$, λ_i (λ_i all of the same sign) such that $\vec{p}^{\,i} = \lambda_i \vec{p}$. On the other hand, $\delta^4 \left(\sum_{i=0}^{n} p^i\right)$ implies that $\vec{p}^{\,i} = \lambda_i \vec{p}^{\,0}$ with $\lambda_i < 0$, and taking the modules we have $E^i = |\vec{p}^{\,i}| = \lambda_i |\vec{p}^{\,0}| = \lambda_i E^0$; this proves (4).

Property 3. – If the initial particle has helicity h and decays into n particles of the same helicity, n must be odd. – The proof follows directly from Property 2 and from the conservation of the angular momentum.

Property 4. – In the decay of a massless particle, all the scalar products $(p^i \cdot p^j)$, $i, j = 0, 1, ..., n$ vanish. This means that the Mandelstam variables vanish. – The proof follows directly from Property 2.

Property 5. – If ε^i represents the polarization vector of the i-th particle involved in the decay, in a gauge such that $(p^i \cdot \varepsilon^i) = 0$, then we have also $(p^i \cdot \varepsilon^j) = 0$ for $i, j = 0, 1, ..., n$. – One more time, the proof follows directly from Property 2.

From Properties 4 and 5 it follows that there are no Lorentz-invariant functions of the external four-momenta and polarizations which can be used in the description of the decay process. The amplitude of the process can only be a constant. As we shall see, in the cases we are examining this constant is zero, except that for QCD.

We define for subsequent use a "decay configuration" as follows: it is a pair of $(n+1)$ four-momenta and $(n+1)$ polarization vectors $(p^i, \varepsilon^i)_{i=0,1,...,n}$ satisfying the properties $(p_i)^2 = 0$, $\sum_{i=0}^{n} p^i = 0$, $(\varepsilon^i \cdot p^i) = 0$, $p_0^0 > 0$, $p_0^l < 0$ for $l = 1, ..., n$. As we have seen, for particles with non-zero helicity n must be odd; in general we have furthermore that $p^l = \lambda_l p^0$ with $l = 1, ..., n$, $\lambda_l \leq 0$; and finally, that $(\varepsilon^i \cdot \varepsilon^j) = 0$ for $i, j = 0, 1, ..., n$.

Property 6. – If a massless particle decays, its lifetime τ in a reference frame where its energy is E^0 has the form

$$\tau = \xi E^0 \tag{8}$$

where ξ is a constant which depends on the dynamics of the process and has dimension $[mass]^{-2}$.
– The proof is based on Lorentz invariance. Let us suppose that in suitable reference system the four-momentum of the particle is

$$p^0 = (E^0,\, 0,\, 0,\, E^0), \tag{9}$$

that is, the particle moves upwards along z, with energy E^0. Consider a Lorentz boost along the z axis, namely of the form

$$L(\beta) = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}. \tag{10}$$

For any four-vector V, we have in the boosted system $V' = L(\beta)V$, that is

$$\begin{aligned} V_0{}' &= \gamma(V_0 - \beta V_z); \\ V_z{}' &= \gamma(V_z - \beta V_0); \\ V_x{}' = V_x; &\qquad V_y{}' = V_y. \end{aligned} \tag{11}$$

At the time $t = 0$, the origins of the two systems coincide. Suppose now that the massless particle is produced at $t = 0$ with the four-momentum p^0 above (eq. (9)) and its decay is observed in the unprimed reference system at a time $t = \tau$, that is, at a coordinate $x = \tau$. Using (11) to transform E^0 and τ in the primed reference system one sees immediately that

$$\frac{E^{0\,\prime}}{\tau'} = \frac{E^0}{\tau}, \tag{12}$$

that is, the lifetime of a massless particle in any reference system is proportional to its energy. This proves (8). Of course, the constant ξ in (8) cannot depend on E^0. We can say that the decay process, if it happens, does not have any characteristic energy scale.

We recall that in quantum field theory the decay probability is given by the formula

$$\tau^{-1} = \frac{1}{2E^0} \sum_{n\geq 2} \int \prod_{i=1}^{n} \frac{d^3p^i}{(2\pi)^3 2E^i}\, \delta^4 \left(p^0 - \sum_{i=1}^{n} p^i\right) |T_n|^2 \tag{13}$$

where T_n is the quantum amplitude for the process with n product particles. Thus the constant ξ of eq. (8) corresponds to half the sum of the integrals in (13), although eq. (8) is more general and does not strictly imply that the probability has the form (13).

We would like now to introduce an infrared regularization in order to allow the concrete computation of the amplitude in the limit in which the Mandelstam variables and the products $(e^i p^j)$ approach zero (compare properties 4, 5 and the subsequent comment). Obtaining such a regularization is not trivial. The most common infrared regularization technique, which consists in giving the soft particles a small mass μ which eventually goes to zero, does not work in the present case, because the (regularized) process in which one particle of mass μ decays into more particles of the same mass has obviously zero probability.

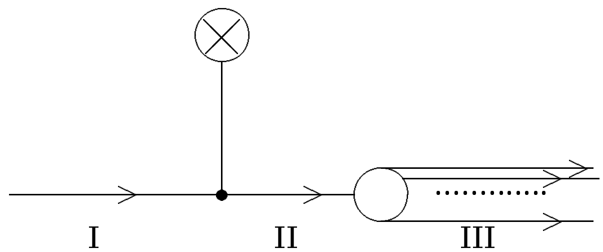

Figure 2: Factorization of the decay amplitude.

Let us instead suppose (Fig. 2) that a very weak external source J gives the decaying particle (state I) an infinitesimal additional energy ω', with probability $f(\omega')$, where f is a function which has a narrow peak around some small value ω. The exact nature of the source and of the particle which carries the energy ω' are not essential. For instance, if J represents a classical field, the energy can be carried by an on-shell boson with four-momentum $(\omega',\ \omega',\ 0,\ 0)$; by absorbing the boson, the initial particle gains a small transversal impulse (state II). Alternatively, the energy ω' could be carried by an off-shell boson produced in J through an annihilation process, with four-momentum $(\omega',\ 0,\ 0,\ 0)$; by absorbing the boson, the initial particle gets off shell too.

Then the decay takes place; the products (state III) have now a small tranversal impulse of order ω and the Mandelstam variables $(p^i \cdot p^j)$ are of order ω^2. The partial decay probability into n product particles is written as a sum over intermediate states (compare (eq. 13))

$$\tau_n^{-1} = \lim_{\omega \to 0} \int d\omega' \, f(\omega') \frac{1}{2E^0_{\omega'}} \int \prod_{i=1}^{n} \frac{d^3 p^i}{(2\pi)^3 2E^i} \, \delta^4 \left(p^{II}_{\omega'} - \sum_{i=1}^{n} p^i \right) |\langle II_{\omega'} | \mathcal{T} | III \rangle|^2. \qquad (14)$$

Let now the probability $f(\omega\prime)$ approach a delta function $\delta(\omega\prime - \omega)$. When $\omega \to 0$, the squared amplitude $|\langle II_{\omega\prime} | \mathcal{T} | III \rangle|^2$ tends to a constant (Properties 4 and 5), in the sense that it does not depend on the final momenta $\vec{p}^{\,1},...,\vec{p}^{\,n}$ over which one integrates in (14), but only on ω and on the couplings and the masses of the theory.

We note that the mass dimension of $|\langle II_{\omega'}|\mathcal{T}|III\rangle|^2$ is equal to $2(3-n)$. Furthermore, let us consider the integral of the phase space. Through the δ^4 it depends on p^{II}, that is, in principle both on ω and E^0. But according to (12) and to the following remark, the integral does not depend on E^0, because this would modify the Lorentz factor $\frac{1}{E^0}$ which already appears in (14). Thus, having dimension $[mass]^{2n-4}$, it must be simply proportional to ω^{2n-4}. We have proved in this way the following property.

Property 7. – The regularized partial decay probability τ_n^{-1} of a massless particle is factorized into three parts: the term $\frac{1}{E^0}$ due to Lorentz invariance; the square of the decay amplitude, which depends on ω and on the couplings and masses of the theory and has dimension $2(3-n)$; the volume of the phase space, proportional to ω^{2n-4}. In formula,

$$\tau_n^{-1} = \lim_{\omega \to 0} \frac{1}{E^0} \times (\text{Phase space} \sim \omega^{2n-4}) \times (\text{Amplitude squared, of dim. } 2(3-n)). \tag{15}$$

Property 8. – Let us finally consider the decay of one massless particle into two particles of the same kind, as it is allowed for spinless particles (then one can generalize to the case of the decay of particles with spin into n particles, with n odd). We would like to find the energy distribution of the product particles; in particular, we wonder if the emission probability of one "infrared" particle is limited, or if this process tends to be dominant. In other words, supposed the decay takes place, does the initial particle prefers (1) to "break into two parts" of comparable energy or rather (2) to loose just a small fraction of its energy through a kind of infrared process? We shall show that (2) is not the case. –

For the proof we recall that according to Property 7 and to the discussion which precedes it the energy distribution of the product particles is determined only by the phase space and not by the amplitude. Let us put the system into a box of volume V. The modes of the massless field in this box have energies which are multiples of some fundamental energy $E_0 = \hbar\omega_0 \sim \hbar c V^{-1/3}$. Since the momenta remain aligned in the decay, we are reduced in practice to a one-dimensional problem.

Let $E = NE_0$ be the energy of the initial particle, and $E_1 = n_1 E_0$, $E_2 = n_2 E_0$ (with $n_1 + n_2 = N$) the energies of the product particles. The number of possible distinct final configurations is clearly given by $N/2$ if N is even and by $(N-1)/2$ if N is odd. Let be $0 \leq x \leq 1$; the number of "infrared" final configurations – those for which one of the two product particles has energy smaller than xNE_0 – divided by the total number of configurations, is a quantity which tends to a constant as $E_0 \to 0$ and thus $N \to \infty$ (that means, $V \to \infty$). For instance, the probability that one of the two product particles carries less than 1/4 of the initial energy equals 1/2 for $N \to \infty$. More generally, the probability that it carries less than a fraction $1/k$ of the initial energy tends to $2/k$.

2 Power counting.

The last factor in eq. (15), namely the probability $|\langle II_\omega|\mathcal{T}|III\rangle|^2$, can be quite easily estimated by dimensional considerations. For instance, in QED the four-photons amplitude is given to lowest order by the four fermions loop (fig. 3a). For small values of the total momentum ω this amplitude – which must be adimensional – behaves like [6]

$$T_4 \sim \alpha^2 \left(\frac{\omega}{m_f}\right)^4, \tag{16}$$

where α is the fine structure constant and m_f is the mass of the fermion. This result can be generalized to the n-fermions loop: the key point is that the fermionic propagators of the loop produce masses in the denominator. The case of the neutrino is analogous: the masses of Z^0 or $W^\pm$ appear at the denominator in the amplitude. In both cases, since the amplitude is proportional to a positive power of the regularizator ω, it vanishes in the infrared limit due to (15).

Figure 3: (a) Fermions square loop. (b) Gravitons or gluons loop.

In the case of pure quantum gravity we have tree and one-loop graviton diagrams with k external legs (fig. 3b). Explicit expressions for the $k = 4$ amplitudes have been given by [7, 8]. In any case, these amplitudes contain positive powers of the constant $\kappa = \sqrt{16\pi G}$ and then, like in QED, they behave always like a positive power of ω and cause the decay probability to vanish.

In the case of QCD the amplitudes do not contain dimensional constants. We expect that the decay amplitude of the gluon into three gluons, being adimensional, tends to a constant when $\omega \to 0$, and this is in fact what happens [8]. The decay amplitudes of a gluon into 5, 7 ... gluons have mass dimensions -2, -4 ... respectively, so they diverge when $\omega \to 0$; but this divergence is compensated in the phase space integral by a bigger positive power of ω (compare (15)), in such a way that the probability behaves like $\omega^2/E^{(0)}$ anf thus vanishes in the limit.

We are not going to apply this power counting argument to all possible theories and couplings, since it is in each case quite immediate. As a last example, we may wonder if a photon can in principle decay due to the gravitational interaction, through diagrams with external photons and one loop of gravitons. Since the coupling constant κ has mass dimension -1, while the fine structure constant α is adimensional and there are no masses involved, we conclude once more that the amplitude of the process vanishes in the infrared limit.

It is clear from eq. (15) that a τ_n^{-1} different from zero can be only obtained when the squared amplitude is proportional to a sufficiently high negative power of ω. If we admit (as is generally true in perturbation theory) that the coupling constants always appear in the numerator, this means that the amplitude must contain a coupling constant with positive mass dimension. We shall return on this point in the conclusions.

3 Diagrammatics: ω-dependence of the decay amplitudes.

The dimensional arguments we used in the previous section allow to determine the ω-dependence of the decay amplitude only for the pure (QED, YM, QG) gauge theories, where the only parameter in the action is the coupling constant (in free QED the latter is absent). If the gauge field is coupled to some matter field, generally speaking new (dimensionful) parameters, like their masses, will appear in the action, and the previous arguments will not be automatically applicable any more. A more explicit analysis of the perturbative expansion and use of Feynman diagrams is therefore needed, in order to determine in full generality the ω-dependence of the decay amplitudes. In this and in the following section we carry it out and show that general results are essentially the same as those found by the dimensional arguments in section 2. We conclude that the decay probability of the gauge bosons of QED, YM, QG vanish.

We will start the analysis of the perturbative expansion from the tree level: a sum of truncated connected tree-diagrams with $(n+1)$ external lines will give the lowest order (in $\hbar$) contribution to the decay amplitude of 1 gauge boson in n gauge bosons. Higher order corrections will involve truncated connected diagrams with one or more loops. To formally compute the "exact" decay amplitude one has to replace in each tree diagram every boson propagator with the corresponding exact boson propagator, and each m-boson vertex with the corresponding m-boson proper vertex (i.e. one-particle-irreducible Green function). To get the $\hbar^r$-order approximation of the decay amplitude, one simply has to retain the terms of order $\leq r$ in this formal " exact" expression. As we will see, the Ward identities imply that when approaching a decay configuration: (1) in QED the decay amplitude of a process with m external photons vanishes; (2) in QG the decay amplitude of a process with m external gravitons or photons vanishes; (3) the decay amplitudes of processes with external Y.M. bosons may be finite or diverge, but in such a way that the corresponding decay probabilities vanish.

3.1 Tree level

Let us start from the Feynman vertices with m gauge massless bosons ($m \geq 3$) [see the actions (26)]: we draw them in fig. (4). The diagrams are to be understood as truncated in the external lines. In QED there is no m-photon vertex. In YM there are only two m-gluon vertices (for $m = 3, 4$). In pure QG there is one m-graviton vertex for every $m \geq 3$; if coupling of gravity with the electromagnetic or the Yang-Mills fields is considered, then there are also vertices with k spin-1 bosons (photons or gluons) and r gravitons, for $k = 2, 3, 4$ and $r \geq 1$. In the figures, a wavy line in the QG case will denote either a graviton or another gauge boson (a photon or a gluon).

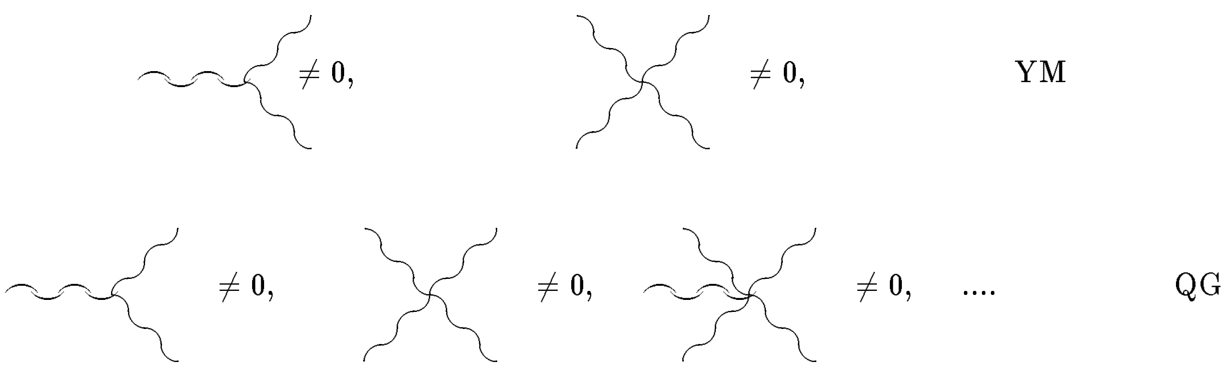

Figure 4: Feynman vertices

At the tree level, the decay amplitudes $T_2^{tree}, T_3^{tree}, T_4^{tree}, T_5^{tree}, ...$ of YM, QG are the sum of the diagrams in fig. (5).

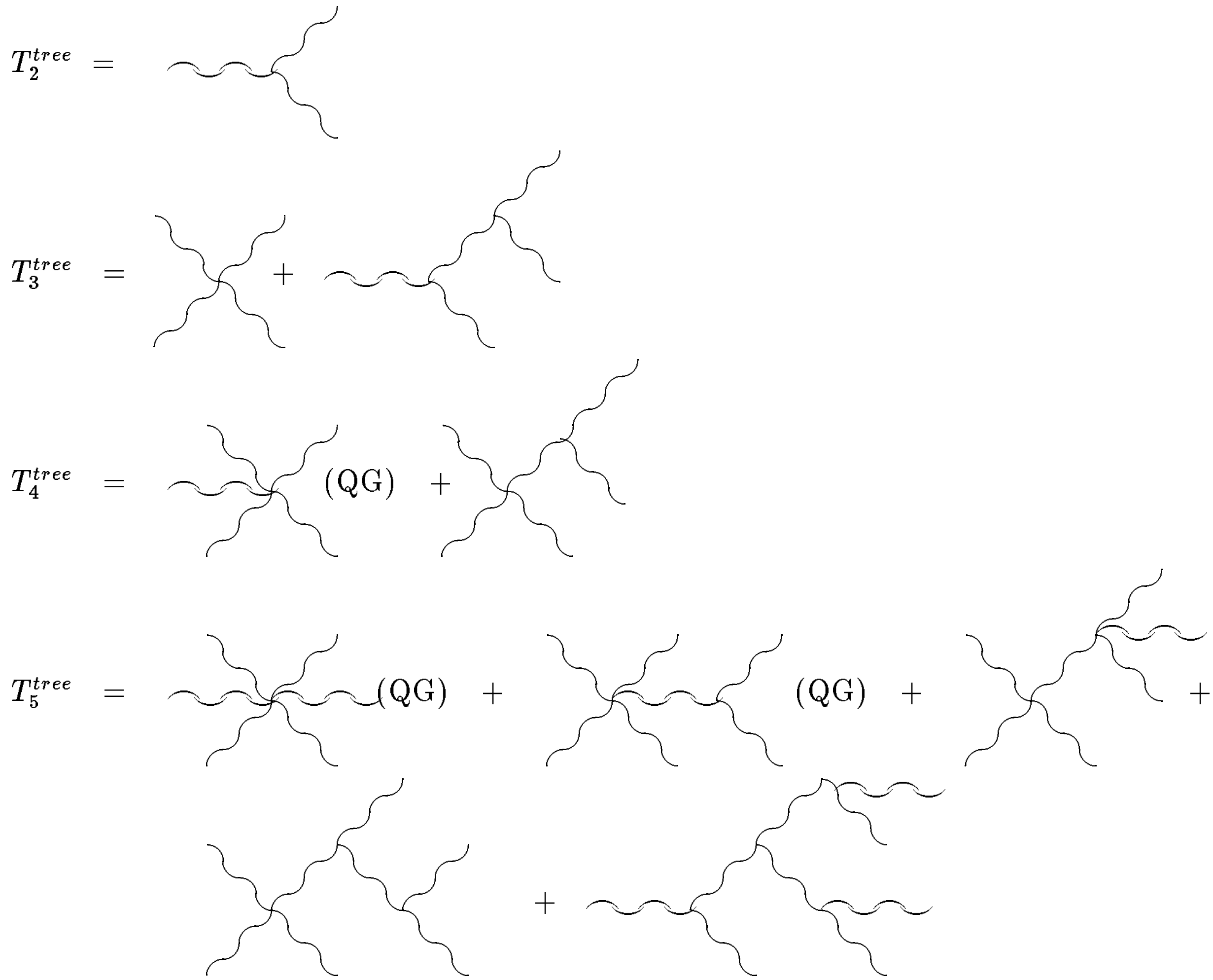

Figure 5: Tree level amplitudes: (QG) means that the diagram in T_5^{tree} is present only in QG.

Tree diagrams involving ghost lines do not contribute to T_n^{tree}. In fact, even though ghosts are massless, diagrams with external ghosts are zero when multiplied by physical polarization vectors, and diagrams with internal ghost lines (propagators) have necessarily also external ghost lines, by ghost number conservation. One can easily verify that in QG the decay amplitudes with only m external gravitons or photons vanish ($T_n^{tree} = 0$) in any decay configuration, because each vertex is quadratic in the momenta k^i, implying an overall $(k)^2$ dependence of each separate diagram in fig. (5); when contracted with the external polarization vectors, this will give zero, since in the decay configuration all 4-momenta are null vectors proportional to each other.

3.2 Higher orders

To formally compute the "exact" decay amplitude one has to replace in each tree diagram every boson propagator with the corresponding exact boson propagator, and each m-boson vertex with the corresponding m-boson proper vertex (i.e. one-particle-irreducible Green function), as depicted in fig. (6); there we have symbolized each proper vertex by a blob. Diagrams involving ghost lines can be excluded for the same reasons as before.

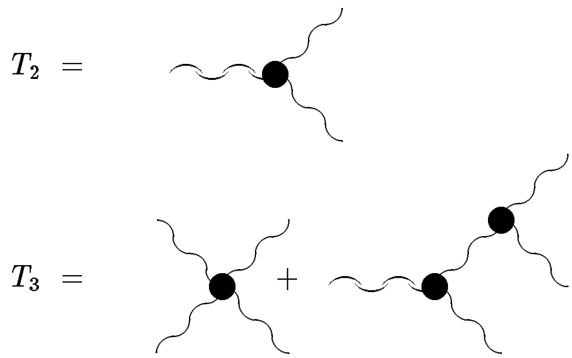

Figure 6: Exact amplitudes

Using Property 2 it is easy to verify that if the external momenta are slightly off-shell, the momenta carried by the propagators in figg. (5), (6) also are, and the scalar products of all momenta are of order ω^2; ω is the infrared regulator (with dimension of a mass) introduced in section 1. The exact propagators for massless particles in the infrared limit have to behave as the naive ones, i.e. are of order ω^{-2}.

Let E_γ, E_y, E_g and I_γ, I_y, I_g denote respectively the number of external and internal photon,YM boson, graviton lines coming out of one of the diagrams in fig. (6). Let m^v_γ, m^v_y, m^v_g denote the numbers of photons,YM bosons, gravitons coming out from the v^{th} proper vertex Γ^v appearing in the same diagram. Clearly,

$$\begin{aligned} E_\gamma &= \sum_v m^v_\gamma - 2I_\gamma \\ E_y &= \sum_v m^v_y - 2I_y \\ E_g &= \sum_v m^v_g - 2I_g. \end{aligned} \tag{17}$$

Moreover,

$$N_p - I_p \geq \theta(E_p) \tag{18}$$

where $\theta(x) := \begin{cases} 0 \text{ if } x = 0 \\ 1 \text{ if } x > 0 \end{cases}$ and N_p denotes the number of proper vertices in the diagram where at least one particle p (p being a YM boson and/or a graviton) comes out; this disequality follow from the fact that $N_p = 0$ if and only if $E_p = 0 = I_p$.

The results of next section (Property 10) can be summarized as follows, that

$$\Gamma^v = o(\omega^{m_\gamma^v + 4\theta(m_y^v) - m_y^v + 2\theta(m_g^v)\delta_0^{m_y^v}}) \tag{19}$$

where in our notation $o(\omega^p)$ will denote an infinitesimal or an infinite of *at least* order p in ω, namely $\lim_{\omega \to 0}[o(\omega^p)\omega^{-p}]$ is zero or finite. The overall ω-dependence of the diagram contribution $D(\omega)$ will be the product of the dependences of each vertex and each propagator:

$$D(\omega) = \left[\prod_v o(\omega^{m_\gamma^v + 4\theta(m_y^v) - m_y^v + 2\theta(m_g^v)\delta_0^{m_y^v}})\right] \omega^{-2(I_\gamma + I_y + I_g)} \tag{20}$$

Using equations (17), the latter becomes

$$D(\omega) = o(\omega^{E_\gamma - E_y + 4(N_y - I_y) + 2(N_g' - I_g)}), \tag{21}$$

where N_g' denotes the number of proper vertices in the diagram where at least one graviton and no YM boson come out. To estimate $4(N_y - I_y) + 2(N_g' - I_g)$ let us distinguish two cases. If $E_y = 0$, then by colour conservation $m_y^v = 0$ for all vertices in the diagram, implying $N_g' = N_g$; using formulae (18) for $p = y$ and $p = g$, we find $4(N_y - I_y) + 2(N_g' - I_g) \geq 4\theta(E_y) + 2\theta(E_g)$. If $E_y > 0$, noting that $(N_y + N_g') = N_p$, $I_y + I_g = I_p$, where now p denotes *either* y *or* g, and using formulae (18) , we find $4(N_y - I_y) + 2(N_g' - I_g) \geq 2\theta(E_y) + 2\theta(E_p) = 4\theta(E_y)$. Summing up, $4(N_y - I_y) + 2(N_g' - I_g) \geq 4\theta(E_y) + 2\theta(E_g)\delta_0^{E_y}$ This expression depends only on the numbers of external bosons of the process, not on the particular diagram we are considering, therefore we find the following

Property 9. – The amplitude T of a decay process with E_γ external photons, E_y external YM boson and E_g gravitons satisfies the condition:

$$T = o(\omega^{E_\gamma - E_y + 4\theta(E_y) + 2\theta(E_g)\delta_0^{E_y}}). \tag{22}$$

This formula is valid at any loop order in all particles different from the gravitons and at least at one loop order in the gravitons, because the matter action with a background metric is multiplicatively renormalizable [11], whereas at first order in the graviton loops pure QG is finite on-shell.

Note that the RHS of formula (22): 1) is independent of the number of external gravitons, provided $E_y > 0$; 2) vanishes if $E_y = 0$.

4 Ward identities

In QED the proper n-photon vertices $\Gamma_n^{\mu_1...\mu_n}(p^1,...,p^n)$ satisfy the Ward identity

$$p_{\mu_1}\Gamma_n^{\mu_1\mu_2...\mu_n}(p^1,p^2,...,p^n)\varepsilon_{\mu_2}(p^2)...\varepsilon_{\mu_n}(p^n)=0, \tag{23}$$

where p^i is the momentum of the i-th photon and $\varepsilon_{\mu_i}(p^i)$ the corresponding polarization vector; this transversality condition amounts to the gauge invariance of any physical process involving n (incoming or outgoing) photons.

In this section we first derive the identity above and its analogues for general Yang-Mills (YM) and Einstein (with $\Lambda=0$) Quantum Gravity (QG) theories in the momentum configuration of decay processes (compare with Property 2). Then we use them and a continuity argument to show that the proper vertex for any decay process with fixed external momenta vanishes in QED and QG, whereas it is finite in YM. The Ward identities are derived formally by using naive functional integration considerations based only on the gauge invariance of the classical action (not on its explicit form). In the case of QED,YM, their validity extends to the true (i.e. renormalized) theories at any order in the loops because renormalization preserves Ward identities. In the case of QG, their validity is guaranteed at any loop order in the matter fields and at least at one loop order in the gravitons, because the matter action with a background metric is multiplicatively renormalizable [11], whereas at first order in the graviton loops pure QG is finite on-shell.

We start by fixing the notation. Let $\mathcal{S}_{inv}(\phi)$ denote the (local) action depending on the classical fields $\{\phi_I\}$ and $R^I_\alpha(\phi)$ corresponding (local) gauge generators:

$$\delta_\xi\mathcal{S}_{inv}=\frac{\delta\mathcal{S}_{inv}}{\delta\phi_I}\delta_\xi\phi^I=0, \tag{24}$$

We employ a condensed notation in which a capital indicex I is a collective index; it represents both discrete indices and a continuous space-time variables x. A repeated index implies summation over discrete indices and integration over x. Explicitly, in the case of QED, YM, QG the fields ϕ_I include

$$\phi_I:=\begin{cases} A_\mu(x),\ \psi(x),\bar\psi(x)\ \text{ and/or } \varphi(x),\ \bar\varphi(x) \text{ in QED;}\\ A^a_\mu(x),\ \ +\text{ possibly } \psi^i(x),\ \bar\psi^i(x) \text{ and/or } \varphi^i(x),\ \bar\varphi^i(x) \text{ in YM;}\\ h_{\mu\nu}(x)\ \ +\text{ possibly any } \phi_I \text{ considered in the two previous cases in QG;}\end{cases} \tag{25}$$

$x\in M^4$ denotes the point in Minkowski spacetime, $A_\mu(x), A^a_\mu(x)$ the gauge potentials corresponding respectively to a $U(1)$ and a semisimple group G, $\psi(x),\bar\psi(x)$ (resp. $\varphi(x),\bar\varphi(x)$) spinors (complex scalars), $\psi^i(x), \bar{\psi}^i(x)$ (resp. $\varphi^i(x), \bar{\varphi}^i(x)$) spinors (complex scalars) making up a finite multiplet belonging to some finite representation $Rep(\mathcal{L}ie(G))$ (in the latter case $(T^a)^i_j$ will denote the matrix representation of the hermitean Lie algebra generators corresponding to A^a_μ), $h_{\mu\nu}(x)$ is the graviton field, $\eta_{\mu\nu}$ denotes the Minkowski metric tensor (which plays the role of background metric) in cartesian coordinates, and $g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ is the the metric tensor. The invariant actions $\mathcal{S}_{inv}$ read

$$\mathcal{S}_{inv} = \begin{cases} -\frac{1}{4}\int_{M^4} d^4x \, (F^{\mu\nu}F_{\mu\nu}) + \mathcal{S}_{mat} & \text{in QED;} \\ -\frac{1}{4}\int_{M^4} d^4x \, (F^{a\ \mu\nu}F^a_{\mu\nu}) + \mathcal{S}_{mat} & \text{in YM;} \\ \int_{M^4} d^4x \, g^{\frac{1}{2}}(\lambda - \frac{1}{16\pi G}R) + \mathcal{S}_{mat} & \text{in QG,} \end{cases} \tag{26}$$

where $F_{\mu\nu}, F^a_{\mu\nu}$ is the field strenght in QED,YM respectively, R is the Ricci scalar of the metric $g_{\mu\nu}$, $g := -det[g_{\mu\nu}]$, f^{abc} are the structure constants of $\mathcal{L}ie(G)$ and e the coupling constant. $\mathcal{S}_{mat}$ is the action of the matter minimally coupled to the gauge potential ¶.

$A_\mu, A^a_\mu, h_{\mu\nu}$ are respectively the gauge potentials for QED, YM, QG, with gauge transformations

$$\begin{aligned} \delta_\xi A_\mu &= \partial_\mu \xi \qquad \text{in QED;} && (27) \\ \delta_\xi A^a_\mu &= (D_\mu \xi)^a := \partial_\mu \xi^a + e f^{abc} A^b_\mu \xi^c \qquad \text{in YM;} && (28) \\ \delta_\xi g_{\mu\nu} &= g_{\nu\rho}\partial_\mu \xi^\rho + g_{\mu\rho}\partial_\nu \xi^\rho + \xi^\rho \partial_\rho g_{\mu\nu}, && \\ \delta_\xi A^a_\mu &= A^a_\rho \partial_\mu \xi^\rho + \xi^\rho \partial_\rho A^a_\mu \qquad \text{in QG.} && (29) \end{aligned}$$

We omit for the sake of brevity the well-known gauge transformations of the other fields.

The quantization of the theory (in a perturbative setting) is performed in the BRST formalism [10, 9]: the set of fields $\{\phi_I\}$ is enlarged to a set $\{\Phi_A\}$ by the introduction of ghosts, antighosts and Stueckelberg fields, and we associate to the action $\mathcal{S}_{inv}$ a gauge-fixed action S_Ψ depending on the gauge-fixing functional Ψ. Index A, like I, represents both discrete indices and the continuous space-time variables x. Let $S_{GF} := S_\Psi(\Phi) - \mathcal{S}_{inv}(\phi)$; in QED and YM, S_{GF} can be constructed as $S_{GF} = s\Psi$, where s denotes the BRST transformation associated to the gauge transformations (27) - (29).

The generating functional $Z(J)$ (depending on the external sources J) for the Green func-

¶Strictly speaking, in the case of QG an action $\mathcal{S}_{mat}$ containing a spinor contribution requires the introduction of vierbeins as dynamical variables instead of the metric. However, the considerations of this section hold also in that case, since they are based on the gauge tranformations (29) of the metric, which can be obtained from the gauge transformations of the vierbeins.

tions of the theory is defined by

$$Z(J) := \int \mathcal{D}\Phi e^{\frac{i}{\hbar}[S_\Psi(\Phi)+J^A\Phi_A]}, \tag{30}$$

where $\mathcal{D}\Phi$ is a gauge invariant functional measure, J^A transforms under diffeomorphisms as the appropriate tensor density.

By performing a gauge ‖ transformation $\phi \to \phi + \delta_\xi \phi$ of the dummy integration variables ϕ in the RHS of eq. (30) the integral $Z(J)$ remains the same (the Jacobian is 1), implying the Ward identities

$$0 = \delta_\xi Z(J) = \frac{i}{\hbar}\int \mathcal{D}\Phi [J^A\, \delta_\xi \Phi_A + \delta_\xi S_{GF}] e^{\frac{i}{\hbar}[S_\Psi(\Phi)+J^A\Phi_A]}, \tag{31}$$

or, in terms of the generating functional $W(J) := \frac{\hbar}{i} ln[Z(J)]$ of the connected Green functions,

$$0 = \left[J^A\, \delta_\xi \Phi_A + \delta_\xi S_{GF}\right]\Big|_{\Phi_A \to \frac{\delta}{\delta J^A}} W(J) + disconnected\ \ terms. \tag{32}$$

The disconnected terms are absent when evaluating the Green function on any decay process, since in this case only *one* initial particle is present. Therefore, as far as we are concerned,

$$0 = \left[J^A\, \delta_\xi \Phi_A + \delta_\xi S_{GF}\right]\Big|_{\Phi_A \to \frac{\delta}{\delta J^A}} W(J). \tag{33}$$

In order to obtain the Ward identities for the proper vertex functions we introduce the usual Legendre transform $\Gamma(\tilde{\Phi}) := [W(J) - J^A\Phi_A]|_{J=J(\tilde{\Phi})}$, where the function $J = J(\tilde{\Phi})$ is obtained by inverting the relations $\tilde{\Phi}_A = \frac{\delta W}{\delta J^A}$; the new independent variables are the "classical fields" $\tilde{\Phi}$. Consequently $J^A(\tilde{\Phi}) = -\frac{\delta\Gamma}{\delta\tilde{\Phi}_A}$.

From identity (33) we draw the following Ward identities for the generating functional of proper vertices Γ

$$0 = \left[\frac{\delta\Gamma}{\delta\tilde{\Phi}_A} \cdot \delta_\xi \tilde{\Phi}_A + \delta_\xi S_{GF}(\tilde{\Phi})\right]. \tag{34}$$

Actually, we are interested in the Ward identities for the proper vertices having only physical gauge bosons as external (incoming or outcoming) particles. The physicality condition is best imposed in momentum space. The proper vertex $\Gamma_n^{12...n}(x^1, x^2, ..., x^n)$ with n external gauge bosons $b_i(x^i)$ (in configuration space) is obtained from Γ through differentiation,

$$\Gamma_n^{12...n}(x^1, x^2, ..., x^n) = \frac{\delta^n \Gamma}{\delta b_1(x^1)...\delta b_n(x^n)}\Bigg|_{\tilde{\Phi}=0}, \tag{35}$$

‖Alternatively, one could perform a BRST transformation; the resulting Ward identities would be the same.

where we have introduced the short-hand notation

$$i \to \begin{cases} \mu_i \\ (\mu_i, a_i) \\ \mu_i\nu_i \ \mathit{or}\ (\mu_i, a_i) \end{cases} \qquad b_i \to \begin{cases} \tilde{A}_{\mu_i} & \text{in QED} \\ \tilde{A}^a_{\mu_i} & \text{in YM} \\ \tilde{h}_{\mu_i\nu_i}\ \mathit{or}\ A^a_{\mu_i} & \text{in QG;} \end{cases} \qquad i = 1, 2, ..., n. \tag{36}$$

The RHS has automatically the required boson symmetry in the identical particles, e.g. if all the b_i's are the same type of fields

$$\Gamma_n^{i_1 i_2 \ldots i_n}(x^{i_1}, x^{i_2}, ..., x^{i_n}) = \Gamma_n^{12\ldots n}(x^1, x^2, ..., x^n), \tag{37}$$

where $(i_1, i_2, ...i_n)$ is a permutation of $(1, 2, ..., n)$. On account of the translation invariance $\Gamma_n^{1\ldots n}(x^1, ..., x^n) = \Gamma_n^{1\ldots n}(x^1 + a, ..., x^n + a)$, its multiple Fourier transform can be written as $\Gamma_n^{1\ldots n}(p^1, ..., p^n)\delta^4(\sum_{i=1}^n p^i)$; it contains a Dirac-$\delta$ implementing the total momentum conservation. Here and below our conventions for the Fourier transform will be $f(p) := \int \frac{d^4x}{(2\pi)^4}\, e^{-ip\cdot x} f(x)$, $f(x) = \int d^4p\, e^{ip\cdot x} f(p)$. As a consequence of the general relation

$$\int \frac{d^4x}{(2\pi)^4}\, e^{-ip\cdot x} \frac{\delta F}{\delta\phi(x)} = (2\pi)^{-4} \frac{\delta F}{\delta\phi(-p)} \tag{38}$$

one finds

$$\delta^4\left(\sum_{i=1}^n p^i\right) \Gamma_n^{12\ldots n}(p^1, p^2, ..., p^n) = (2\pi)^{-4n} \left.\frac{\delta^n \Gamma}{\delta b_1(-p^1)...\delta b_n(-p^n)}\right|_{\tilde{\Phi}=0}. \tag{39}$$

Differentiating relation (34) with respect to $b_1(-p^1), ..., b_n(-p^n)$ and setting thereafter $\tilde{\Phi} = 0$, we obtain

$$\begin{aligned} 0 \;=\; & \int d^4q \left[(2\pi)^4 \delta^4\left(q + \sum_{j=1}^l p^j\right) \Gamma_{n+1}^{01\ldots n}(q, p^1, ..., p^n)\delta_\xi b_0(q) \right. \\ & + \sum_{h=1}^n \delta^4\left(q + \sum_{j=1,\ j\neq h}^n p^j\right) \Gamma_n^{01\ldots,h-1,h+1,\ldots n}(q, p^1, ..., p^{h-1}, p^{h+1}, ..., p^n) \frac{\delta(\delta_\xi b_0(q))}{\delta b_h(-p^h)} \\ & \left.\left. + \frac{\delta^n \delta_\xi S_{GF}(\tilde{\Phi})}{\delta b_1(-p^1)...\delta b_n(-p^n)} \right] \right|_{\tilde{\Phi}=0}. \end{aligned} \tag{40}$$

In fact, only the terms with $\tilde{\Phi}_A = b$ in the first term in eq. (34) contribute to eq (40), since when $\tilde{\Phi}_A \neq b$ then $\frac{\delta^m(\delta_\xi \tilde{\Phi}_A)}{\delta b_1 \ldots \delta b_m}|_{\tilde{\Phi}=0} = 0$ (indeed, for any Φ_A $\delta_\xi \Phi_A$ is of degree ≥ 1 in Φ_A).

To get identities involving proper vertices with *physical* external bosons we will have to contract their Lorentz indices with the ones of transverse polarization tensors/vectors (we will choose them with well-defined helicity) $e^1(p^1)...e^n(p^n)$, where

$$e(p) = e^{\pm}(p) := \begin{cases} \varepsilon^{\pm}_\mu(p) & \text{when } b = \tilde{A}_\mu, \tilde{A}^a_\mu \\ (\varepsilon^{\pm}_\mu(p)\varepsilon^{\pm}_\nu(p)) & \text{when } b = \tilde{h}_{\mu\nu}, \end{cases} \qquad \text{with } \varepsilon^{\pm}_\mu(p)p^\mu = 0. \tag{41}$$

Now it is easy to realize that in all cases the following property holds:

$$\left.\frac{\delta^n \delta_\xi S_{GF}(\tilde{\Phi})}{\delta b_1(p^1)...\delta b_n(p^n)}\right|_{\tilde{\Phi}=0} e_1^1(-p^1)...e_n^n(-p^n) = 0; \tag{42}$$

where contraction of the Lorentz indices hidden in the symbols $1, ..., n$ and $e^1, ..., e^n$ is understood. In fact, the terms of non-zero degree in the ghosts contained in $\delta_\xi S_{GF}$ vanish after setting $\tilde{\Phi}' = 0$; the other terms depend on the longitudinal modes of the bosons, and vanish after contraction with the polarization vectors/tensors. We prove explicitly this statement in the appendix, for the Feynman (harmonic) gauge fixings.

Introducing the notation

$$\Gamma_n^{1...e^i...n} := \Gamma_n^{1...i...n} \cdot e^i, \tag{43}$$

where again contraction of the Lorentz indices hidden in the symbols i and e^i is understood, the Ward identities (44) will therefore reduce to

$$\begin{aligned} 0 \quad = \quad & \int d^4q \left[(2\pi)^4 \delta^4 \left(q + \sum_{j=1}^{l} p^j \right) \Gamma_{n+1}^{0e^1...e^n}(q, p^1, ..., p^n) \delta_\xi b_0(q) \right. \\ & + \sum_{h=1}^{n} \delta^4 \left(q + \sum_{j=1,\ j\neq h}^{n} p^j \right) \end{aligned}$$

$\Gamma_n^{0e^1...,e^{h-1},h+1,...e^n}(q, p^1, ..., p^{h-1}, p^{h+1}, ..., p^n) \frac{\delta(\delta_\xi b_0(q))}{\delta b_h(-p^h)} e^h_{\tilde{\Phi}=0}$.(44)

The identity above is one essential ingredient that we need in order to prove the main property of this section. In order to formulate this property, we need now a notion of " vicinity " of a "decay configuration" parametrized by one regularization parameter ω. Therefore, we introduce some useful definitions.

A configuration ω-converging to the decay configuration $(\hat{k}^i, \hat{\varepsilon}^i)_{i=0,\dots,n}$ $(\omega \geq 0)$ is a one-parameter family $(k^i(\omega), \varepsilon^i(\omega))_{i=0,\dots,n}$ such that $\varepsilon^i(\omega) \cdot k^i(\omega) = 0$, $k^i(\omega) - \hat{k}^i = o(\omega)$, $\varepsilon^i(\omega) - \hat{\varepsilon}^i = 0(\omega)$, $k^i \cdot k^{i'} = o(\omega^2)$ $\forall i, i' = 0, 1, ..., n$. Examples of these families will be given in formulae (74), (81).

It is easy to show that in the mentioned hypotheses the 3-momenta are in general no more collinear, but form angles $\lesssim \omega$; consequently,

$$\varepsilon^i(k^i) \cdot \varepsilon^j(k^j) = \begin{cases} either \quad o(1) \\ or \quad o(\omega) \end{cases} \qquad\qquad \varepsilon^i(k^i) \cdot k^j = o(\omega). \tag{45}$$

We are now able to prove the following fundamental property of the vertices, which is the main result of this Section and adds to the kinematical properties of Section 1:

Property 10. – On any configuration $(k^i(\omega),\varepsilon^i(\omega))_{i=0,...,n}$ ω-converging to the decay configuration $(\hat{k}^i,\hat{\varepsilon}^i)_{i=0,...,n}$

$$\Gamma_{n+1}^{e^0,...,e^n}(k^0,...,k^n)=o(\omega^{n+1})\qquad \text{in QED;} \tag{46}$$

$$\Gamma_{n+1}^{e^0a_0...e^na_n}(k^0,...,k^n)=o(\omega^{4-n-1})\qquad \text{in YM;} \tag{47}$$

$$\Gamma_{n+1}^{e^0...e^n}(k^0,...,k^n)=o(\omega^{m_\gamma+\theta(m_y)(4-m_y)+2\theta(m_g)\delta_0^{m_g}})\qquad \text{in QG.} \tag{48}$$

where in the third equation m_γ, m_y, m_g denote the number of external photons, YM bosons and gravitons respectively $(m_\gamma+m_y+m_g=n+1)$, and $\theta(x):=\begin{cases}0 \text{ if } x=0\\ 1 \text{ if } x>0\end{cases}$.

Proof.

The claim is evidently true when $n=0$. In fact, $\Gamma_1^{\mu_0}\propto(k^0)^{\mu_0}$ in QED, YM, but this vanishes since momentum conservation imposes the condition $k^0=0$; in QG still it could be $\Gamma_1^{\mu_0\nu_0}=const\times\eta^{\mu_0\nu_0}$, but this vanishes after contraction with $e^{\mu_0\nu_0}$ (which is a traceless tensor).

The rest of the proof is by induction and divided in three parts. Let us assume that the claim is true when $n=m-1$. We will prove that it is true when $n=m$. For the sake of simplicity, we explicitly prove the claim (48), which is the most general possible, in the simpler case $m_\gamma=0=m_y$,

$$\Gamma_{n+1}^{e^0...e^n}(k^0,...,k^n)=o(\omega^2)\qquad \text{in QG;} \tag{49}$$

at the end of this section we will briefly sketch how the proof goes in the general case.

Part 1 Here we prove the equations

$$\Gamma_{n+1}^{e^0...e^{i-1},\mu_i,e^{i+1}...e^n}(k^0,...,k^n)k^i_{\mu_i}=0\qquad \text{in QED;} \tag{50}$$

$$\Gamma_{n+1}^{e^0a_0...e^{i-1}a_{i-1},\mu_ia_i,e^{i+1}a_{i+1}...e^na_n}(k^0,...,k^n)k^i_{\mu_i}=o(\omega^{4-n})\qquad \text{in YM;} \tag{51}$$

$$\Gamma_{n+1}^{e^0...e^{i-1},\mu_i\nu_i,e^{i+1}...e^n}(k^0,...,k^n)k^i_{\mu_i}=o(\omega^2)\qquad \text{in QG.} \tag{52}$$

We drop in the sequel the tilde and write $A_\mu, A^a_\mu, g_{\mu\nu}$ instead of $\tilde{A}_\mu, \tilde{A}^a_\mu, \tilde{g}_{\mu\nu}$. We treat separately the cases of QED, YM and QG.

– QED. From $\delta_\xi A_\mu(p)=ip_\mu\xi(p)$ (eq. (27)), and eq. (44), from differentiating w.r.t. q it immediately follows

$$p^0_{\mu_0}\Gamma_{n+1}^{\mu_0e^1...e^n}(p^0,p^1,...,p^n)=0 \tag{53}$$

(we have factored out $\delta^4(\sum_{i=0}^n p^i)$), whence formula (50) follows at once (using boson symmetry), if we choose p^i so that the sets $\{p^0,...,p^n\}$, $\{k^0,...,k^n\}$ coincide. Actually we can derive directly from eq. (40) the stronger property

$$k^i_{\mu_i}\Gamma^{\mu_0\ldots\mu_i\ldots\mu_n}_{n+1}(k^0,\ldots,k^i,\ldots,k^n)=0,\qquad n\geq 2 \tag{54}$$

– YM. From

$$\delta_\xi A^a_\mu(p)=ip_\mu\xi^a(p)+ef^{abc}\int d^4q\,A^b_\mu(p-q)\xi^c(q) \tag{55}$$

(eq. (28) in momentum space), and from differentiating formula (44) (with $n=m$) w.r.t. $\xi(p^0)$, it immediately follows

$$\begin{aligned}&ip^0_{\mu_0}\Gamma^{\mu_0 a_0, e^1 a_1,\ldots,e^m a_m}_{m+1}(p^0,p^1,\ldots,p^m)+\\&+\sum_{l=1}^m ef^{b_l a_l a_0}\Gamma^{e^1 a_1,\ldots e^{l-1}a_{l-1},e^l b_l,e^{l+1}a_{l+1},\ldots,e^m a_m}_m(p^1,\ldots,p^{l-1},p^l+p^0,p^{l+1},\ldots,p^m)=0\end{aligned} \tag{56}$$

(again, we have factored out $\delta^4(\sum\limits_{i=0}^m p^i)$). This formula holds for any configuration $\sum\limits_{i=0}^m p^i=0$, $e^i(p^i)\cdot p^i=0$. On a configuration ω-converging to the decay configuration we deduce from the induction hypothesis that the second term is $o(\omega^{4-m})$.

– QG. The gauge transformation (29) in momentum space reads

$$\delta_\xi g_{\mu\nu}(p)=i\int d^4r\Big\{g_{\rho\nu}(p-r)r_\mu\xi^\rho(r)+g_{\rho\mu}(p-r)r_\nu\xi^\rho(r)+\xi^\rho(p-r)r_\rho g_{\mu\nu}(r)\Big\}, \tag{57}$$

implying

$$\delta_\xi g_{\mu\nu}(p)\big|_{g_{\mu\nu}(p)=\eta_{\mu\nu}\delta^4(p)}=i\{p_\mu\xi^\rho(p)\eta_{\rho\nu}+p_\nu\xi^\rho(p)\eta_{\rho\mu}\}. \tag{58}$$

Moreover, we note that

$$\frac{\delta g_{\alpha\beta}(p)}{\delta g_{\mu\nu}(-q)}=\delta^4(p+q)\left[\delta^\mu_\alpha\delta^\nu_\beta+\delta^\nu_\alpha\delta^\mu_\beta\right]. \tag{59}$$

After differentiation w.r.t. $\xi^{\nu_0}(p^0)$, Eq. (44) with $n=m$ reads:

$$\begin{aligned}0\;=\;&\Gamma^{\mu\nu,e^1,\ldots,e^m}_{m+1}(p^0,p^1,\ldots,p^m)2(p^0)_\mu\eta_{\nu\nu_0}\\&+\sum_{h=1}^m\Big[\Gamma^{e^1\ldots e^{h-1},\mu\nu,e^{h+1}\ldots e^n}_m(\ldots,p^{h-1},p^0+p^h,p^{h+1},\ldots)4(p^0)_\mu(\varepsilon^h)_\nu(\varepsilon^h)_{\nu_0}\\&\quad+(p^h)_{\nu_0}\Gamma^{e^1\ldots e^n}_m(\ldots,p^{h-1},p^0+p^h,p^{h+1},\ldots)\Big]\Big|_{g_{\alpha\beta}(p)=\eta_{\alpha\beta}\delta^4(p)}.\end{aligned} \tag{60}$$

(once again, we have factored out $\delta^4(\sum\limits_{i=0}^m p^i)$). This formula holds for any configuration $\sum\limits_{i=0}^m p^i=0$, $e^i(p^i)\cdot p^i=0$. On a configuration ω-converging to the decay configuration we deduce from the induction hypotheses (52), (49) that the second, third terms are $o(\omega^2)$, which proves eq. (52) for $n=m$.

Part 2: We prove the factorization formulae

$$\Gamma_{n+1}^{e^0....e^n}(k^0,...,k^n) = \sum_P A_{i_0 i_1 ... i_n} E^{i_0 i_1} ... E^{i_{n-1} i_n} \qquad \text{in QED, (n+1) even:} \tag{61}$$

$$\Gamma_{n+1}^{e^0 a_0 ... e^n a_n}(k^0,...,k^n) = \sum_P A_{i_0 i_1 ... i_n}^{a_0 ... a_n} E^{i_0 i_1} ... E^{i_{n-1} i_n} + o(\omega^{3-n}) \qquad \text{in YM, (n+1) even;} \tag{62}$$

$$\Gamma_{n+1}^{e^0 ... e^n}(k^0,...,k^n) = \sum_P A_{i_0 i_1 ... i_{2n} i_{2n+1}} E^{i_0 i_1} ... E^{i_{2n} i_{2n+1}} + o(\omega) \qquad \text{in QG.} \tag{63}$$

and

$$\Gamma_{n+1}^{e^0....e^n}(k^0,...,k^n) = \sum_P \sum_{j_0=0}^{n} A_{i_0 i_1 ... i_n}^{j_0} (k^{j_0} \cdot \varepsilon^{i_0}) E^{i_1 i_2} ... E^{i_{n-1} i_n} \qquad \text{in QED, (n+1) odd;} \tag{64}$$

$$\Gamma_{n+1}^{e^0 a_0 ... e^n a_n}(k^0,...,k^n) = \sum_P \sum_{j_0=0}^{n} A_{i_0 i_1 ... i_n}^{j_0; a_0 ... a_n} (k^{j_0} \cdot \varepsilon^{i_0}) E^{i_1 i_2} ... E^{i_{n-1} i_n} + o(\omega^{3-n}) \qquad \text{in YM (n+1), odd;} \tag{65}$$

where:

1) $\sum_P$ means the sum over all the permutations P ($P \equiv (i_0, i_1, ..., i_n)$ is a permutation of $(0, 1, ..., n)$ in QED and YM, whereas $P \equiv (i_0, i_1, ..., i_{2n+1})$ is a permutation of $(0, 1, ..., 2n+1)$ in QG);

2) the A's are scalar functions depending on the scalar products $k^i \cdot k^j$ (and, in the Y.M. case, on $2m$ Lie algebra indices a_i);

3) we have introduced the shorthand notation

$$E^{ij} := \left(\varepsilon^i \cdot \varepsilon^j \; k^i \cdot k^j - \varepsilon^i \cdot k^j \; \varepsilon^j \cdot k^i \right). \tag{66}$$

In the RHS of eq. (63) it is tacitly understood that $\varepsilon^{2s+1} \equiv \varepsilon^{2s}$, $k^{2s+1} = k^{2s}$, $s = 0, ..., 2n$.

We prove explicitly the first three (the proof of formulae (64), (65), is completely analogous): let $n+1 = 2m$. We look for the most general $\Gamma_{n+1}^{\mu_1 ... \mu_{n+1}}(k^1, ..., k^{n+1})$ satisfying:

1) the constraint

$$\Gamma_{n+1}^{\varepsilon^0 ... \varepsilon^{i-1}, \mu_i, \varepsilon^{i+1} ... \varepsilon^n}(k^0, ..., k^n) k^i_{\mu_i} = o(\omega^d) \tag{67}$$

in any configuration $(k^i(\omega), \varepsilon^i(\omega))_{i=0,...,n}$ ω-converging to the decay configuration $(\hat{k}^i, \hat{\varepsilon}^i)_{i=0,...,n}$;

2) symmetry under any replacement $(\mu_i, k^i) \leftrightarrow (\mu_l, k^l)$, $i, l = 0, ..., n$.

If we set $o(\omega^d) \equiv 0$ this amounts to solving eq. (50) equipped with boson symmetry for the $(n+1)$-photons vertex function of Q.E.D.; if we set $d = 3 - n$, this amounts to solving eq. (51) equipped with boson symmetry for the $(n+1)$-gluons vertex function of Y.M., provided we understand an implicit dependence of Γ_{n+1} on the Lie algebra indices a_i and remind that the latter have to be permuted along with the indices μ_i and the momenta k^i when boson symmetry is imposed; if we choose $n+1 = 4r$, $d = 2$, and add the additional symmetry conditions $k^{2i+1} = k^{2i}$, $\varepsilon^{2i+1} = \varepsilon^{2i}$ $(i = 0, ..., 2r-1)$, this will amount to solving eq. (52) equipped with boson symmetry for the $2r$-gravitons vertex function of Q.G. In this way, we can formally deal with eq.'s (50), (51), (52) simultaneously, by just dealing with one.

The dependence of $\Gamma_{n+1}^{\mu_0 \dots \mu_n}(k^0, ..., k^n)$ on Lorentz indices can only occur through the metric tensors $\eta^{\mu_i \nu_j}$ and the 4-vectors k^{μ_l}. Compactly, the most general dependence can be written in the following way

$$\Gamma_{n+1}^{\mu_0 \dots \mu_n} = \sum B^0 \underbrace{\eta ... \eta}_{m\ times} + \sum B^1 kk \underbrace{\eta ... \eta}_{(m-1)\ times} + ... + \sum B^m \underbrace{k ... k}_{2m\ times}, \tag{68}$$

where the B's denote Lorentz scalar functions. For our purposes, it will be more convenient to expand Γ_{n+1} in terms of the 4-vectors k^{μ_l} and of the tensors $E^{\mu_i \mu_j}(k^i, k^j) := \eta^{\mu_i \mu_j} k^i \cdot k^j - (k^i)^{\mu_j}(k^j)^{\mu_i})$, which satisfy the relation

$$(k^i)_{\mu_i} E^{\mu_i \mu_j} = 0 = (k^j)_{\mu_j} E^{\mu_i \mu_j} \tag{69}$$

The general expansion (68) can be replaced by

$$\Gamma_{n+1}^{\mu_0 \dots \mu_n}(k^0, ..., k^n) = \sum_P \sum_{l=0}^{m} \sum_{j_0, \dots j_{2l-1}=0}^{n} A_{i_0 \dots i_n}^{l;\ j_0 \dots j_{2l-1}} (k^{j_0})^{\mu_{i_0}} ... (k^{j_{2l-1}})^{\mu_{i_{2l-1}}} E^{\mu_{i_{2l}} \mu_{i_{2l+1}}} .... E^{\mu_{i_{n-1}} \mu_{i_n}} \tag{70}$$

where $\sum\limits_P$ means the sum over all the permutations $P \equiv (i_0, i_1, ..., i_n)$ of $(0, 1, ..., n)$ and $A_{i_0 \dots i_n}^{l;\ j_1 \dots j_{2l}}$ are scalar functions depending on the scalar products $k^i \cdot k^j$ (and, in the Y.M. case, on $2m$ Lie algebra indices a_i).

We have introduced a quite redundant set of scalars $\{A_{i_0 \dots i_n}^{l;\ j_1 \dots j_{2l}}\}$ to make formula (70) more compact. The set is redundant in the sense that $A_{i_0 \dots i_n}^{l;\ j_0 \dots j_{2l-1}}$ and $A_{\hat{i}_0 \dots \hat{i}_n}^{l;\ \hat{j}_0 \dots \hat{j}_{2l-1}}$ will both contribute to the same term $(k^{j_0})^{\mu_{i_0}} ... (k^{j_{2l-1}})^{\mu_{i_{2l-1}}} E^{\mu_{i_{2l}} \mu_{i_{2l+1}}} .... E^{\mu_{i_{n-1}} \mu_{i_n}}$ in the expansion (70), whenever

1) there exists a permutation P_{2l} of $2l$ objects such that $(\hat{i}_0, \hat{i}_1, ..., \hat{i}_{2l-1}) = P_{2l}(i_0, i_1, ..., i_{2l-1})$, $(\hat{j}_0, \hat{j}_1, ..., \hat{j}_{2l-1}) = P_{2l}(j_0, j_1, ..., j_{2l-1})$;

2) $(\hat{i}_{2l}, \hat{i}_{2l+1}, ..., \hat{i}_n) = P_{n+1-2l}(i_{2l}, i_{2l+1}, ..., i_n)$, where P_{2m-2l} is a permutation of $n+1-2l = 2m-2l$ objects which is the product: 2.a) of transpositions between the $(2s)^{th}$ and the $(2s+1)^{th}$ object $(s = 1, ..., m-l)$; 2.b) of transpositions between different pairs $(2s, 2s+1)$, $(2r, 2r+1)$, $r, s = 1, ..., m-l$.

We are free to set $A_{i_0 \dots i_n}^{l;\ j_0 \dots j_{2l-1}} = A_{\hat{i}_0 \dots \hat{i}_n}^{l;\ \hat{j}_0 \dots \hat{j}_{2l-1}}$ in these cases.

Finally, boson symmetry (37) implies that the scalars A^l satisfy the relations

$$A_{\dots j \dots i \dots}^{l;\ \tilde{j}_0 \dots \tilde{j}_{2l-1}}(k^i \leftrightarrow k^j) = A_{\dots i \dots j \dots}^{l;\ j_0 \dots j_{2l-1}} \qquad \tilde{h} := \begin{cases} j & \text{if } h = i \\ i & \text{if } h = j \\ h & \text{if } h \neq i, j \end{cases} \tag{71}$$

for any pair of indices i, j.

Plugging the general expansion (70) into Eq. (67) and using relation (69) we find

$$
\begin{aligned}
o(\omega^d) \;=\; & \sum_{P'} \sum_{l=1}^{m} \sum_{j_0,\ldots j_{2l-1}=0}^{n} \left[A^{l;\; j_1\ldots j_{2l}}_{ii_1\ldots i_n} + A^{l;\; j_1\ldots j_{2l}}_{i_1 i\ldots i_n} + \ldots + A^{l;\; j_0\ldots j_{2l-1}}_{i_1\ldots i_{2l-1} i i_{2l}\ldots i_n}\right] \times \\
& (k^i \cdot k^{j_0})\; \varepsilon^{i_1} \cdot k^{j_1} \ldots \varepsilon^{i_{2l-1}} \cdot k^{j_{2l-1}} E^{i_{2l} i_{2l+1}} \ldots E^{i_{n-1} i_n}, \qquad (72)
\end{aligned}
$$

where $\sum_{P'}$ means the sum over all the permutations $P' \equiv (i_1, \ldots, i_n)$ of $(0, 1, \ldots, i-1, i+1, \ldots, n)$, whereas

$$
\begin{aligned}
\Gamma^{\varepsilon^0 \ldots \varepsilon^n}_{n+1}(k^0, \ldots, k^n) \;=\; & \sum_{P'} \sum_{l=0}^{m} \sum_{j_0,\ldots j_{2l-1}=0}^{n} \left[A^{l;\; j_0\ldots j_{2l-1}}_{ii_1\ldots i_n} + A^{l;\; j_0\ldots j_{2l-1}}_{i_1 i\ldots i_n} + \ldots + A^{l;\; j_0\ldots j_{2l-1}}_{i_1\ldots i_{2l-1} i i_{2l}\ldots i_n}\right] \times \\
& (\varepsilon^i \cdot k^{j_0})\; \varepsilon^{i_1} \cdot k^{j_1} \ldots \varepsilon^{i_{2l-1}} \cdot k^{j_{2l-1}} E^{i_{2l} i_{2l+1}} \ldots E^{i_{n-1} i_n}. \qquad (73)
\end{aligned}
$$

Note that the term $l = 0$ has completely disappeared from the sum in eq. (72), due to eq. (69).

Let us fix the xyz axes so that $k^0 = (k^0_0, 0, 0, k^0_0)$ [according to property 2 this implies $k^j = \lambda^j(k^0_0, 0, 0, k^0_0)$, $j = 1, 2, \ldots n$]; we can always assume that the polarization vectors $\hat{\varepsilon}^i$ are real and have the form $\hat{\varepsilon}^i = (0, cos\theta^i, sen\theta^i, 0)$. We now start exploiting the available freedom in the choice (1) of the angles θ^i characterizing the polarization vectors $\hat{\varepsilon}^i$; (2) of the configuration (k^i, ε^i) ω-converging to $(\hat{k}^i, \hat{\varepsilon}^i)_{i=0,\ldots,n}$. A family of possible choices of the latter is

$$
\begin{aligned}
& k^i \equiv \hat{k}^i + \omega b^i \hat{\varepsilon'}^i \qquad \hat{\varepsilon'}^i := (0, -sen\theta^i, cos\theta^i, 0) \qquad i = 0, 1, \ldots, n, \\
& \varepsilon^i \equiv \hat{\varepsilon}^i; \qquad (74)
\end{aligned}
$$

the family is parametrized by the $2n+2$ parameters (b^i, θ^i), which are only constrained by the condition $\sum_{i=0}^{n} b^i \hat{\varepsilon'}^i = 0$ (so that $\sum_{i=0}^{n} k^i = \sum_{i=0}^{n} \hat{k}^i = 0$). As a consequence

$$
k^i \cdot k^j = -\omega^2 b^i b^j cos(\theta^i - \theta^j), \qquad \varepsilon^i \cdot k^j = -\omega b^j sin(\theta^i - \theta^j) \qquad \varepsilon^i \cdot \varepsilon^j = -cos(\theta^i - \theta^j) \quad (75)
$$

$$
\left(\varepsilon^i \cdot \varepsilon^j \;\, k^i \cdot k^j - \varepsilon^i \cdot k^j \;\, \varepsilon^j \cdot k^i\right) = \omega^2 b^i b^j \qquad (76)
$$

By plugging these (k^i, ε^i) into Eq. (72) we find

$$
\begin{aligned}
& o(\omega^d) = \omega^{n+2} \sum_{P'} \sum_{l=1}^{m} \sum_{j_0,\ldots j_{2l-1}=0}^{n} \sum_{i=1}^{n} \left[A^{l;\; j_0\ldots j_{2l-1}}_{ii_1\ldots i_n} + A^{l;\; j_0\ldots j_{2l-1}}_{i_1 i\ldots i_n} + \ldots + A^{l;\; j_0\ldots j_{2l-1}}_{i_1\ldots i_{2l-1} i i_{2l}\ldots i_n}\right] \\
& b^i b^{j_0} \ldots b^{j_{2l-1}} b^{i_{2l}} \ldots b^{i_n} cos(\theta^i - \theta^{j_0}) sin(\theta^{i_1} - \theta^{j_1}) \ldots sin(\theta^{i_{2l-1}} - \theta^{j_{2l-1}}). \qquad (77)
\end{aligned}
$$

The coefficients in the square brackets can depend on the angles θ^i only through the cosines $cos(\theta^i - \theta^j)$ (since $k^i \cdot k^j = -\omega^2 b^i b^j cos(\theta^i - \theta^j)$); since the above equation has to hold for all θ^i's then all terms in the square brackets have to satisfy the relation

$$
\left[A^{l;\; j_0\ldots j_{2l-1}}_{ii_1\ldots i_n} + A^{l;\; j_0\ldots j_{2l-1}}_{i_1 i\ldots i_n} + \ldots + A^{l;\; j_0\ldots j_{2l-1}}_{i_1\ldots i_{2l-1} i i_{2l}\ldots i_n}\right] = o(\omega^{d-n-2}) \qquad l = 1, \ldots, m \qquad (78)
$$

independently.

Replacing the above results in formula (73) we find the factorization formula

$$\Gamma_{n+1}^{\varepsilon^0 \dots \varepsilon^n}(k^0, \dots, k^n) = \sum_P A^{0;}_{i_0 i_1 \dots i_n} E^{i_0 i_1} \dots E^{i_{n-1} i_n} + o(\omega^{d-1}) \tag{79}$$

whence formulae (61), (62), (63) follow.

Part 3: On any configuration $(k^i(\omega), \varepsilon^i(\omega))_{i=0,\dots,n}$ ω-converging to the decay configuration $(\hat{k}^i, \hat{\varepsilon}^i)_{i=0,\dots,n}$ we have $E^{ij} = o(\omega^2)$. To prove formulae (46), (47), (49) it remains to show that the scalar functions A's appearing in eq.'s (61), (62), (63) can show poles in ω at most of degree so high to yield the global ω-dependence reported in the former formulae. For this purpose we use a continuity argument, i.e. we argue that the claimed ω-dependence is the only one compatible with equations (61), (62), (63) if we require the LHS to be independent of the particular configuration $(k^i(\omega), \varepsilon^i(\omega))_{i=0,\dots,n}$ ω-converging to the decay configuration $(\hat{k}^i, \hat{\varepsilon}^i)_{i=0,\dots,n}$.

For the sake of brevity we continue to use the factorization formula (79) to deal at once with all three cases. We choose two different multi-parameter families $(k^i(\omega), \varepsilon^i(\omega))$, $(\tilde{k}^i(\omega), \tilde{\varepsilon}^i(\omega))$ of configurations ω-converging to the decay configuration, and we require that

$$\lim_{\omega \to 0} \Gamma_{n+1}^{e^0(k^0) \dots e^n(k^n)}(^0, \dots, k^n) = \lim_{\omega \to 0} \Gamma_{n+1}^{\tilde{e}^0(k^0) \dots \tilde{e}^n(q^n)}(\tilde{k}^0, \dots, \tilde{k}^n) \tag{80}$$

In the xyz axes as before, the first is the family (74), the second is

$$\tilde{k}^i := \hat{k}^i + \omega(0, 0, c^i, 0) \qquad \tilde{\varepsilon}^i := [(\hat{k}^i_3)^2 + (\omega c^i)^2]^{-\frac{1}{2}}(0, 0, \hat{k}^i_3, \omega c^i) \qquad i = 0, 1, \dots, n, \tag{81}$$

where $\sum_{i=0}^{n} c^i = 0$. This implies in particular $\tilde{k}^i \cdot \tilde{k}^j = -\omega^2 c^i c^j$.

With the first family we find

$$\Gamma_{n+1}^{\varepsilon^0 \dots \varepsilon^n}(k^0, \dots, k^n) = \omega^{n+1} b^0 \dots b^n \sum_P A^{0;}_{i_0 i_1 \dots i_n} + o(\omega^{d-1}). \tag{82}$$

Now we specialize our discussion to the case of QED and QG, where $d - 1 \geq 1$, so that the second term vanishes when $\omega \to 0$. Let us consider *per absurdum* the hypothesis that the functions A's have poles of degree $(n+1)$ in ω. In order that the RHS has a limit independent of the b^i's when $\omega \to 0$, the A's must have the form

$$A^{0;}_{i_0 i_1 \dots i_n} = \left[\sum_P a_{i_0 i_1 \dots i_n} k^{i_0} \cdot k^{i_1} \dots k^{i_{n-1}} \cdot k^{i_n} \right]^{-1}, \tag{83}$$

where $a_{i_0 i_1 ... i_n}$ are constants, so that

$$A^{0;}_{i_0 i_1 ... i_n} = \left[\omega^{n+1} b_0 ... b^n\right]^{-1} \times const. \tag{84}$$

On the other hand, plugging the family (81) into eq. (83) and replacing the result into formula (79), we find

$$\Gamma_{n+1}^{\tilde{\varepsilon}^0(k^0)...\tilde{\varepsilon}^n(q^n)}(\tilde{k}^0, ..., \tilde{k}^n) = const. \times \sum_P \left(\frac{d^{i_0}}{d^{i_1}} + \frac{d^{i_1}}{d^{i_0}} - 1\right) ... \left(\frac{d^{i_{n-1}}}{d^{i_n}} + \frac{d^{i_n}}{d^{i_{n-1}}} - 1\right) + o(\omega^2) \tag{85}$$

where we have defined $d^i := \frac{k^i_3}{c^i}$. This expression depends on the choice of the coefficients c^i, i.e. depends on the way the family $(\tilde{k}^i(\omega), \tilde{\varepsilon}^i(\omega))$ approaches $(\hat{k}^i(\omega), \hat{\varepsilon}^i(\omega))$, against the hypothesis. In a similar way, one can exclude the hypothesis that the functions A's have poles in ω of degree $> (n+1)$, otherwise the RHS would diverge to either $+\infty$ or $-\infty$ according to the way the families approach the decay configuration

Summing up, we have discarded the possibility that the A's have poles in ω of degree $\geq n+1$, so that consequently in QED,QG

$$\Gamma_{n+1}^{\varepsilon^0...\varepsilon^n}(k^0, ..., k^n) = o(\omega) \tag{86}$$

In QED we can improve the bound (86) into the stronger bound (46). In fact, if one plugs the general expansion (70) into eq. (54) [instead of eq. (67)] and argues as in part 2, one ends up with a stronger form of the factorization,

$$\Gamma_{n+1}^{\mu_0 ... \mu_n}(k^0, ..., k^n) = \sum_P A_{i_0 i_1 ... i_n} E^{\mu_{i_0} \mu_{i_1}} ... E^{\mu_{i_{n-1}} \mu_{i_n}}. \tag{87}$$

Looking at the Feynman diagrams contributing at each order in the loops to $\Gamma_{n+1}^{\mu_0 ... \mu_n}(k^0, ..., k^n)$, it is easy to understand that they are continuous and finite for all values of k^i's, since the fermion/scalar masses are infrared cutoffs[see fig. (3)]. Hence, the scalars A can have no poles in $k^i \cdot k^j$, because otherwise at least the terms

$A_{i_0 i_1 ... i_n}(k^{i_0})^{\mu_{i_1}} ... (k^{i_n})^{\mu_{i_{n-1}}}(k^{i_{n-1}})^{\mu_{i_n}}$ would diverge. The A's have dimension $[mass]^{4-2(n+1)}$, since Γ_{n+1} has dimension $[mass]^{4-(n+1)}$. This can be accounted for without introducing poles in $k^i \cdot k^j$, but using the mass parameters of the charged particle interacting with the photon. For instance, if the only charged particle is a fermion with mass m, then $A = m^{4-2(n+1)} o(1)$. We have completed the proof of the claim (46).

In QG the $o(\omega)$ in the RHS of (86) can be improved into a $o(\omega^2)$, since $\Gamma_{n+1}^{e^0 ... e^n}(k^0, ..., k^n)$ can be only of even degree in ω, if we assume that the proper vertices depend analitically on the momenta k^i. This follows from formula (45), because the LHS of eq. (86) has to be a function of the Lorentz scalars $k^i \cdot k^j$, $\varepsilon^i(k^i) \cdot k^j$, of even degree in the latter. This completes the proof of the claim (49).

In YM formula (79) and the continuity argument do not exclude that there exists a limit $\lim_{\omega\to 0} \Gamma_{n+1}^{\varepsilon^0...\varepsilon^n}(k^0, ..., k^n) =: L \neq 0$ independent on the way the family $(k^i(\omega), \varepsilon^i(\omega))$ approaches $(\hat{k}^i(\omega), \hat{\varepsilon}^i(\omega))$. In fact, if the functions A's have a pole of degree $\geq (n+1)$ in ω, the second term in formula (79) (which in principle can be finite or divergent) could compete with the first, and $\Gamma_{n+1}^{\varepsilon^0...\varepsilon^n}(k^0, ..., k^n)$ could have a family-independent limit even though the first term has not. This is exactly what happens with the 4-gluon proper vertex, as one can already check at the tree level

$$\Gamma_{4,tree}^{\hat{\varepsilon}^0 a_0 ... \hat{\varepsilon}^3 a_3} \propto \left[(\hat{\varepsilon}^0 \cdot \hat{\varepsilon}^1)(\hat{\varepsilon}^2 \cdot \hat{\varepsilon}^3) f^{a_0 a_3 e} f^{a_1 a_2 e} + perm.\right] \neq 0. \tag{88}$$

By an explicit analysis of the general expansion (70) one can easily realize that a family-independent limit $L \in \mathbf{R} \cup \{\pm\infty\}$ can be obtained only if equation (47) is satisfied.

Finally, the proof of the general claim (48) can be done by an induction procedure in the number of external photons (resp. of YM bosons) which mimics the one sketched so far for QED (resp. YM), with the only difference that as starting input we do not use the value of proper vertex with zero photons, zero YM bosons and zero gravitons, but the proper vertex with $m_g > 0$ gravitons or $m_y > 0$ YM bosons (resp. with $m_g > 0$ gravitons or $m_\gamma > 0$ photons).

We have thus completed the proof of property 10 $\diamondsuit$.

5 Concluding remarks.

We have seen that the decay probabilities for the photon, the graviton and the Yang-Mills boson all vanish. The decay amplitudes involving only photons and/or gravitons are themselves zero; we have first shown these properties by a simple power counting argument and then proved them rigorously through the Ward identities, assuming only continuity of the Greens functions in the infrared limit. In the case of the Yang-Mills boson, the power counting shows that the amplitude does not vanish in the infrared limit; the decay probability is however suppressed by the phase-space factor.

As mentioned at the end of Section 2, a partial decay probability τ_n^{-1} different from zero can be only obtained when the squared amplitude is proportional to a sufficiently high hegative power of ω. If we admit (as is generally true in perturbation theory) that the coupling constants appear in the numerator, this means that the amplitude must contain a coupling constant with positive mass dimension.

The only theory we are aware of, in which such a coupling occurs is gravity in the presence of a cosmological constant. In this case the action of the gravitational field is written as

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{g(x)} \, [\Lambda - R(x)] \tag{89}$$

or, redefinig the metric in the form $g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa \tilde{h}_{\mu\nu}(x)$, with $\kappa = \sqrt{16\pi G}$,

$$S = \int d^4x \sqrt{1 + \kappa\tilde{h} + \kappa^2\tilde{h}^2 + \kappa^3\tilde{h}^3 + ...} \left[\frac{\Lambda}{\kappa^2} - \tilde{R}^{(2)}(x) + ... \right] \tag{90}$$

We have denoted symbolically with $\tilde{h}$, $\tilde{h}^2$, $\tilde{h}^3$ ... in the square root terms which are linear, quadratic, cubic ... in $\tilde{h}$, omitting the indices and the exact algebraic structure. $\tilde{R}^{(2)}(x)$ denotes the part of the curvature quadratic in $\tilde{h}$. The term $\kappa^3\tilde{h}^3$, when is multiplied by Λ/κ^2, gives rise to a vertex $\kappa\Lambda\tilde{h}^3$ which couples three gravitons with a coupling constant $\kappa\Lambda$ of mass dimension 1 (unlike the corresponding three-vertex of the pure Einstein action, which is proportional to κ^3 and contains 4 four-momenta, so that the infrared processes are strongly suppressed).

It is then possible to construct gravitonic loops with n external legs using these vertices; the amplitudes will be proportional to positive powers of $\kappa\Lambda$ and – in our regularization scheme – to negative powers of ω. This means that τ_n^{-1} would be finite in the limit $\omega \to 0$, or even diverge. But we should not forget the terms which are linear and quadratic in $\tilde{h}$ in the square root of eq. (90). In particular, the quadratic term gives rise to some graviton mass (if $\Lambda < 0$) or to instability (if $\Lambda > 0$) [4]. In the first case, we end up with gravitons which are not massless any more, so that all our preceding formalism does not apply.

On the other hand, it is known that the cosmological constant Λ, although possibly very big in principle, is limited by astronomical observations to be less than $|\Lambda| \leq 10^{120} G^{-1}$. In order to explain this vanishing, many mechanisms have been proposed [12]. In the non-perturbative quantum Regge calculus [5] the effective value of the adimensional product $|\Lambda| G$ depends on the length scale and vanishes with a power law as the scale grows. In this sense, the constant Λ could be non-vanishing on small scale, while the graviton would maintain asymptotically zero mass.

The detailed dynamics of the decay would however in this hypotesis be unknown, and we limit ourselves to apply the general kinematical considerations. Eq. (8) for the mean life takes the form – supposed that only the constants G and Λ enter in the process [1] –

$$\tau^{-1} = \frac{1}{GE} \sum_{j=1,2,...} c_j (\Lambda G)^j . \tag{91}$$

We have also admitted that it is possible to define a Lorentz-invariant scale for the process, and that such scale enters into (91) only by determining the effective value of Λ. The scale could be given, for instance, by the transversal size of the vawe packet describing the graviton, which in turn is connected to the features of the measuring apparatus. The coefficients c_j are unknown.

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Appendix

We prove eq. (42). In the case of QED with (for instance) Feynman's gauge-fixing $\frac{1}{2\alpha}\int d^4x(\partial_\mu A_\mu)^2$, the LHS is zero when $n > 1$ because the gauge variation of the gauge-fixing above is of first degree in A_μ, and is zero in the case $n = 1$ because

$$-\frac{1}{\alpha}p_1^\mu (p_1)^2 \xi(p_1), \tag{92}$$

vanishes after contraction with the polarization vector $\varepsilon_\mu^\pm(p_1)$. In the case of YM with (for instance) Feynman's gauge-fixing $\frac{1}{\alpha}\int d^4x(\partial^\mu A_\mu)^2$, the LHS is zero if $n > 2$ because the gauge variation of the gauge-fixing above is of second degree in A_μ; if $n = 1$ it is zero for the same reason as in the preceding case (92) ; if $n = 2$ it is zero because

$$-\frac{2}{\alpha}(p_1)^{\mu_1}(p_2)^{\mu_2}\xi^c(p_1+p_2)f^{a_1a_2c}, \tag{93}$$

vanishes after contractions with the polarization vector $\varepsilon_{\mu_1}^\pm(p_1)\varepsilon_{\mu_2}^\pm(p_2)$. In the case of QG with harmonic gauge-fixing $\frac{1}{2\alpha}\int d^4x(\partial^\mu h_{\mu\nu})^2$ we have

$$\delta_\xi\left[\frac{1}{2\alpha}\int d^4x(\partial^\mu h_{\mu\nu})^2\right] = -\frac{1}{\alpha}\int d^4p(p^\mu h_{\mu\nu}(p))p^\rho(\hat{\xi^\nu_{;\rho}} + \hat{\xi^{;\nu}_\rho})(-p) \tag{94}$$

(here $\hat{}$ means Fourier transform). When some $\frac{\delta}{\delta g_{\mu_i\nu_i}(-p_i)}$ acts on $p^\mu h_{\mu\nu}(p)$ we get a factor $\delta^4(p+p_i)[\delta^{\mu_i}_\mu\delta^{\nu_i}_\nu + \delta^{\mu_i}_\nu\delta^{\nu_i}_\mu]$ (see formula (59)), which makes zero after contraction with the polarization tensor $e^{\mu_i\nu_i}(p_i)$. $\diamondsuit$