Critical Behavior of Simplicial Chiral Models

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The large-N saddle-point equations for the principal chiral models defined on a $d-1$ dimensional simplex are derived from the external field problem for unitary integrals. The saddle point equation are studied analytically and numerically in many relevant instances, including $d = 4$ and $d \to \infty$, with special attention to the critical domain, which is found to correspond to $\beta_c = 1/d$ for all d. Related models (chiral chains) are discussed and large-N solutions are analyzed.

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I. INTRODUCTION

The $1/N$ expansion of matrix-valued field theories is probably the most important non-perturbative and non-numerical theoretical tool presently available in the study of such models as non-Abelian gauge theories and two-dimensional quantum gravity. A resolution of the above-mentioned models in the large-N limit would be the starting point for many analytical developments. In particular when a lattice formulation is involved one must consider different possibilities in the search for the continuum limit; for the case of asymptotically free theories one must explore the limit of vanishing coupling $g = 0$ (trivial fixed point) while keeping a physical mass scale fixed, while in the case of quantum gravity one must search for a nontrivial fixed point g_c and reach the limit with a specific power-law dependence on N of $g - g_c$, which is known as "double scaling limit" [1–3]. Therefore it is useful to achieve a full knowledge of the coupling dependence of such models, from extreme weak coupling to strong coupling, in order to explore those regions that may turn out to be physically most interesting.

As a matter of fact, notwithstanding many recent efforts toward an understanding of the possible properties of large-N solutions to nontrivial quantum field theories, our present analytical knowledge is limited to a small number of few-matrix systems. This number is even smaller if we restrict our attention to the case of unitary matrix fields, which is especially relevant to the problem of lattice QCD. To the best of our knowledge, the only solved examples are Gross-Witten's single-link problem [4] and its generalizations, the external field problem [5,6] and $L = 3, 4$ chiral chains [7,8].

We stress that extending the number of solved few-matrix systems is not at all a pointless exercise. Indeed apart from purely theoretical informations that might be achieved, not only does every few-matrix system have a reinterpretation, via the double scaling limit, as some different kind of matter coupled to 2-dimensional quantum gravity, but also every few-matrix system involving unitary matrices can be reinterpreted as the generating functional for a class of integrals over unitary groups, and these integrals in turn are the essential missing ingredient in the context of a complete algorithmization of the strong coupling expansion of many interesting models [9].

Following Ref. [9], we may introduce the notion of a "superskeleton", that is a graph whose vertices are joined by at most one link (simple graph). As has been shown, knowledge of all the group integrals involved in the strong coupling expansion of a lattice model with nearest-neighbor interactions defined on such a graph provides sufficient information for the algorithmic reconstruction of the strong coupling series for a model enjoying the same global symmetry and defined on an arbitrary lattice.

These were basic motivations for us to begin the study of the class of lattice chiral models which we termed "simplicial chiral models" [10]. In particular we focused on principal

chiral models, with a global $U(N) \times U(N)$ symmetry, defined on a $d-1$ dimensional simplex formed by connecting d vertices by $(d-1)(d-2)/2$ links, and explored specifically the large-N limit of such models, whose relevance we have just been discussing.

Our fundamental result is the reduction of the above problem to that of solving a single inhomogeneous integral equation for the eigenvalue distribution of a single Hermitian semi-positive definite matrix. Although we could not find a closed form solution to this equation for arbitrary d, we are able to solve it in several interesting special cases and we set up a systematic numerical approach to the solutions which led us to a conjecture about the location of the critical surface as a function of d. We have also studied in detail the related topic of chiral chains, their strong coupling expansion and critical behavior. As a result of these analyses, we are confident that the critical surface is defined by $\beta_c = 1/d$ for all d.

Moreover, by treating d as a continuous parameter, there are two distinct regions. For small d, $0 < d < 4$, the models exhibit the third order Gross-Witten transition. Indeed for $d = 1, 2, 3$ they coincide exactly with the chiral chains studied earlier by Brower, Rossi and Tan [7]. In this region, the criticality is related to that of $O(n)$ spin models on random surfaces, as discussed by Gaudin-Kostov [15]. For $d > 4$, however, there is a first order transition ending at the "upper critical" dimensions $d = 4$, which we scrutinize in some detail.

This paper is organized as follows:

- In Section II we set up our formalism for simplicial chiral model and derive the large-N effective action and a representation for the internal energy. We begin in Section II B by illustrating the formalism for the simpler case of vector $O(N)$ spin models on a simplicial lattice, deriving the closed form large-N solution for arbitrary values of d and studying its properties. Sec II C gives the large-N effective action for the simplicial chiral model and Section II D, the saddle-point equation (large-N Schwinger-Dyson equation) for the eigenvalue distribution, discussing its features and converting it into a standard inhomogeneous Fredholm equation of the second kind.

- In Section III we analyze the solvable examples. We begin with integer values of $d < 4$ (which correspond to $L < 4$ chiral chains), followed in Section III B by a detailed discussion of the $d = 4$ case, including features of the weak and strong coupling expansions and the asymptotic expansion around the critical point $\beta_c = 1/4$. We present in Section III C the exact result for the limit $d \to \infty$ and develop in Section III D a treatment based on the $1/d$ expansion. This large-d analysis, which works best outside the critical region: $\beta \gg \beta_c$, has provided us with the first numerical confirmation for the conjecture that $\beta_c = 1/d$, and is discussed in greater details in Appendix D.

- In Section IV we solve the models at criticality for arbitrary $0 < d \leq 4$ and sketch the peculiar features of the critical behavior when $d > 4$. Also in Section IV B we

present numerical methods based on the Gaussian integration techniques for $d > 4$ on the critical surface $\beta_c = 1/d$.

- Some technical extensions are included in other appendices. Appendix A is devoted to a discussion of the double-scaling limit of critical chiral chains $L \leq 4$ and Appendix B extends the discussion of chiral chains to $L \geq 4$ by an analysis of strong coupling expansion. Whereas Appendix C is devoted to details of the weak and strong coupling expansions and series analysis for the $d = 4$ simplicial chiral model.

II. SIMPLICIAL CHIRAL MODELS

A $d - 1$ dimensional simplex is formed by connecting in a fully symmetric way d vertices by $(d - 1)(d - 2)/2$ links. Let us assign a $U(N)$ matrix to each vertex. The partition function for principal chiral models on a simplicial lattice is obtained by integrating over unitary $N \times N$ matrices with a normalized invariant Haar measure:

$$Z_d \;=\; \int \prod_{i=1}^{d} dU_i \, \exp\left\{ N\beta \sum_{i>j=1}^{d} \mathrm{Tr} \left[U_i U_j^{\dagger} + U_j U_i^{\dagger} \right] \right\} \; . \tag{1}$$

Thus the d-matrix simplicial model has an underlying permutation symmetry instead of the cyclic symmetry of the d-matrix chiral chains. For $d = 1$, 2 and 3 these two symmetries and the associated models are equivalent. We shall explore the $U(N) \times U(N)$ symmetry of the system and, in particular, study its critical behavior in the large-N limit.

For this purpose, it is sufficient to study the bulk "thermodynamic" properties, $e.g.$, the free energy density, internal energy, and specific heat, which are respectively given by

$$\begin{aligned}
F_N(\beta, d) &\;=\; \frac{1}{N^2} \ln Z_d \; , \\
U_N(\beta, d) &\;=\; \frac{1}{2} \frac{\partial F_N}{\partial \beta} \; , \\
C_N(\beta, d) &\;=\; \beta^2 \frac{\partial U_N}{\partial \beta} \; .
\end{aligned} \tag{2}$$

We focus in this paper on computing the free energy and determine the critical point β_c for all values of the parameter d in the large N-limit. We find that there is a sequence of critical theories with $\beta_c = 1/d$, which exhibit a third order Gross-Witten singularities for $d < 4$ and a first order transition for $d > 4$. Special attention will be give to the marginal dimension at $d = 4$. Scaling exponents and finite size (or double scaling) properties will be presented in some special cases, but a thorough investigation for all d is beyond the scope of this paper.

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A. External Field Method

Because of the the permutation symmetry of the vertices, the simplicial chiral models can be reformulated in terms of a Lagrange multiplier field which decouples the original degrees of freedom. The resulting effective theory is very reminiscent of the mean field approximation to standard lattice models, but in contrast with mean field this reformulation is exact.

We therefore replace the direct interaction between unitary matrices with the coupling to an auxiliary field, which in this case is a complex $N \times N$ matrix, A, introduced in the following representation of the identity,

$$
1 = \frac{\int dA \exp \left\{ -N\beta \operatorname{Tr} \left[\left(A - \sum_{i=1}^{d} U_i \right) \left(A^\dagger - \sum_{i=1}^{d} U_i^\dagger \right) \right] \right\}}{\int dA \exp \left\{ -N\beta \operatorname{Tr} \left[AA^\dagger \right] \right\}} \,. \tag{3}
$$

Again by exchanging the order of integrations and representing the partition function in the form $Z_d = \tilde{Z}_d / \tilde{Z}_0$, we may obtain

$$
\tilde{Z}_d = \int dA \int \prod_{i=1}^{d} dU_i \exp \left\{ N\beta \operatorname{Tr} \left[-\left[AA^\dagger \right] + \left[A \sum_i U_i^\dagger \right] + \left[A^\dagger \sum_i U_i \right] - d \right] \right\} \,. \tag{4}
$$

By performing the single-link external field integral, we may introduce the auxiliary function,

$$
F_N(BB^\dagger) = \frac{1}{N^2} \ln \int dU \exp \left\{ \frac{N}{2} \operatorname{Tr} \left[BU^\dagger + UB^\dagger \right] \right\} \,, \tag{5}
$$

and re-express $\tilde{Z}_d$, up to an irrelevant multiplicative factor, in the form

$$
\tilde{Z}_d = \int dB \exp \left[-\frac{N}{4\beta} \operatorname{Tr} BB^\dagger + N^2 dF(BB^\dagger) - N^2 \beta d \right] \,, \tag{6}
$$

where B replaces $2\beta A$.

A first crucial point in our analysis is the observation that the integrand in Eq. (6) is a function of the eigenvalues x_i of the Hermitian semi-positive definite matrix $BB^\dagger$. Moreover Morris [12] has shown that, when integrating over complex matrices, a proper parameterization may offer in specific cases, like ours, the possibility of performing the "angular" integrations exactly and reducing the problem to that of integrating over the N variables x_i.

Referring to Morris' paper for a proof of the angular integration, we apply it to our Eq. (6) thus obtaining, again up to irrelevant numerical factors,

$$\tilde{Z}_d = \int_0^\infty \prod_{i=1}^d dx_i \prod_{j>i}(x_i - x_j)^2 \exp\left[-\frac{N}{4\beta}\sum_i x_i + N^2 dF(x_i) - N^2\beta d\right] . \qquad (7)$$

The second crucial observation concerns the function $F(x_i)$. It is known exactly for all $U(N)$ groups [7], while integral representations exist for $SU(N)$ groups [13], and it takes on a relatively simple form in the large-N limit. Before proceeding further, we shall first provide an even simpler illustrative example whose large-N solution can be obtained fairly straightforwardly.

B. Pedagogical example: simplicial spin models.

Consider instead of our simplicial chiral models, an example of an $O(N)$ symmetric nonlinear model defined on a simplex. The same basic methods used for the chiral models are easily illustrated in this much simpler context.

The partition function is obtained by integrating over the $N-1$ independent components of d vectors,

$$Z_d = \int \prod_{i=1}^d d\mathbf{s}_i\, \delta(\mathbf{s}_i^2 - 1) \exp\left[N\beta \sum_{i>j=1}^d \mathbf{s}_i \cdot \mathbf{s}_j\right] . \qquad (8)$$

The effective field is a single unconstrained N-component vector, $\vec{z}$, which again can be introduced as Lagrange multiplier field via an identity,

$$1 \equiv \frac{\int d\mathbf{z}\, \exp-\frac{N\beta}{2}(\mathbf{z} - \sum_{i=1}^d \mathbf{s}_i)^2}{\int d\mathbf{z}\, \exp-\frac{N\beta}{2}\mathbf{z}^2} . \qquad (9)$$

Upon substituting this identity into Eq. (8) and inverting the order of the integrations, we may then represent the partition function by $Z_d = \tilde{Z}_d/\tilde{Z}_0$, where

$$\tilde{Z}_d = \int d\mathbf{v} \prod_{i=1}^d \int d\mathbf{s}_i\, \delta(\mathbf{s}_i^2 - 1) \exp\left[-\frac{N\beta}{2}\mathbf{v}^2 + N\beta\, \mathbf{v} \cdot \left(\sum_{i=1}^d \mathbf{s}_i\right) - \frac{N\beta}{2}d\right] . \qquad (10)$$

It is now possible to perform the decoupled constrained integrations. To this end we may define the auxiliary function

$$F_N(z^2) = \frac{1}{N} \ln \int d\mathbf{s} \; \delta(\mathbf{s}^2 - 1) \exp N \mathbf{s} \cdot \mathbf{z} , \tag{11}$$

where $z^2 = \mathbf{z} \cdot \mathbf{z}$. This function is known explicitly for all values of N and admits a large N limit [11]:

$$
\begin{aligned}
F_N(z^2) &= \frac{1}{N} \ln \; \Gamma \left(\frac{N}{2} \right) \; I_{\frac{N}{2}-1}(Nz) \left(\frac{Nz}{2} \right)^{1 - \frac{N}{2}} \\
&\xrightarrow[N \to \infty]{} \frac{1}{2} \left[\sqrt{1 + 4z^2} - 1 - \ln \left(\frac{1}{2}\sqrt{1 + 4z^2} + \frac{1}{2} \right) \right] .
\end{aligned}
\tag{12}
$$

As a consequence for large N we have the following representation of $\tilde{Z}_d$,

$$\tilde{Z}_d = \int d\mathbf{z} \exp \left\{ -\frac{N}{2} \left[\frac{z^2}{\beta} - d \left(\sqrt{1 + 4z^2} - 1 - \ln \left(\frac{1}{2}\sqrt{1 + 4z^2} + \frac{1}{2} \right) \right) + \beta d \right] \right\} . \tag{13}$$

The large-N value of the integral in Eq. (13) may be obtained by a saddle-point estimate. After some simple manipulations, the saddle-point equation may be reduced to

$$1 - \frac{z^2}{\beta} = \frac{d}{2} \left(1 - \sqrt{1 + 4z^2} \right) . \tag{14}$$

The solution of this equation is

$$z^2 = \beta \left[1 - \frac{d}{2} + \beta \frac{d^2}{2} + \frac{d}{2}\sqrt{(1 - \beta d)^2 + 4\beta} \right] . \tag{15}$$

By taking the logarithmic derivative of the partition function with respect to β, we obtain an expression for the internal energy (per unit link) U_d of simplicial spin models in the large-N limit

$$\frac{d(d-1)}{2} U_d = \frac{1}{2} \left(\frac{z^2}{\beta^2} - d - \frac{1}{\beta} \right) = \frac{d}{2} \left[\frac{\beta d - 1}{2\beta} + \frac{1}{2\beta}\sqrt{(1 - \beta d)^2 + 4\beta} - 1 \right] . \tag{16}$$

We may check many special cases of this result, and in particular we may notice that the r.h.s. of Eq. (16) is zero when $d = 1$, while when $d = 2$

$$U_2 = \frac{1}{2\beta} \left[\sqrt{1 + 4\beta^2} - 1 \right] , \tag{17}$$

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consistent with the single-link model result.

Finally let us notice that in the large-d limit, as a trivial consequence of the structure of the model, the solution we found coincides with the mean field solution, which is exact in this limit. It is worth observing that, while Eq. (16) is formally correct for all values of β, in order to recover the standard strong and weak coupling expansions of the solution we must separately consider the two different regimes $\beta < \beta_c$ and $\beta > \beta_c$, where for all d we obtain $\beta_c = 1/d$.

C. Large-N limit

Returning to the simplicial chiral models, we are again interested in the large-N limit. For the free energy function $F(x_i)$ resulting from a one-link integral over a $U(N)$ matrix, the limiting form can be extracted by solving the Schwinger-Dyson equations and written in a simple closed form [5,6],

$$F(x_i) \;=\; \frac{1}{N}\sum_i \sqrt{r+x_i} - \frac{1}{2N^2}\sum_{ij}\ln\left(\frac{\sqrt{r+x_i}+\sqrt{r+x_j}}{2}\right) - \frac{r}{4} - \frac{3}{4} \,. \tag{18}$$

We must distinguish two different phases, a weak coupling regime where $r = 0$ and

$$\frac{1}{N}\sum \frac{1}{\sqrt{x_i}} \;\leq\; 1, \tag{19}$$

and a strong coupling regime where r is dynamically determined by the condition,

$$\frac{1}{N}\sum_i \frac{1}{\sqrt{r+x_i}} \;=\; 1 \,. \tag{20}$$

It is important for future developments to observe that Eq. (20) also leads to the condition

$$\frac{\partial F(x_i, r)}{\partial r} \;=\; 0 \,. \tag{21}$$

It is completely legitimate to apply the above results to a saddle-point evaluation of the large-N limit of the integral appearing in Eq. (7). To this end we may define an effective action

$$S_d = \frac{N}{4\beta} \sum_i x_i - N^2 dF(x_i) - \sum_{i \neq j} \ln(x_i - x_j) + N^2 \beta d \ , \tag{22}$$

and derive a saddle-point equation

$$\frac{1}{N} \frac{\partial S_d}{\partial x_i} = \frac{1}{4\beta} - N d \frac{\partial F}{\partial x_i} - \frac{2}{N} \sum_{i \neq j} \frac{1}{x_i - x_j} = 0 \ . \tag{23}$$

Very simple manipulations, including the use of Eq. (21), lead to a reformulation of Eq. (23), which can be turned into the relationship

$$\frac{\sqrt{r + x_i}}{2\beta} - d = \frac{1}{N} \sum_{i \neq j} \frac{(4 - d)\sqrt{r + x_i} + d\sqrt{r + x_j}}{x_i - x_j} \ . \tag{24}$$

This equation, supplemented with the condition $x_i \geq 0$ and with the constraint $r = 0$ (weak coupling) or Eq. (20) (strong coupling) is the fundamental saddle-point equation of principal chiral models on a simplicial lattice. It is the starting point of most of the developments presented in the following sections.

We recall that, once Eq. (24) is solved, knowledge of the saddle-point value of the eigenvalues $\bar{x}_i$ allows the large-N evaluation of $\tilde{Z}_d$ via the relationship,

$$\tilde{Z}_d \longrightarrow \exp[-S_d(\bar{x}_i)] \ , \tag{25}$$

and we can also extract the internal energy per unit link by taking a logarithmic derivative of Z_d with respect to β which leaves us with the relationship,

$$d(d - 1)U = \frac{1}{4\beta^2} \sum_i \bar{x}_i - d - \frac{1}{\beta} \ . \tag{26}$$

D. Saddle-point equation.

In order to study Eq. (24) we shall start by applying well-established techniques, and in particular by introducing an eigenvalue density function. It is however convenient first to introduce a new variable z_i whose formal definition is

$$z_i = \sqrt{r + x_i} \ . \tag{27}$$

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subject to the condition $0 \leq \sqrt{r} \leq z_i$. We may assume that the eigenvalue variable x_i lies in a single interval $[x_a, x_b]$, $0 \leq x_a \leq x_b$. In terms of the new variable z_i, one has $z_i \epsilon [a, b]$ where $a = \sqrt{r + x_a}$, $b = \sqrt{r + x_b}$ and

$$0 \leq \sqrt{r} \leq a \leq b. \tag{28}$$

For weak coupling, $r = 0$, and we expect in general $a = \sqrt{x_a} > 0$. For strong coupling, one expects $x_a = 0$ so that $a = \sqrt{r} \neq 0$.

We shall be interested in the weak-strong transition as one varies d and β. In a third-order transition, typical of large-N transition previously studied, $a_c = \sqrt{r_c} = 0$. In a first-order transition, which we will encounter for $d > 4$, $a_c = \sqrt{r_c} \neq 0$ when approached from the strong coupling regime.

Denoting the large-N eigenvalue density by $\rho(z)$; it vanishes outside the interval $[a, b]$. We may now turn Eq. (23) into the following integral equation,

$$\frac{z}{2\beta} - d \ = \ \int_a^b dz' \rho(z') \left(\frac{2}{z - z'} - \frac{d - 2}{z + z'} \right) . \tag{29}$$

The function $\rho(z)$, and therefore also the extremes a and b of the integration region, are thus determined dynamically. In particular the normalization condition,

$$\int_a^b dz' \rho(z') \ = \ 1 , \tag{30}$$

must be satisfied.

In addition to the positivity requirement, $\rho(z) \geq 0$ over the interval $[a, b]$, the desired solution to Eq. (29) must also satisfy either the weak coupling inequality, Eq. (19) or the strong coupling constraint, Eq. (20). In the large-N limit, Eq. (19) becomes

$$\int_a^b dz' \frac{\rho(z')}{z'} \ \leq \ 1 , \tag{31}$$

whereas Eq. (20) becomes

$$\int_a^b dz' \frac{\rho(z')}{z'} \ = \ 1 . \tag{32}$$

The determination of the transition point, β_c, and of the critical behavior around this value is one of the interesting physical problems concerning this model.

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Eq. (29) has a somewhat unconventional form when compared to other integral equations, because of the special structure of its kernel. We may however perform a few manipulations in order to obtain a more familiar relationship. Our starting point is the introduction of an analytic function of z, by the definition

$$f(z) \equiv \int_a^b \frac{\rho(z')}{z - z'} dz' \,. \tag{33}$$

By construction, the analyticity domain of $f(z)$ is the complex z plane with the exception of a cut on the positive real axis in the interval $[a, b]$. The discontinuity on the cut may be parameterized by writing

$$f(z \pm i\epsilon) = R(z) \mp i\pi\rho(z) \,, \tag{34}$$

when $z \in [a, b]$ and it is easy to recognize that

$$R(z) = \frac{z}{4\beta} - \frac{d}{2} + \frac{d-2}{2} \int_a^b dz' \frac{\rho(z')}{z + z'} = \frac{z}{4\beta} - \frac{d}{2} - \frac{d-2}{2} f(-z) \,. \tag{35}$$

It follows that $R(z)$ is itself an analytic function of z, with a cut on the negative real axis in the interval $[-b, -a]$.

Let us now notice that the normalization condition implies

$$f(z) \xrightarrow[|z| \to \infty]{} \frac{1}{z} \,. \tag{36}$$

As a consequence in the same limit we obtain

$$R(z) \longrightarrow \frac{z}{4\beta} - \frac{d}{2} + \frac{d-2}{2z} \tag{37}$$

and in turn

$$\rho(z) = \frac{1}{i\pi} [R(z) - f(z)] \longrightarrow \frac{1}{i\pi} \left(\frac{z}{4\beta} - \frac{d}{2} + \frac{d-4}{2z} \right) + O\left(\frac{1}{z^2}\right) \,. \tag{38}$$

This equation can in principle be used in order to determine relationships between the constants a and b in place of the normalization condition.

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We must now distinguish between weak and strong coupling regimes. In both cases, by exploiting analyticity properties of the function $f(z)$ and defining appropriate auxiliary functions, it is relatively easy to reduce Eq. (29) to the following forms

$$\rho(z) = \frac{\sqrt{(b-z)(z-a)}}{\pi} \left[\frac{1}{4\beta} - \frac{d-2}{2} \int_a^b \frac{dy}{y+z} \frac{\rho(y)}{\sqrt{(b+y)(y+a)}} \right] \qquad \text{for} \quad \beta > \beta_c \,, \quad (39)$$

$$\rho(z) = \frac{z}{\pi} \sqrt{\frac{b-z}{z-a}} \left[\frac{1}{4\beta} - \frac{d-2}{2} \int_a^b \frac{dy}{y+z} \sqrt{\frac{y+a}{y+b}} \frac{\rho(y)}{y} \right] \qquad \text{for} \quad \beta < \beta_c \,. \qquad (40)$$

The values of a and b as functions of β are determined by enforcing the asymptotic condition (38). Eqs. (39-40) are inhomogeneous Fredholm equations of the second kind. It is therefore in principle possible to apply standard methods of (approximate) resolution by expressing the kernels in terms of appropriate orthonormal sets of eigenfunctions.

The weak and strong coupling constraints, Eq. (19) and Eq. (20), can be expressed in terms of the analytic function $f(z)$ as $f(0) \geq -1$ and $f(0) = -1$ respectively. Alternatively, writing $f(z) = R(z) - i\pi\rho(z)$ and analytically continue this expression outside of the interval $[a,b]$, Eq. (19) and Eq. (20), can also be expressed as

$$-i\rho(0) \geq 0 \,, \qquad (41)$$

and

$$-i\rho(0) = 0 \,, \qquad (42)$$

respectively. Note that Eq. (40) is parameterized so that the strong coupling constraint (42) is automatically satisfied. The transition point β_c can be determined approaching from the weak coupling regime by enforcing the equality $\rho(0) = 0$. We shall return to a general discussion of this criticality in Section IV.

A final comment concerns the explicit evaluation of $\tilde{Z}_d$. Instead of directly substituting $\rho(z)$ in the expression of the partition function, it is convenient to apply Eq. (26) in the form

$$d(d-1)U = \frac{1}{4\beta^2} \int_a^b dz' \rho(z')(z'^2 - r) - d - \frac{1}{\beta} \,, \qquad (43)$$

and perform an integration with respect to β to recover the free energy.

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III. EXACT LARGE N SOLUTIONS

Here we present the solutions at $N = \infty$ as a function of d. They can be broken into three classes for $d < 4$, $d = 4$ and $d > 4$ respectively. For $d < 4$, they are equivalent to chiral chain models with $L < 4$ studied earlier [7] all of which exhibit the third order Gross-Witten transition at β_c. For $d > 4$ there is a first order transition, which ends exactly at $d = 4$, consequently the end point at $d = 4$ is of special interest.

A. Solutions for $d < 4$

As we mentioned in the introduction, when d is integer and less than 4 simplicial chiral models are only reformulation of trivial or already solved models. It is however quite instructive to consider even these examples in our new language. Let us begin with the only apparently trivial case $d = 0$. Obviously $Z_0 = 1$, however $\tilde{Z}_0$ is nontrivial and we need to know its value in order to compute Z_d. As a matter of fact Eq. (6) already implies that, up to a constant

$$\tilde{Z}_0 \propto \exp N^2 \ln \beta \ . \tag{44}$$

We would like, as a consistency check, to derive this result from the saddle point equation. A straightforward manipulation of Eq. (29) leads to

$$\frac{1}{8\beta} = \fint_a^b dz' \frac{\rho(z')}{z^2 - z'^2} \ . \tag{45}$$

This equation is solved by

$$\rho(z) = \frac{z}{4\pi\beta} \frac{\sqrt{b^2 - z^2}}{\sqrt{z^2 - a^2}} \ , \tag{46}$$

with the only constraint $b^2 - a^2 = 16\beta$. However by keeping in mind that only the combination $x = z^2 - a^2$ is physically meaningful because of Eq. (27), we recognize that

$$\rho(z)dz = \frac{dz^2}{8\pi\beta} \frac{\sqrt{16\beta - (z^2 - a^2)}}{\sqrt{z^2 - a^2}} = \frac{1}{8\pi\beta} \frac{dx}{\sqrt{x}} \sqrt{16\beta - x} \ , \tag{47}$$

and the physical solution is unique and leads by a trivial integration, to Eq. (44).

Because of our definitions $Z_1 = 1$, $\tilde{Z}_1 = \tilde{Z}_0$. The eigenvalue distribution $\rho(z)$ however is the generating function for the moments of the linear combination of a complex and a unitary matrix, and these moments can be highly nontrivial, even if the complex matrix itself has a Gaussian probability distribution, as a consequence of the averaging over unitary matrices. As a matter of fact we were not able to solve explicitly the saddle-point equation associated to the $d = 1$ models, even though the solution probably has reasonably simple mathematical properties.

Let us now turn to the $d = 2$ case. As a straightforward application of Eqs. (39-40), we immediately find both the weak and strong coupling solutions,

$$\rho_{\mathrm{w}}(z) = \frac{1}{4\pi\beta}\sqrt{8\beta - (z - 4\beta)^2} \qquad \text{for} \quad \beta \geq \frac{1}{2}\,, \tag{48}$$

$$\rho_{\mathrm{s}}(z) = \frac{z}{4\pi\beta}\sqrt{\frac{1 + 6\beta - z}{z - 1 + 2\beta}} \qquad \text{for} \quad \beta \leq \frac{1}{2}\,. \tag{49}$$

For the strong coupling region, $r = (1 - 2\beta)^2$.

It is easy to recognize that the $d = 2$ model corresponds to the Gross-Witten single-link problem, which in turn is equivalent to large-N QCD$_2$ with Wilson action on the lattice [4]. The properties of this model are well known, and in particular it is known that $\beta_c = 1/2$, consistent with Eqs. (48-49). Another consistency check is easily made by applying Eq. (43) and verifying that the known expressions for U_2 are reproduced.

Finally let us comment about the $d = 3$ case. This model in its original formulation is completely equivalent to the three-link chiral chain studied in Refs. [7,8]. We therefore know that it must possess a third-order phase transition at the critical value $\beta_c = 1/3$. However, as we already observed, our reformulation leads to exploring quite different classes of correlation functions and there is no obvious relationship between old and new results apart from bulk thermodynamical properties. Again we have no analytical solution for the $d = 3$ model equation, whose known properties stand as a benchmark for future attempts.

B. Solution at $d = 4$

Turning to $d = 4$ leads us to a new situation, where we are no longer guided by known results, since the 3-dimensional simplex (tetrahedron) is distinct from the solved four-link chain. Actually it would be instructive and convenient to embed both models in a more general case interpolating between them and including many more interesting situations.

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We are studying the most general four-site system with bilinear interactions of four unitary matrices, which turns out to be reducible to an interacting two-complex matrix system. A separate paper will be devoted to a discussion of this system. Here we only discuss the solutions of the saddle point equation obtained from Eq. (29) in the $d = 4$ case,

$$\frac{z}{8\beta} - 1 = \fint dz' \frac{z'\rho(z')}{z^2 - z'^2} . \tag{50}$$

In order to solve this equation, let us separately consider the weak and the strong coupling regimes, while changing variables for convenience to $x = z^2$ and defining the distribution $\tilde{\rho}(x)$ by $\rho(z)dz = \tilde{\rho}(x)dx$. The special structure of Eq. (50) makes it convenient to follow a special procedure not directly related to Eqs. (39-40) derived for the general case.

In the weak coupling phase, we define the functions

$$f_{\mathrm{w}}(x) = \int_{a^2}^{b^2} dx' \frac{\tilde{\rho}_{\mathrm{w}}(x')\sqrt{x'}}{x - x'} , \tag{51}$$

and

$$g_{\mathrm{w}}(x) = \frac{f_{\mathrm{w}}(x)}{\sqrt{(x - b^2)(x - a^2)}} , \tag{52}$$

subject to the normalization constraint

$$\int_{a^2}^{b^2} dx' \, \tilde{\rho}_{\mathrm{w}}(x') = 1 . \tag{53}$$

The functions $f(x)$ and $g(x)$ are real analytic, with a cut along the interval $[a^2, b^2]$ on the real axis. On this interval the relationship,

$$\mathrm{Im}\, g_{\mathrm{w}}(x \pm i\epsilon) = \mp \frac{\mathrm{Re}\, f_{\mathrm{w}}(x)}{\sqrt{(x - a^2)(b^2 - x)}} , \tag{54}$$

holds, while analyticity and Eq. (50) imply

$$g_{\mathrm{w}}(x) = \frac{\frac{\sqrt{x}}{8\beta} - 1}{\sqrt{(x - b^2)(x - a^2)}} - \frac{1}{8\pi\beta} \int_0^\infty dy \frac{\sqrt{y}}{x + y} \frac{1}{\sqrt{(y + b^2)(y + a^2)}} . \tag{55}$$

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However Eqs. (51-52) imply that

$$\tilde{\rho}_{\mathrm{w}}(x) \;=\; -\frac{\mathrm{Im}\, f_{\mathrm{w}}(x + i\epsilon)}{\pi\sqrt{x}} \;=\; \frac{1}{8\pi^2\beta} \int_0^\infty \frac{dy}{x+y} \sqrt{\frac{y}{x}} \sqrt{\frac{(b^2 - x)(x - a^2)}{(b^2 + y)(y + a^2)}} \;. \tag{56}$$

In order to determine a^2 and b^2 we may use Eq. (53) and the observation that in the complex x plane when $|x| \to \infty$

$$g_{\mathrm{w}}(x) \;\longrightarrow\; O\left(\frac{1}{x^2}\right) \;. \tag{57}$$

As a consequence one obtains that

$$\oint dx'\, g_{\mathrm{w}}(x') \;=\; 2 \int_{a^2}^{b^2} dx'\, \mathrm{Im}\, g_{\mathrm{w}}(x') \;=\; 0 \tag{58}$$

around $[a^2, b^2]$, that is

$$\int_{a^2}^{b^2} dx'\, \frac{\sqrt{x'}}{\sqrt{(b^2 - x')(x' - a^2)}} \;=\; 8\pi\beta \;. \tag{59}$$

In strong coupling we adopt a similar strategy by defining

$$f_{\mathrm{s}}(x) \;=\; \int_{a^2}^{b^2} \frac{dx'}{\sqrt{x'}} \frac{\tilde{\rho}_{\mathrm{s}}(x')}{x - x'} \;, \tag{60}$$

and

$$g_{\mathrm{s}}(x) \;=\; f_{\mathrm{s}}(x) \sqrt{\frac{x - a^2}{x - b^2}} \;, \tag{61}$$

with the constraint

$$\int_{a^2}^{b^2} \tilde{\rho}_{\mathrm{s}}(x')\, dx' \;=\; 1 \;, \tag{62}$$

and the boundary condition

$$g_{\mathrm{s}}(x) \;\xrightarrow[|x|\to\infty]{}\; \frac{1}{x} \;. \tag{63}$$

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We then find

$$g_{\rm s}(x) \;=\; \frac{1}{8\beta\sqrt{x}}\sqrt{\frac{x-a^2}{x-b^2}} - \frac{1}{8\pi\beta}\int\frac{dy}{x+y}\frac{1}{\sqrt{y}}\sqrt{\frac{y+a^2}{y+b^2}}\;, \tag{64}$$

and

$$\tilde{\rho}_{\rm s}(x) \;=\; \frac{1}{8\pi^2\beta}\int_{a^2}^{b^2}\frac{dy}{x+y}\sqrt{\frac{x(b^2-x)(y+a^2)}{y(b^2+y)(x-a^2)}}\;. \tag{65}$$

The boundary condition (63) leads to the relationship

$$-\int_{a^2}^{b^2} dx'\,{\rm Im}\,g_{\rm s}(x'+i\epsilon) \;=\; \pi\;, \tag{66}$$

where

$${\rm Im}\,g_{\rm s}(x+i\epsilon) \;=\; -\frac{1}{8\beta\sqrt{x}}\sqrt{\frac{x-a^2}{b^2-x}}\;. \tag{67}$$

All the integrals appearing in Eqs. (56), (59), (65) and (66) are elliptic integrals. It is therefore possible to re-express both the weak and the strong coupling results in terms of known functions. In particular it is convenient to re-express everything in terms of "natural" rescaled variables, by defining

$$k \;=\; \sqrt{1-\frac{a^2}{b^2}}\;, \tag{68}$$

$$\zeta \;=\; \sqrt{1-\frac{z^2}{b^2}}\;, \tag{69}$$

and setting $\bar{\rho}(\zeta)d\zeta \;=\; \rho(z)dz$. It is not too difficult to eliminate completely the parameters a and b in favor of k by making use of Eqs. (59) and (66) respectively. As a consequence we obtain the weak coupling expression

$$\bar{\rho}_{\rm w}(\zeta) \;=\; \frac{8\beta}{E(k)^2}\left[\frac{\sqrt{k^2-\zeta^2}}{\sqrt{1-\zeta^2}}K(k) - \sqrt{k^2-\zeta^2}\sqrt{1-\zeta^2}\,\Pi(\zeta^2,k)\right]\;, \tag{70}$$

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and the strong coupling counterpart

$$\bar{\rho}_s(\zeta) = \frac{8\beta}{[E(k) - (1-k^2)K(k)]^2} \left[k^2 \frac{\sqrt{1-\zeta^2}}{\sqrt{k^2-\zeta^2}} K(k) - \sqrt{k^2-\zeta^2}\sqrt{1-\zeta^2}\Pi(\zeta^2,k) \right] , \quad (71)$$

where K, E, Π are the elliptic integrals of the first, second and third kind respectively, and the domain of ζ is the interval $[0,k]$, $0 \le k \le 1$.

Obviously, in order for the problem to be completely solved, one must try expressing k as a function of β. This is achieved in principle by enforcing the normalization condition, which takes the form

$$\int_0^k \bar{\rho}(\zeta)d\zeta = 1 . \quad (72)$$

By symbolically writing

$$\bar{\rho}(\zeta) = \beta D(\zeta, k) , \quad (73)$$

in agreement with Eqs. (70) and (71), it is actually possible to express all results as functions of k by the relationship

$$\beta = \frac{1}{\int_0^k d\zeta\, D(\zeta, k)} . \quad (74)$$

In practice this form of our results is sufficient for both numerical evaluation and asymptotic expansions, not to mention the possibility of exploring the region around the criticality. Criticality is characterized by the limit $k \to 1$, where simple mathematical properties of elliptic integrals allow us to show that both weak and strong coupling results lead to $\beta_c = 1/4$ and

$$\bar{\rho}_c(\zeta) = \zeta \ln \frac{1+\zeta}{1-\zeta} . \quad (75)$$

In order to obtain the usual weak and strong coupling expansion of physical quantities, like the internal energy, as power series in $1/\beta$ and β respectively, one must consider in turn the $k \to 0$ limit and the expansion in powers of k. Obviously the different structure of $\bar{\rho}(\zeta)$ in the two phases will lead to different expressions. In particular we have the asymptotic behaviors

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$$\beta_{\mathrm{w}} \xrightarrow[k \to 0]{} \frac{2}{k^4} - \frac{2}{k^2} + O(1) \,, \tag{76}$$

$$\beta_{\mathrm{s}} \xrightarrow[k \to 0]{} \frac{k^2}{16} + \frac{k^4}{32} + O(k^6) \,, \tag{77}$$

and it is conceptually straightforward to obtain power series expansions in the powers of k for such quantities as the internal energy and to convert them into standard weak and strong coupling series. A few details will be discussed in Appendix C.

The expansion around the critical point $\beta_c = 1/4$, $k_c = 1$, is slightly subtler because the expansion of elliptic integrals around $k = 1$ is asymptotic. However by exploiting a few known or previously derived results, we have managed to obtain the following relationships, holding in weak coupling near the criticality:

$$\bar\rho_{\mathrm{w}}(\zeta) \approx 4\beta \left[1 - k'^2 \left(\ln \frac{4}{k'} - \frac{1}{2} \right) + O(k'^2) \right] \left[\zeta \ln \frac{1+\zeta}{1-\zeta} - k'^2 \left(\ln \frac{4}{k'} - \frac{1}{2} \right) \frac{\zeta^2}{1-\zeta^2} + O(k'^2) \right] \,, \tag{78}$$

where $k'^2 = 1 - k^2 \to 0$. From Eq. (74) we then obtain

$$4\beta \approx 1 + k'^2 \left[\ln \frac{4}{k'} \ln \frac{2}{k'} + \frac{1}{2} \ln \frac{2}{k'} + \frac{1}{2} \right] \,, \tag{79}$$

and as a consequence

$$\beta - \beta_c \approx \frac{k'^2}{4} \left[\ln \frac{4}{k'} \ln \frac{2}{k'} + \frac{1}{2} \ln \frac{2}{k'} + \frac{1}{2} \right] \,, \tag{80}$$

therefore $k'^2 \sim \beta - \beta_c$ apart from logarithms. By properly applying Eq. (43) we may also extract the result

$$U_{\mathrm{w}} \xrightarrow[k' \to 0]{} \frac{\pi^2 - 6}{9} + O(k'^2 \ln^2 k') \,, \tag{81}$$

and by simple manipulations, from the specific heat relationship

$$C = \beta^2 \frac{\partial U}{\partial \beta} \,, \tag{82}$$

we may obtain near the criticality

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$$C_{\mathrm{w}} \xrightarrow[k' \to 0]{} \frac{\pi^2 + 3}{36} - \frac{\pi^2}{12 \ln(4/k')} + O\left(\frac{1}{\ln^2 k'}\right) . \tag{83}$$

A similar analysis can be performed in strong coupling near criticality,

$$\bar{\rho}_{\mathrm{s}}(\zeta) \approx 4\beta \left[1 + k'^2 \left(\ln \frac{4}{k'} + \frac{1}{2}\right) + O(k'^2)\right] \left[\zeta \ln \frac{1+\zeta}{1-\zeta} + k'^2 \left(\ln \frac{4}{k'} + \frac{1}{2}\right) \frac{\zeta^2}{1-\zeta^2} + O(k'^2)\right] , \tag{84}$$

where again $k'^2 = 1 - k^2 \to 0$, and

$$\beta_c - \beta \approx \frac{k'^2}{4} \left[\ln \frac{4}{k'} \ln \frac{2}{k'} - \frac{1}{2} \ln \frac{2}{k'} - \frac{1}{2}\right] . \tag{85}$$

We then find

$$C_{\mathrm{s}} \xrightarrow[k' \to 0]{} \frac{\pi^2 + 3}{36} - \frac{\pi^2}{12 \ln(4/k')} + O\left(\frac{1}{\ln^2 k'}\right) . \tag{86}$$

The strong and weak coupling expressions of C near the criticality show that the critical behavior around $\beta_c = \frac{1}{4}$ corresponds to a limiting case of a third order phase transition with critical exponent of the specific heat

$$\alpha = 0^- , \tag{87}$$

near the boundary with weak second order critical behavior. Notice that in terms of double scaling limit $\alpha = 0^-$ would correspond to a central charge $c = 1$.

For the interested readers we mention that in the derivation of Eqs. (78) and (84) we made use of the following formula (which appeared with some misprints in Ref. [16])

$$\Pi(\zeta^2, k) \approx \frac{1}{1-\zeta^2} \left(\ln \frac{4}{k'} + \frac{\zeta}{2} \ln \frac{1-\zeta}{1+\zeta}\right) + \frac{k'^2}{4(1-\zeta^2)^2} \left(-1 + (1+\zeta^2) \ln \frac{4}{k'} + \zeta \ln \frac{1-\zeta}{1+\zeta}\right) , \tag{88}$$

for the asymptotic expansion of the elliptic integral of the third kind $\Pi(\zeta^2, k)$ in the region $k' = \sqrt{1-k^2} \to 0$.

C. The $d = \infty$ solution

While at present we are not aware of any general method to get an analytic solution of the saddle-point equation (29) for arbitrary d, the $d \to \infty$ limit provides another interesting instance in which the equation is solvable.

It is easy to show that for larger and larger values of d the distribution $\rho(z)$ becomes narrower and narrower, with a width decreasing like $d^{-1/2}$ and a peak value $\bar{z}$ which can easily be determined by replacing in Eq. (29)

$$\rho(z) \longrightarrow \delta(z - \bar{z}) \,, \tag{89}$$

and obtaining the large-N, large-d equation

$$\frac{\bar{z}}{2\beta} - d \;=\; -\frac{d - 2}{2\bar{z}} \,. \tag{90}$$

A consistent solution is obtained by assuming the limit to be taken at a fixed value of βd, in which case

$$\bar{z} \longrightarrow \beta d \left(1 + \sqrt{1 - \frac{1}{\beta d}} \right) k \tag{91}$$

with the obvious restriction $\beta d \geq 1$ (weak coupling phase). When $\beta d \leq 1$ one must recognize that the saddle-point condition, when correctly applied to the original expression for the effective action Eq. (22) in the large-d limit, unambiguously leads to the prediction $\bar{z} \to 1$, $r \to 1$ (strong coupling phase).

The most interesting features of this result are:

- The large-d prediction for the location of the critical point, $\beta_c \longrightarrow 1/d$, amazingly enough, seems to be satisfied for all values of d.

- The complete equivalence with the mean field solution of infinite volume principal chiral models on a D-dimensional hypercubic lattice such that $D = d/2$ [14], where we may observe that this last relationship enforces the constraint that corresponding models have the same coordination number.

It is easy to compute the large-d expression for the internal energy in the weak coupling,

$$U \longrightarrow \frac{1}{4(\beta d)^2} \bar{z}^2 = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{\beta d}} - \frac{1}{4\beta d} \ . \tag{92}$$

At the criticality. the weak coupling value of U is $\frac{1}{4}$, while the strong coupling value is $U = 0$. Therefore the large-N, large-d prediction for the nature of the criticality is that of a first-order phase transition. It is however important to notice that the large-d prediction for the specific heat in the weak-coupling phase,

$$dC = \frac{1}{4}\left[\frac{1}{\sqrt{1 - \frac{1}{\beta d}}} + 1\right] , \tag{93}$$

shows a divergence at the phase transition, with no indication for the existence of a metastable phase.

It is interesting to compare the specific heat behavior for $d = 2, 3, 4, \infty$. In Fig. 1 we plotted dC versus $\bar{\beta} \equiv d\beta$.

D. The $1/d$ expansion

The large-d result may also be the starting point for a systematic $1/d$ expansion of Eq. (29), and for a numerical approximation scheme which turns out to be quite efficient at least in the weak coupling domain away from criticality. The essential ingredient for both these developments is the observation that, substituting the definition of $f(z)$, Eq. (33), into Eq. (29) we obtain the functional equation,

$$\frac{z}{2\beta} - d = 2\mathrm{Re}\, f(z) + (d-2)f(-z) , \tag{94}$$

subject to the following constraints: (a) Eq. (94) is satisfied in the interval $[a, b]$, with a and b dynamically determined; (b) $f(z)$ is real analytic outside the interval $[a, b]$; (c) the asymptotic behavior of $f(z)$ when $|z| \to \infty$ is $f(z) \longrightarrow 1/z$.

Let us now define

$$\zeta \equiv z - \tilde{z} , \quad \phi(\zeta) \equiv f(\tilde{z} + \zeta) , \tag{95}$$

where $\tilde{z}$ is a constant whose value lies in the interval $[a, b]$ and will be dynamically generated. Substituting the definition (95) into Eq. (94), we obtain

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$$\frac{\tilde{z}}{2\beta} - d + \frac{\zeta}{2\beta} \; = \; 2\mathrm{Re}\,\phi(\zeta) + (d-2)\phi(-2\tilde{z}-\zeta) \;. \tag{96}$$

Postponing the discussion of the numerical approximation scheme, let us illustrate here the procedure for a systematic $1/d$ expansion of Eq. (96). We introduce the following Ansatz for the function $\phi(\zeta)$,

$$\phi(\zeta) \; = \; dQ(\zeta)\left(1 - \sqrt{1 - \frac{c_1}{d\zeta} - \frac{c_2}{d\zeta^2}}\right) + R(\zeta) \;, \tag{97}$$

and assume the functions $Q(\zeta)$, $R(\zeta)$ to be real analytic in the interval between the roots of the polynomial

$$d\zeta^2 - c_1\zeta - c_2 \;, \tag{98}$$

including the point $\zeta = 0$. Moreover we require $Q(0) = R(0) = 0$ and the boundary condition

$$\phi(\zeta) \xrightarrow[|\zeta|\to\infty]{} \frac{1}{\zeta} \;. \tag{99}$$

These requirements fix the constants $\tilde{z}$, c_1 and c_2 dynamically, which in turn determine a and b. Finally we assume all functions and constants to be expandable in $1/d$, with non vanishing leading order.

As an illustration let us find the solution to first nontrivial order. All leading order quantities will be independent of d and labeled by a subscript 0. After an expansion of Eq. (97) in powers of $1/d$ we obtain

$$\phi(\zeta) \; \approx \; \frac{1}{2}Q_0(\zeta)\left(\frac{c_1}{\zeta} + \frac{c_2}{\zeta^2}\right) + R_0(\zeta) + O\left(\frac{1}{d}\right) \;. \tag{100}$$

By imposing the asymptotic boundary conditions we have the stricter condition (forced by analyticity of Q_0, R_0),

$$\frac{1}{2}Q_0(\zeta)\left(\frac{c_1}{\zeta} + \frac{c_2}{\zeta^2}\right) + R_0(\zeta) \; = \; \frac{1}{\zeta} \;. \tag{101}$$

As a consequence we may also predict the $O\left(\frac{1}{d}\right)$ asymptotic behavior,

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$$\phi(\zeta) \longrightarrow \frac{1}{\zeta} + \frac{1}{d}\frac{1}{4\zeta^2}\left(c_1 + \frac{c_2}{\zeta}\right) + O\left(\frac{1}{d^2}\right) . \tag{102}$$

Now by substituting the above results into Eq. (96) we obtain, after expansion in powers of $1/d$,

$$\frac{\tilde{z}_0 + \zeta}{2\beta d} - 1 = 2Q_0(\zeta) - \frac{1}{2\tilde{z}_0 + \zeta} , \tag{103}$$

where we always assume the large-d limit to be taken while keeping βd finite. Substituting the condition $Q_0(0) = 0$ into Eq. (103) we may solve it in the form,

$$\frac{1}{2\beta d} = \frac{1}{\tilde{z}_0} - \frac{1}{2\tilde{z}_0^2} , \tag{104}$$

$$Q_0(\zeta) = \frac{\zeta}{2\tilde{z}_0}\left[\left(1 - \frac{3}{4\tilde{z}_0}\right) + \frac{1}{4\tilde{z}_0}\frac{\zeta}{2\tilde{z}_0 + \zeta}\right] , \tag{105}$$

while the implementation of Eq. (101) fixes

$$\begin{aligned}
c_1 &= -\frac{1}{2\tilde{z}_0}\frac{1}{\left(1 - \frac{3}{4\tilde{z}_0}\right)^2} , \\
c_2 &= \frac{4\tilde{z}_0}{\left(1 - \frac{3}{4\tilde{z}_0}\right)} ,
\end{aligned} \tag{106}$$

and in conclusion we may also write

$$R_0(\zeta) = \frac{\left(1 - \frac{1}{2\tilde{z}_0}\right)}{8\tilde{z}_0^2\left(1 - \frac{3}{4\tilde{z}_0}\right)^2}\frac{\zeta}{2\tilde{z}_0 + \zeta} . \tag{107}$$

Let us notice that the leading order is completely determined in terms of the parameter $\tilde{z}_0$, which in turn is fixed through Eq. (104) to take the value

$$\tilde{z}_0 = \bar{z} = \beta d\left(1 + \sqrt{1 - \frac{1}{\beta d}}\right) . \tag{108}$$

Hence we recognize that $\bar{z}$ is nothing but a generalization of the mean field parameter which (roughly speaking) describes the center of the eigenvalue distribution, while the width of the

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distribution itself is $O\left(1/\sqrt{d}\right)$ as one may easily see by studying the roots of the polynomial under the square root sign.

The eigenvalue distribution itself may be recovered (order by order in $1/d$) by the relationship

$$\rho(z) = \frac{dQ(z-\tilde{z})}{\pi(z-\tilde{z})}\sqrt{\frac{c_2}{d} + \frac{c_1}{d}(z-\tilde{z}) - (z-\tilde{z})^2}\,, \tag{109}$$

and it is not too difficult to check that the large-d limit of Eq. (109) may be taken and the result is

$$\rho(z) \longrightarrow \lim_{d\to\infty} \frac{d}{\pi}\frac{2}{c_2}\sqrt{\frac{c_2}{d} - (z-\tilde{z}_0)^2} = \delta(z-\tilde{z}_0)\,, \tag{110}$$

as expected. We have worked out higher orders of the $1/d$ expansion.

The Ansatz (97) can also be used as a starting point for numerical approximations, and this will be described in Appendix D. For instance, with the precision of about 1%, one can quickly verify numerically the conjecture that $\beta_c = 1/d$.

IV. SIMPLICIAL MODELS AT CRITICALITY

We would like to be able to understand in more detail the behavior of the critical properties as a function of d. Eq. (29), to the best of our knowledge, does not lend itself to an exact treatment for arbitrary values of β and d. However, on the critical surface for weak-strong transition, we find that it is possible to turn Eqs. (39)-(40) into a homogeneous (eigenvalue) equation. For $d \leq 4$, this eigenvalue problem can be solved analytically. For $d > 4$, the problem can be solved numerically with great precision.

It is also worth pointing out that at criticality for $d \leq 4$, Eq. (29) can be solved by applying a method of Gaudin and Kostov for the study of $O(n)$ spins on random surfaces. There is in fact an exact mapping of our weak-coupling critical saddle point equation to that of Ref. [15], with $d = n + 2$. However, for $d > 4$, the Gaudin-Kostov's solution become pathological. In contrast, for the simplicial models, we find that a consistent solution exists for $d > 4$.

The determination of β_c can be achieved by considering Eq. (39) in the limit when the equality in the weak coupling constraint, Eq. (41), is reached. On the weak coupling side of criticality the condition $\rho(0) = 0$ implies

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$$\sqrt{a_c b_c} \left[\frac{1}{4\beta_c} - \frac{d-2}{2} \int_{a_c}^{b_c} \frac{dy}{y} \frac{\rho_c(y)}{\sqrt{(b_c+y)(y+a_c)}} \right] = 0 \ . \tag{111}$$

There are two possible solutions, (i) $a_c = 0$, and (ii) $a_c \neq 0$, with

$$\frac{1}{\beta_c} = 2(d-2) \int_{a_c}^{b_c} \frac{dy}{y} \frac{\rho_c(y)}{\sqrt{(b_c+y)(y+a_c)}} \ . \tag{112}$$

If Eq. (111) is solved by $a_c = 0$ at $\beta = \beta_c$, Eqs. (39) and (40) both reduce to

$$\rho_c(z) = \frac{\sqrt{(b_c-z)z}}{\pi} \left[\frac{1}{4\beta_c} - \frac{d-2}{2} \int_0^{b_c} \frac{dy}{y+z} \frac{\rho_c(y)}{\sqrt{(b_c+y)y}} \right] \ . \tag{113}$$

This indeed applies for $0 \leq d \leq 4$, and we shall find an explicit solution of Eq. (113), which agrees with a result previously found by Gaudin and Kostov [15]. In particular, we find that $\beta_c = 1/d$ for $0 \leq d \leq 4$.

When $d > 4$, Eq. (111) must be solved with $a_c > 0$. Substituting Eq. (112) into Eqs. (39)-(40), one arrives at a homogeneous equation

$$\lambda \rho_c(z) = z \sqrt{(b_c-z)(z-a_c)} \int_{a_c}^{b_c} \frac{dy}{y(y+z)} \frac{\rho_c(y)}{\sqrt{(b_c+y)(y+a_c)}} \ , \tag{114}$$

where $\lambda = \frac{2\pi}{d-2}$. Eq. (114) can be solved numerically with great accuracy. Surprisingly, the relationship $\beta_c = 1/d$ was found to be satisfied within machine precision. The essential feature of the solution for $d > 4$ is the strong coupling relationship $r_c = a_c{}^2 \neq 0$, while in weak coupling necessarily $r = 0$. Since $\rho_c(z)$ is the same on both sides of the transition, at criticality one finds from Eq. (43)

$$U_{\mathrm{w}}^{(c)} - U_{\mathrm{s}}^{(c)} = \frac{1}{4\beta_c^2} \int_{a_c}^{b_c} dz' \frac{\rho_c(z')r_c}{d(d-1)} = \frac{d}{4(d-1)} a_c^2 \ , \tag{115}$$

and as a consequence a first order phase transition is observed.

A. Critical Solution for $0 \leq d \leq 4$

When $0 \leq d \leq 4$, Eq. (29) at criticality can be solved by assuming that $a_c = 0$, as suggested by our analytic results discussed in Section III. Let us therefore focus on

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Eq. (113). This equation on the first sight suggests that $\rho(z)$ would vanish at $z = 0$ as $\sqrt{z}$. However, it is easy to verify that, upon substituting this behavior into the right hand side of the equation, this square-root behavior is in fact inconsistent. Based on our earlier exact analytic solutions, we assume that $\rho_c(z)$ vanishes at $z = 0$ faster the $\sqrt{z}$; it follows that the square-bracket in Eq. (113) must also vanish at $z = 0$. As a consequence, we have

$$\frac{1}{\beta_c} = 2(d-2) \int_{a_c}^{b_c} \frac{dy}{y^{3/2}} \frac{\rho_c(y)}{\sqrt{b_c + y}} \ , \tag{116}$$

and we again arrive at a homogeneous equation

$$\lambda \rho_c(z) = z^{3/2} \sqrt{b_c - z} \int_0^{b_c} \frac{dy}{y+z} \frac{\rho_c(y)}{y^{3/2} \sqrt{b_c + y}} \ . \tag{117}$$

Note that this homogeneous equation connects smoothly with that appropriate for $d > 4$, Eq. (114), with $a_c = 0$.

It is convenient to change variable from z to ω, $\omega + \omega^{-1} = 2b_c/z$. In solving for ω in terms of z, we shall choose the branch $1 \leq \omega < \infty$ so that Eq. (117) becomes

$$\lambda f(\omega) = (\omega^{1/2} - \omega^{-1/2}) \int_1^\infty \frac{d\omega'(\omega'^{1/2} - \omega'^{-1/2})}{(\omega' + \omega)(\omega' + \omega^{-1})} f(\omega') \ , \tag{118}$$

where $f(\omega) = (1/2)(\omega + \omega^{-1})\rho_c(z)$.

Although $f(\omega)$ is originally defined only for the interval $1 \leq \omega < \infty$, the right-hand side of Eq. (118) provides a natural extension to the region $0 \leq \omega \leq 1$. With this extension, one finds that, over the positive axis, $0 \leq \omega < \infty$,

$$f(\omega) = -f(\omega^{-1}) \ . \tag{119}$$

It is then straightforward to verify that this extension can also be made for the right-hand side of Eq. (118) so that it becomes

$$\lambda f(\omega) = \frac{(\omega^{1/2} - \omega^{-1/2})}{2} \int_0^\infty \frac{d\omega'(\omega'^{1/2} - \omega'^{-1/2})}{(\omega' + \omega)(\omega' + \omega^{-1})} f(\omega') \ . \tag{120}$$

Let us next treat Eq. (120) as an eigenvalue problem, and consider the Ansatz where $f(\omega) = c[\omega^\theta - \omega^{-\theta}]$. Using the technique of contour-integration, it is easy to verify that

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this indeed is an eigenvector with eigenvalue $\lambda = \pi / \cos \pi\theta$. Since $\lambda = \frac{2\pi}{d-2}$, it follows that $d = 4\cos^2\frac{\pi\theta}{2}$. With $0 \le d \le 4$, one has θ real and $0 \le \theta \le 1$, which allows a solution where $\rho_c(z)$ is positive definite!

Using the normalization condition for ρ_c together with the criticality condition $\int_0^{b_c} dz'/z' \rho_c(z') = 1$, we can fix the normalization constant $c = \frac{1}{\pi\theta}\cos\frac{\pi\theta}{2}$ and the end point $b_c = \frac{2}{\theta}\text{tg}\frac{\pi\theta}{2}$. One then obtains

$$\rho_c(z) = \frac{\cos\frac{\pi\theta}{2}}{\frac{\pi\theta}{2}}\frac{\text{sh}\theta u}{\text{ch}u} , \qquad (121)$$

where $e^u = \omega$ and $0 \le u < \infty$. One can show that Eq. (121) reproduces the known critical solution when $d = 2$ and 4. When substituted into the critical equation at $d = 3$ Eq. (121) is numerically found to be a satisfactory solution.

To determine the critical value β_c, we can re-express Eq. (116) in terms of $f(\omega)$ as

$$\frac{1}{\beta_c} = \frac{(d-2)}{\sqrt{2}b_c}\int_0^\infty d\omega \frac{(\omega^{1/2} - \omega^{-1/2})}{1 + \omega^2}f(\omega) . \qquad (122)$$

Again, by an contour integration, one arrives at the remarkable result

$$\beta_c = \frac{1}{4\cos^2\frac{\pi\theta}{2}} = \frac{1}{d} . \qquad (123)$$

B. Criticality for $d > 4$

The solution discussed above does not apply to the case $d > 4$ because the analytic continuation of Eq. (121) for the critical density would no longer be positive-definite in the interval $[0, b_c]$; we must choose the alternative, $a_c \ne 0$. We have previously seen, with $a_c \ne 0$, how the criticality condition for β_c, Eq (112), and the homogeneous integral equation for ρ_c, Eq. (114), can be obtained, approaching from the weak coupling regime, by enforcing the condition $\rho(0) = 0$ with $r = 0$. It is instructive to see how these equations can be similarly derived from the strong coupling regime.

Starting with Eq. (40), one is working within the strong coupling regime where the constraint, Eq. (20), is automatically satisfied with $a = \sqrt{r} \ne 0$. It can be shown that the criticality condition, Eq. (112), corresponds to a situation where a zero of $\rho(z)$ enters at

$z = a$. That is, as one increases β beyond β_c, the positivity of $\rho(z)$ would be violated, thus terminating the validity of the strong coupling solution. As pointed earlier, with Eq. (112), the strong coupling equation, Eq. (40), again leads to Eq. (114). As a consequence, for $d > 4$, when one approaches β_c from the strong coupling regime, one finds that $a_c = \sqrt{r_c} \neq 0$ and a first-order phase transition occurs. The solution to Eq. (114), on the critical point, subject to Eq. (30) and Eq. (32), can be found numerically. In order to have a better behaved kernel when d is close to 4 we define a new function H,

$$H_c(z) = \frac{\rho_c(z)}{z\sqrt{(b-z)(z-a)}} \ . \tag{124}$$

With $\lambda = \frac{2\pi}{d-2}$, the integral equation Eq. (114), and the constraints, Eqs. (30)-(32), and the equation which determines β_c, Eq. (112), become

$$\lambda H_c(z) = \int_a^b \frac{dy}{y+z} \, H_c(y) \, \frac{\sqrt{(b-y)(y-a)}}{\sqrt{(b+y)(y+a)}} \ , \tag{125}$$

$$\int_a^b dz \, H_c(z) \, z\sqrt{b-y)(y-a} = 1 \ , \tag{126}$$

$$\int_a^b dz \, H_c(z) \, \sqrt{b-y)(y-a} = 1 \ , \tag{127}$$

$$\frac{1}{\beta_c} = 2(d-2)\int_a^b dz \, H_c(z) \, \frac{\sqrt{(b-z)(z-a)}}{\sqrt{(b+z)(z+a)}} \ . \tag{128}$$

The solution of the integral equation Eq. (125) can be done numerically by discretizing the kernel. After the discretization, the problem is reduced to an eigenvalue problem of a real non symmetric matrix. There are several ways to discretize the kernel. Any rule of numerical integration is a discretization rule for the kernel. It is known that for integral equations the best discretization rules are the Gauss quadrature rules [28]. There are several Gauss quadrature rules. We used the simplest possible: the Gauss-Chebyshev rule. All these rules require to map the integration interval to $[-1, 1]$. Thus we perform the following change of variables

$$z = \frac{b-a}{2}\zeta + \frac{a+b}{2} \ , \tag{129}$$

or

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$$\zeta = \frac{2}{b-a}z - \frac{a+b}{b-a} \ . \tag{130}$$

Under this change of variables the integral equation becomes

$$\lambda H_c(\zeta) = \int_{-1}^{1} d\xi \ \frac{H_c(\xi)}{\xi + \zeta + 2\frac{1+\kappa}{1-\kappa}} \ \frac{\sqrt{1-\xi^2}}{\sqrt{(\xi + \frac{3+\kappa}{1-\kappa})(\xi + \frac{1+3\kappa}{1-\kappa})}} \ , \tag{131}$$

where $\kappa = a/b$. The constraints take the following form

$$\int_{-1}^{1} d\xi \ H_c(\xi) \ (\xi + \frac{1+\kappa}{1-\kappa})\sqrt{1-\xi^2} = (\frac{2}{1-\kappa})^3 \frac{1}{b^3} \ , \tag{132}$$

$$\int_{-1}^{1} d\xi \ H_c(\xi) \ \sqrt{1-\xi^2} = (\frac{2}{1-\kappa})^2 \frac{1}{b^2} \ , \tag{133}$$

and the equation for β_c is

$$\frac{1}{\beta_c} = b(d-2)(1-\kappa) \int_{-1}^{1} d\xi \ H_c(\xi) \ \frac{\sqrt{1-\xi^2}}{\sqrt{(\xi + \frac{3+\kappa}{1-\kappa})(\xi + \frac{1+3\kappa}{1-\kappa})}} \ . \tag{134}$$

The solution to Eq. (131) can be found up to an overall constant C, (assuming that λ is a nondegenerate eigenvalue). This constant and the upper bound, b, of the support of ρ_c can be computed using the constrains Eq. (132), Eq. (133). Let's denote an eigenfunction of (131) by $\hat{H}_c$. Then H_c, which is the function that is positive in [-1,1] and satisfies the constraints, (132) and (133), is related to $\hat{H}_c$ by $H_c = C\hat{H}_c$. If we now define

$$I_1 = \int_{-1}^{1} d\xi \ \hat{H}_c(\xi) \ (\xi + \frac{1+\kappa}{1-\kappa})\sqrt{1-\xi^2} \ , \tag{135}$$

$$I_2 = \int_{-1}^{1} d\xi \ \hat{H}_c(\xi) \ \sqrt{1-\xi^2} \ , \tag{136}$$

then

$$b = \frac{I_2}{I_1} \frac{2}{1-\kappa} \ , \tag{137}$$

$$C = \frac{I_1^2}{I_2^3} \ . \tag{138}$$

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From the above formulas it is obvious that one can fix κ, solve Eq. (131) and then find the eigenvalue λ which has a positive definite eigenfunction. Because the problem is well defined one expects that there exists only one such function. This expectation is confirmed by the numerical results. It turns out that the eigenfunction with the largest eigenvalue is the one which is positive definite in $[-1, 1]$. The nth eigenfunction has $n - 1$ zeros in $[-1, 1]$. Thus for a given κ one computes $b(\kappa)$, $a(\kappa)$, $\lambda(\kappa)$, $\beta_c(\kappa)$, $d(\kappa) = 2\pi\lambda(\kappa)^{-1} + 2$.

Using the above numerical method we have computed a_c, b_c with great precision for d in the interval (4.4,250). Combining the numerical results with the analytical for $d \leq 4$, a_c and b_c are plotted in Fig. 2 as functions of $1/d$ for $d \geq 2$. The functions $a_c(d)$, $b_c(d)$ are continuous functions of d at $d = 4$.

Several interesting features now emerge from an analysis of this data. On the one hand we can fit the functions $a_c(d)$, $b_c(d)$ with great accuracy as power series of $d - 4$ around $d = 4$ and they agree with the corresponding weak coupling expressions up to very high orders. On the other hand, if one does a careful extrapolation of a_c, b_c to $d = 4$ a new feature is seen (Figure. 3). The upper limit, $b_c(d)$, extrapolates linearly in $d - 4$ to π, consistent with analyticity in the $d - 4$ series expansion. The data alone determines the intercept $b_c(4)$ to be π to an accuracy of 10^{-8}. When one examines the lower limit a_c, it also approaches zero as d approaches 4. However it does not go to zero as a simple power. The more we improved our data near $d = 4$ the higher the effective power became. It appears that a_c may have an essential singularity at $d = 4$ vanishing faster than any power. Since the discontinuity of the internal energy on the first order line is given by $da_c^2/(4(d - 1))$, this is pertinent to the critical properties at the end of the first order transition. The log-log plots of $a_c(d)$ and $\pi - b_c(d)$ in Fig. 3 clearly support these observations.

In Fig. 2 we have also been helped by an expansion of the functions $(a_c + b_c)/2$ and $a_c b_c$ in powers of $1/d$. The coefficients of this expansion have been determined by best fits on the numerical results and found to be consistent with integer numbers within the precision of our determination. This result is also consistent with the results of the $1/d$ expansion which we shall discuss in the next section. In particular we found

$$\frac{a_c + b_c}{2} = 1 + \frac{2}{d} + \frac{2}{d^2} - \frac{6}{d^4} - \frac{20}{d^5} - \frac{48}{d^6} - \frac{92}{d^7} - \frac{118}{d^8} - \frac{42}{d^9} + \dots \tag{139}$$

$$a_c b_c = 1 - \frac{12}{d} + \frac{40}{d^2} - \frac{8}{d^3} - \frac{40}{d^4} - \frac{128}{d^5} - \frac{328}{d^6} - \frac{694}{d^7} - \frac{1112}{d^8} + \dots \tag{140}$$

The last integer term in both equations is uncertain.

Furthermore from the numerical computation of the β_c for $d > 4$, we can see the $d < 4$ result $\beta_c = 1/d$ still holds above the critical point. We performed the numerical calculations in double precision and we see no deviation at all from the $1/d$ law. The deviation of $|\beta_c - 1/d|$ from zero is determined to be less than 10^{-16} for d in the interval (4.4,250).

V. SUMMARY AND CONCLUSION

The 1/N expansion of matrix models has recently been used as a discrete representation for summing over random surfaces and, through the "double-scaling" limit, for studying low-dimensional string theories. Even more importantly, the large-N expansion has provided us with a scheme for addressing non-perturbative issues in non-Abelian gauge theories. For instance, many qualitative features of QCD, *e.g.*, confinement, the OZI rule, etc, can best be understood in a large-N setting.

However, quantitative progress in these directions has been slow due partly to the technical difficulties associated with the large number of independent "loop" variables in this limit. Nevertheless, it has been possible to gain useful insights into various interesting situations by utilizing as guides solvable models involving a small number of matrices, *e.g.*, models involving two Hermitian matrices. Much less is known for models involving unitary matrices. Our current work not only adds to the list of solvable models in this category but also introduces new techniques for addressing matrix model studies in the large-N limit.

In this paper, we have studied the large-N structure of simplicial chiral models defined on a $d - 1$ dimensional simplex as one varies d and the coupling β. By exploring the global $U(N) \times U(N)$ symmetry and by introducing an auxiliary complex matrix field, we are able to reduce the problem to that of solving for the eigenvalue distribution of a single Hermitian semi-positive-definite matrix in the large-N limit. In addition to providing exact large-N solutions for several specific values of d, we are able to identify and solve the strong-weak criticality problem for all values of d, $0 \leq d < \infty$.

For $0 \leq d \leq 4$, analytic solutions for ρ_c can be found. Interestingly we find that the criticality occurs precisely at $\beta_c = 1/d$, as suggested by our previous studies [10]. We find that the transition is third order. For $4 < d < \infty$, the criticality can also be studied by solving a homogeneous integral equation. However, we are only able to carry this out numerically. Within numerical accuracy, we have shown that the criticality again takes place at $\beta_c = 1/d$, but with a first order transition.

Since we are able to reduce a $d - 1$ dimensional simplicial chiral model to a model involving a single complex matrix, with d entering as a parameter in the effective action, the large-N limit can thus be solved by finding a density function for the eigenvalue distribution. Unlike usual matrix models where all eigenvalues lie in a single connected band in the large-N limit, this model effectively involves two bands, a "right-band" where $\lambda_i > 0$ and a "left-band" where $\lambda_j < 0$. However, unlike other two-band problems [29], the distribution over these two bands are correlated. This new feature presents a challenge which cannot be handled by a conventional large-N treatment.

Our key result is the reduction of the above problem to that of solving a single inhomogeneous integral equation for the eigenvalue distribution of a single Hermitian semi-positive definite matrix. Although we could not find a closed form solution to this equation for arbitrary d, we are able to solve it in several interesting special cases and we set up a systematic numerical approach to the solutions. We have found that the critical surface is defined by $\beta_c = 1/d$ for all d.

For small d, $0 < d < 4$, the models exhibit the third order Gross Witten transition. Indeed for $d = 1, 2, 3$ they coincide exactly with the chiral chains studied earlier by Brower, Rossi and Tan [7]. In this region, the criticality is related to that of $O(n)$ spin models on random surfaces, as discussed by Gaudin-Kostov [15]. For $d > 4$, however, there is a first order transition ending at the "upper critical" dimensions $d = 4$. It therefore appears that, from the perspective of the double-scaling limit, the most interesting situation corresponds to $0 < d \leq 4$. We have found that the point $d = 4$, having a logarithmic singularity, corresponds to $\alpha = 0$. In the language of the double-scaling limit, this corresponds to having a vanishing "string susceptibility", $\alpha \equiv \gamma_{string} = 0$, where $C_{sing} \sim (\beta - \beta_c)^{-\gamma_{string}}$, which formally correspond to that resulted from a $c = 1$ CFT theory. This calls for further studies around $d = 4$ which can provide further insight into possible different mechanisms for generating $c = 1$ physics. One way is to vary d near $d = 4$. Another approach is to stay at $d = 4$, and embellish the model by relaxing the "permutation symmetry" of the original $d = 4$ simplicial chiral model. This will be presented in a subsequent publication.

ACKNOWLEDGMENTS

APPENDIX A: CRITICAL PROPERTIES OF CHIRAL CHAINS FOR $L = 2, 3, 4, \infty$

Chiral chain models are defined by the partition function

$$Z_L = \int \prod_{i=1}^{L} dU_i \, \exp\left[N\beta \sum_{i=1}^{L} \text{Tr}(U_i U_{i+1}^\dagger + U_i^\dagger U_{i+1}) \right] , \qquad (A1)$$

with $U_i = U_{i+L}$. Free energy density, internal energy and specific heat are given by

$$
\begin{aligned}
F_L &= \frac{1}{LN^2}\ln Z_L \ , \\
U_L &= \frac{1}{2}\frac{\partial F_L}{\partial \beta} \ , \\
C_L &= \beta^2 \frac{\partial U_L}{\partial \beta} \ .
\end{aligned}
\tag{A2}
$$

When $L \to \infty$, Z_L can be reduced to the partition function of the Gross-Witten single-link problem [4]

$$
Z = \int dU \, \exp\left[N\beta \, \mathrm{Tr}(U + U^\dagger) \right] \ ,
\tag{A3}
$$

thus sharing the same thermodynamic properties. The free energy density at $N = \infty$, $F = \frac{1}{N^2}\ln Z$, is piecewise analytic with a third order transition at $\beta = \beta_c = 1/2$ between the strong coupling and weak coupling domains. The large-N limit of the specific heat is

$$
\begin{aligned}
C_\infty &= \beta^2 && \text{for} \quad \beta \le \beta_c \ , \\
C_\infty &= \frac{1}{4} && \text{for} \quad \beta \ge \beta_c.
\end{aligned}
\tag{A4}
$$

The behavior of C_∞ around β_c can be characterized by a specific heat critical exponent $\alpha = -1$. It is worth noting that an analysis of the double scaling limit, $N \to \infty$ and $\beta \to \beta_c$, allows the determination of the correlation length critical exponent, $\nu = 3/2$ [17,18], and that α and ν satisfy a hyperscaling relationship associated to a two-dimensional critical phenomenon, $2\nu = 2 - \alpha$. This fact is related to the equivalence of the double scaling limit with the continuum limit of a two-dimensional gravity model with central charge $c = -2$.

It has been shown that in the context of single-matrix models the parameter N plays a role quite analogous to the volume in ordinary systems, and double scaling limit turns out to be very similar to finite size scaling in a two-dimensional critical phenomenon [19,20]. As a manifestation of this fact, it has been observed that in the Gross-Witten single-link problem, (i) the asymptotic approach of the complex $Z(N,\beta)$ zeroes closest to β_c, $\bar{\beta}(N)$, toward the real axis occurs at a rate determined by the correlation length exponent [21],

$$
\mathrm{Im}\,\bar{\beta}(N) \ \propto \ N^{-\frac{1}{\nu}} \ ,
\tag{A5}
$$

and, (ii) for sufficient large N the position of the peak of the specific heat, $\beta_{peak}(N)$, behaves as [22]

$$
\beta_{peak}(N) - \beta_c \ \propto \ N^{-\frac{1}{\nu}} \ .
\tag{A6}
$$

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We recall that in ordinary critical behaviors finite size scaling leads to relations of the type (A5-A6) with N replaced by the size of the system.

The $L = 2$ chiral chain again corresponds to a Gross-Witten model with β replaced by 2β, thus obtaining $\beta_c = 1/4$ and the same critical exponents.

Solutions of the models with $L = 3, 4$ have been found in Ref. [7]. The results we present in the following without details on the derivations are easily obtainable from the analysis of Ref. [7].

We recall that the chiral chain with $L = 3$ is equivalent to the simplicial model with $d = 3$. In this case $\beta_c = 1/3$ and the phase transition is still third order. In the weak coupling region, $\beta \geq \beta_c$, the $N = \infty$ specific heat is given by

$$C_3^{(\mathrm{w})} = \beta^2 + \frac{1}{8} - \beta^2 \left(1 + \frac{1}{6\beta} \right) \left(1 - \frac{1}{3\beta} \right)^{1/2} . \qquad (A7)$$

Therefore close to β_c,

$$C_3^{(\mathrm{w})} = \frac{17}{72} - \frac{1}{2\sqrt{3}} (\beta - \beta_c)^{1/2} + O(\beta - \beta_c) . \qquad (A8)$$

Similarly in the strong coupling region, $\beta \leq \beta_c$, and close to β_c,

$$C_3^{(\mathrm{s})} = \frac{17}{72} - \frac{1}{2\sqrt{3}} (\beta_c - \beta)^{1/2} + O(\beta_c - \beta) . \qquad (A9)$$

Then the strong and weak coupling expressions of C_3 show that the critical point $\beta_c = 1/3$ is third order and $\alpha = -1/2$.

For $L = 4$ the study of the critical behavior around $\beta_c = \pi/8$ is slightly subtler. In the weak coupling domain the $N = \infty$ internal energy can be expressed as

$$U_4^{(\mathrm{w})} = 2\beta - \frac{1}{8\beta} - 2\beta\delta^2 , \qquad (A10)$$

where δ is implicitly determined by the equation

$$\frac{8\beta}{\pi} \left[E \left(\sqrt{1 - \delta^2} \right) - \delta^2 F \left(\sqrt{1 - \delta^2} \right) \right] = 1 . \qquad (A11)$$

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(This equation comes from the normalization of the eigenvalue distribution $\rho(\theta)$ introduced in Ref. [7]). Since $\delta = 0$ at $\beta_c = \pi/8$, in order to study the critical behavior close to β_c we expand Eq. (A11) around $\delta = 0$, obtaining the following relation

$$\frac{\beta - \beta_c}{\beta_c} = \frac{1}{2}\delta^2 \left(\ln\frac{4}{\delta} + \frac{1}{2} \right) + O\left(\delta^4\right) , \tag{A12}$$

and therefore $\delta^2 \sim \beta - \beta_c$ apart from logarithms. Furthermore we have

$$\frac{d\delta^2}{d\beta} = \frac{16}{\pi \ln(4/\delta)} + O(\delta^2) . \tag{A13}$$

We then obtain for the specific heat

$$C_4^{(\mathrm{w})} = \frac{\pi^2}{32} + \frac{1}{8} - \frac{\pi^2}{16\ln(4/\delta)} + O\left(\beta - \beta_c\right) \tag{A14}$$

when $\beta \to \beta_c^+$.

A similar analysis can be performed in strong coupling, where the internal energy can be written as

$$U_4^{(\mathrm{s})} = 2\beta - \frac{1}{8\beta} + 2\beta\frac{\zeta^2}{1 - \zeta^2} , \tag{A15}$$

with ζ implicitly defined by the equation

$$\frac{8\beta}{\pi} \frac{1}{\sqrt{1 - \zeta^2}} E\left(\sqrt{1 - \zeta^2}\right) = 1 . \tag{A16}$$

Since $\zeta = 0$ at β_c, we expand Eq. (A16) around $\zeta = 0$ obtaining

$$\frac{\beta_c - \beta}{\beta_c} = \frac{1}{2}\zeta^2 \left(\ln\frac{4}{\zeta} + \frac{1}{2} \right) + O\left(\zeta^4\right) . \tag{A17}$$

Consequently, $\zeta^2 \sim \beta_c - \beta$ apart from logarithms and

$$\frac{d\zeta^2}{d\beta} = -\frac{16}{\pi \ln(4/\zeta)} + O(\zeta^2) . \tag{A18}$$

We then obtain when $\beta \rightarrow \beta_c^-$,

$$C_4^{(s)} = \frac{\pi^2}{32} + \frac{1}{8} - \frac{\pi^2}{16\ln(4/\zeta)} + O(\beta_c - \beta) . \qquad (A19)$$

A comparison of Eqs. (A14) and (A19) leads to the conclusion that the phase transition is again third order with a critical exponent $\alpha = 0^-$. The critical exponent ν could then be determined by using the 2-d hyperscaling relationship, obtaining $\nu = 1$. This value of ν was confirmed by a numerical Monte Carlo study of the scaling of the specific heat peak position at finite N, we indeed observed a behavior like Eq. (A6) compatible with $\nu = 1$ within a few per cent of uncertainty.

In Fig. 4 we plot the specific heat versus β for $L = 2, 3, 4, \infty$. In conclusion we have seen that $L = 2, 3, 4, \infty$ have a third order phase transition at increasing critical values $\beta_c = \frac{1}{4}, \frac{1}{3}, \frac{\pi}{8}, \frac{1}{2}$, with specific heat critical exponents $\alpha = -1, -1/2, 0^-, -1$, respectively. Notice the behavior of α with respect to L, which is increasing for $L = 2, 3, 4$ reaching the limit of a third order critical behavior, but then in large-L limit it returns to $\alpha = -1$.

APPENDIX B: STRONG COUPLING EXPANSION OF CHIRAL CHAIN MODELS

Strong coupling series of the free energy density of chiral chain models are best generated by means of the character expansion, which leads to the following result

$$F_L(\beta) = F(\beta) + \tilde{F}_L(\beta) , \qquad (B1)$$

where $F(\beta)$ is the free energy of the single unitary matrix model ($F(\beta) = \frac{1}{N^2}\ln Z$ and Z is given by Eq. (A3)), and

$$\tilde{F}_L = \frac{1}{LN^2} \ln \sum_{(r)} d_{(r)}^2 \, z_{(r)}(\beta)^L , \qquad (B2)$$

$\sum_{(r)}$ denotes the sum over all irreducible representations of $U(N)$, $d_{(r)}$ and $z_{(r)}(\beta)$ are the corresponding dimensions and character coefficients. The calculation of the strong coupling series of $F_L(\beta)$ is much simplified in the large-N limit, due to the following relationships [9]

$$F(\beta) = \beta^2 + O\left(\beta^{2N+2}\right) , \qquad (B3)$$

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and

$$z_{(r)}(\beta) \; = \; \bar{z}_{(r)}\beta^n \, + \, O\left(\beta^{2N}\right) \, , \tag{B4}$$

where $\bar{z}_{(r)}$ is independent of β and n is the order of the representation (r). Explicit expressions of $d_{(r)}$ and $\bar{z}_{(r)}$ are given in Ref. [9]. Notice that the large-N strong coupling expansion of $\widetilde{F}_L(\beta)$ is actually a series in β^L,

$$\widetilde{F}_L \; = \; \sum_n c(n, L)\beta^{nL} \, . \tag{B5}$$

$\widetilde{F}_L$ represents also the generating functional for the "potentials" $W(n, L)$ introduced in Ref. [9], in the context of the strong coupling expansion of more general models, indeed the following relationship holds

$$W(n, L) \; = \; \frac{L}{2}c(n, L) \, . \tag{B6}$$

It is important to recall that the large-N character coefficients have jumps and singularities at $\beta = \frac{1}{2}$ [26], and therefore the relevant region for a strong-coupling character expansion is $\beta < \frac{1}{2}$.

We have analyzed the strong coupling series of chiral chain models in order to investigate their large-N critical behaviors for $L > 4$. Given the simple behavior of the large-N limit of $F(\beta)$, we considered only the contributions from $\widetilde{F}(\beta)$, thus working with series in β^L. We generated about 15 terms for each $L < 10$ and analyzed, as series in β^L, the specific heat derivative, which diverges at the critical point in a third phase transition. We employed the integral approximant technique [23–25], which at present seems to be one of the most powerful method of resummation. In particular we considered integral approximants obtained from first order linear differential equations.

Let us begin with the results obtained for the known cases $L = 3, 4$. For $L = 3$ already 15 terms in the series (in β^L) suffice to get $\beta_c = \frac{1}{3}$ and $\alpha = -\frac{1}{2}$ with a precision of about 10^{-9} and 10^{-7} respectively. However it is worth noticing that in the analysis of the specific heat derivative we found spurious non-diverging singularities on the positive real axis for $\beta < \beta_c$.

Concerning the $L = 4$ case, it is known that the integral approximant resummation analysis cannot reproduce an $\alpha = 0^-$ singularity type [24] and therefore it is not really suitable to this case. Anyway, we obtained a good determination of β_c, we found $\frac{\pi}{8}$ up to about 10^{-5}, and a rather stable but wrong exponent, $\alpha \simeq -0.18$, which should somehow simulate the logarithmic corrections found in the Appendix A, given that they cannot be

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generated by the differential equation solution. Again we found spurious non-diverging singularities for $\beta < \beta_c$.

The strong-coupling analysis starts giving new information when $L > 4$. Due to the persistent presence of spurious singularities, guided by the $L = 3, 4$ analysis, in all cases we considered the first diverging singularity on the positive real axis as an estimate of the true critical point. For $L = 5$ we obtained quite stable results: $\beta_c \simeq 0.43756$ and $\alpha \simeq -0.17$. We should say that the $L = 4$ analysis suggests some caution in accepting this estimate of α, it could still be a masked $\alpha = 0^-$. The analysis of $L = 6$ series gave a rather stable estimate of the critical point $\beta_c \simeq 0.4737$, but unstable exponents (although negative and small). Similar results were found for $L \geq 7$: $\simeq 0.504$ for $L = 7$, $\simeq 0.526$ for $L = 8$, $\simeq 0.57$ for $L = 10$. Notice that, unlike the $L \leq 6$ cases, these values cannot be considered as an estimate of the critical point. They are indeed larger than $\frac{1}{2}$, that is out of the region where a strong coupling analysis can be predictive, and therefore something else must happen before, breaking the validity of the strong coupling expansion. An example of this phenomenon comes from the Gross-Witten single-link model (recovered when $L \to \infty$), where the strong coupling expansion of the $N = \infty$ free energy leads to an analytical function not having singularity at all, $F(\beta) = \beta^2$, thus $\beta_c = \frac{1}{2}$ cannot be determined from a strong coupling analysis.

Of course we cannot consider this analysis satisfactory, but from it we may hint at a possible scenario. As for $L \leq 4$, for $L = 5, 6$, that is when the estimate of β_c coming from the above strong coupling analysis is smaller than $\frac{1}{2}$ and therefore acceptable, the term $\widetilde{F}(\beta)$ in Eq. (B1) should be the one relevant for the critical properties, determining the critical points and giving $\alpha \neq -1$ (maybe $\alpha = 0^-$ as in the $L = 4$ case). For $L \geq 7$ the critical point may not be a singular point in strong or weak coupling, but just the point where weak coupling and strong coupling curves meet each other. This would cause a softer phase transition with $\alpha = -1$, as for the Gross-Witten single-link problem. We expect $\beta_c < \frac{1}{2}$ also for $L \geq 7$.

This scenario would be consistent with the analysis of Green and Samuel [27], who studied the behavior of the link determinant (i.e. $\langle \det U_i U_{i+1}^\dagger \rangle$) to determine the critical points. The values of β_c we found for $L = 5, 6$ are consistent with their estimates.

APPENDIX C: $d = 4$ MODEL: WEAK AND STRONG COUPLING EXPANSION AND SERIES ANALYSIS.

We have briefly discussed in Section III B the possibility of performing weak and strong coupling expansion in the $d = 4$ model starting from an expansion in the powers of k^2. Here we want to give more details on the concrete implementation of this program.

Let us first of all for convenience define a few auxiliary functions of k^2, the labels s and w are to remind the strong and weak coupling expansions, which we shall treat on the same footing. In terms of standard elliptic integrals we define

$$n_\mathrm{w}(k^2) \;\equiv\; \frac{2}{\pi} \int_0^k \frac{\sqrt{1-\zeta^2}}{\sqrt{k^2-\zeta^2}} d\zeta \;=\; \frac{2}{\pi} E(k) \;, \tag{C1}$$

$$n_\mathrm{s}(k^2) \;\equiv\; \frac{2}{\pi} \int_0^k \frac{\sqrt{k^2-\zeta^2}}{\sqrt{1-\zeta^2}} d\zeta \;=\; \frac{2}{\pi} \left[E(k) - (1-k^2)K(k) \right] \;, \tag{C2}$$

$$d_\mathrm{w}(k^2) \;\equiv\; \frac{2}{\pi} K(k) n_\mathrm{s}(k^2) \;, \tag{C3}$$

$$d_\mathrm{s}(k^2) \;\equiv\; \frac{2}{\pi} K(k) k^2 n_\mathrm{w}(k^2) \;. \tag{C4}$$

let us introduce the function

$$p(k^2) \;=\; \left(\frac{2}{\pi}\right)^2 \int_0^k \sqrt{k^2-\zeta^2}\sqrt{1-\zeta^2}\,\Pi(\zeta^2,k)d\zeta \;, \tag{C5}$$

and notice that, because of the properties of elliptic integrals and of Eq. (74) we may obtain from Eqs. (70-71)

$$8\beta \;=\; \frac{n(k^2)^2}{d(k^2) - p(k^2)} \tag{C6}$$

holding for the proper choice of indices both in the weak and in the strong coupling phase. In order to set up our expansion we therefore need power series representation for the functions $\frac{2}{\pi}E(k)$, $\frac{2}{\pi}K(k)$ and $p(k^2)$. The elliptic integrals of the first and second kind have simple known expansions,

$$\frac{2}{\pi} K(k) \;=\; 1 + \sum_{n=1}^{\infty} c_{0n} \left(k^2\right)^n \;, \tag{C7}$$

$$\frac{2}{\pi} E(k) \;=\; 1 - \sum_{n=1}^{\infty} \frac{c_{0n}}{2n-1} \left(k^2\right)^n \;, \tag{C8}$$

where

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$$c_{0n} = \frac{(2n!)^2}{2^{4n}(n!)^4} \, .$$ (C9)

The elliptic integral of the third kind, needed in the construction of $p(k^2)$, admits the following expansion in the powers of the first argument,

$$\frac{2}{\pi}\Pi(\zeta^2, k) = \sum_{l=0}^{\infty} \zeta^{2l} I_l(k^2) \, ,$$ (C10)

where

$$I_l(k^2) = \frac{2}{\pi} \int_0^1 \frac{\xi^{2l}}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}} d\xi$$ (C11)

satisfy the recursion equation

$$(2l+1)k^2 I_{l+1}(k^2) - 2l(1+k^2)I_l(k^2) + (2l-1)I_{l-1}(k^2) = 0 \, ,$$ (C12)

with boundary condition,

$$I_0(k^2) = \frac{2}{\pi}K(k) \, ,$$
$$I_1(k^2) = \frac{2}{\pi k^2}\left[K(k) - E(k)\right] \, .$$ (C13)

In turn we may show that

$$J_l(k^2) \equiv (k^2)^{-(l+1)}\frac{2}{\pi} \int_0^k \sqrt{k^2 - \zeta^2}\sqrt{1 - \zeta^2}\, \zeta^{2l} d\zeta$$ (C14)

are related to I_l by

$$J_l(k^2) = I_l(k^2) - (1+k^2)I_{l+1}(k^2) + k^2 I_{l+2}(k^2) \, .$$ (C15)

As a consequence the function $p(k^2)$ is completely determined once $I_l(k^2)$ are known, by the relationship

$$p(k^2) = \sum_{l=0}^{\infty} I_l(k^2)J_l(k^2)(k^2)^{l+1} \, .$$ (C16)

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The recursion Eq. (C12) may be solved if we expand the functions $I_l(k^2)$ in a power series of k^2,

$$I_l(k^2) \equiv \sum_{n=0}^{\infty} c_{ln}(k^2)^n . \tag{C17}$$

It is tedious but straightforward to verify that

$$c_{ln} = \frac{(2n + 2l)! \, (2n)!}{[(n + l)! \, n! \, 2^{2n+l}]^2} \tag{C18}$$

satisfy the recursion.

One may also for convenience define

$$J_l(k^2) \equiv \sum_{n=0}^{\infty} d_{ln}(k^2)^n , \tag{C19}$$

and finds from Eq. (C15) that

$$d_{ln} = -\frac{c_{ln}}{2(2n - 1)(n + l + 1)} . \tag{C20}$$

Direct substitution of the series expansions thus obtained in Eq. (C6) allows to construct an expansion of β (or $1/\beta$ respectively) in powers of k^2 which a simple symbolic manipulation program can easily extend to extremely high orders.

In order to make our analysis complete we must manage to extend our discussion to the evaluation of a physical observable. By recalling Eq. (43), we notice that the expression for the internal energy in the $d = 4$ case is

$$U = \frac{1}{48\beta^2} \int_a^b dz' \, \rho(z')(z'^2 - r) - \frac{1}{3} - \frac{1}{12\beta} , \tag{C21}$$

which gives rise in weak coupling to

$$U_{\mathrm{w}} = \frac{4}{3n_{\mathrm{w}}(k^2)^2} \int_0^k d\zeta \, \rho_{\mathrm{w}}(\zeta)(1 - \zeta^2) - \frac{1}{3} - \frac{2}{3} \frac{d_{\mathrm{w}}(k^2) - p_{\mathrm{w}}(k^2)}{n_{\mathrm{w}}(k^2)^2} , \tag{C22}$$

while in strong coupling we obtain

$$U_{\mathrm{s}} \;=\; \frac{4}{3 n_{\mathrm{s}}(k^2)^2} \int_0^k d\zeta \, \rho_{\mathrm{s}}(\zeta)(k^2 - \zeta^2) - \frac{1}{3} - \frac{2}{3} \frac{d_{\mathrm{s}}(k^2) - p_{\mathrm{s}}(k^2)}{n_{\mathrm{s}}(k^2)^2} \;. \tag{C23}$$

By considering the explicit form of the functions $\rho_{\mathrm{w}}(\zeta)$ and $\rho_{\mathrm{s}}(\zeta)$ which we obtain from Eqs. (70-71), it is straightforward to parameterize U by

$$U \;=\; \frac{4}{3} \frac{k^2 r(k^2)}{[d(k^2) - p(k^2)] \, n(k^2)^2} - \frac{1}{3} - \frac{2}{3} \frac{d(k^2) - p(k^2)}{n(k^2)^2} \;, \tag{C24}$$

where the functions $r(k^2)$ can in turn be reconstructed, by using the results presented in this Appendix, in terms of the functions I_l and J_l. The results for the two regimes are

$$r_{\mathrm{w}}(k^2) \;=\; \sum_{l=1}^{\infty} J_l(k^2) \left[I_{l-1}(k^2) - I_l(k^2) \right] (k^2)^l \tag{C25}$$

and

$$r_{\mathrm{s}}(k^2) \;=\; \sum_{l=1}^{\infty} J_l(k^2) \left[I_{l-1}(k^2) - k^2 I_l(k^2) \right] (k^2)^l \tag{C26}$$

respectively. By expanding Eq. (C24) in power series of k^2, inverting Eq. (C6) and substituting k^2 as a function of $1/\beta$ or β respectively, we obtain the standard weak and strong coupling series for the internal energy.

In conclusion we present some terms of the strong and weak coupling series of the internal energy, obtained by implementing the procedure outlined in this appendix:

$$\begin{aligned} U_{\mathrm{s}} = \; & \beta + 2\,\beta^2 + 2\,\beta^3 + 4\,\beta^5 + 28\,\beta^6 + 38\,\beta^7 + 8\,\beta^8 + 440\,\beta^9 + 1936\,\beta^{10} + 1712\,\beta^{11} \\ & + 4160\,\beta^{12} + 62160\,\beta^{13} + 178072\,\beta^{14} + 101038\,\beta^{15} + 1215704\,\beta^{16} + 9259720\,\beta^{17} \\ & + 17052880\,\beta^{18} + 15519376\,\beta^{19} + 291277184\,\beta^{20} + 1351546592\,\beta^{21} + \ldots \end{aligned} \tag{C27}$$

$$\begin{aligned} U_{\mathrm{w}} = \; & 1 - \frac{0.5}{4\beta} - \frac{0.03125}{(4\beta)^2} - \frac{0.012695}{(4\beta)^3} - \frac{0.006744}{(4\beta)^4} - \frac{0.004131}{(4\beta)^5} - \frac{0.002767}{(4\beta)^6} \\ & - \frac{0.001971}{(4\beta)^7} - \frac{0.001469}{(4\beta)^8} - \frac{0.001133}{(4\beta)^9} - \frac{0.000898}{(4\beta)^{10}} - \frac{0.000727}{(4\beta)^{11}} - \frac{0.000600}{(4\beta)^{12}} + \ldots \end{aligned} \tag{C28}$$

The strong coupling series was also generated by using the more general approach of Ref. [9], finding the same results.

APPENDIX D: NUMERICAL APPROACH BASED ON LARGE d EXPANSION

The Ansatz (97) can also be used as a starting point for numerical approximations based on the very simple consideration that real analytic functions of ζ can be approximated with any assigned precision by polynomials of sufficiently high degree in ζ itself. We may therefore introduce the n-th truncations of $Q(\zeta)$ and $R(\zeta)$ respectively by the definitions

$$
Q_n(\zeta) = \sum_{i=1}^{n} q_i \zeta^i \ ,
$$
$$
R_n(\zeta) = \sum_{i=1}^{n} r_i \zeta^i \ . \tag{D1}
$$

We may now explicitly perform a Laurent series expansion around the point $\zeta = 0$ in the form,

$$
dQ_n(\zeta) \left[1 - \sqrt{1 - \frac{c_1^{(n)}}{d\zeta} - \frac{c_2^{(n)}}{d\zeta^2}} \right] \equiv \sum_{i=1}^{n} p_i \zeta^i + p_0 + \sum_{j=1}^{\infty} p_{-j} \zeta^{-j} \ , \tag{D2}
$$

where the coefficients p_i, p_0 and p_{-j} are completely determined in terms of q_i, c_1 and c_2.

We may now notice that the asymptotic condition on $\phi_n(\zeta)$ forces us to impose the relationships

$$
p_i + r_i = 0 \qquad i > 0 \ ,
$$
$$
p_0 = 0 \ ,
$$
$$
p_{-1} = 1 \ . \tag{D3}
$$

At this stage $R_n, c_1^{(n)}, c_2^{(n)}$ are completely determined in terms of the coefficients q_i and the parameter $\tilde{z}^{(n)}$. We may now consider the effect of substituting $\phi_n(\zeta)$ into Eq. (96) and power-series expanding in ζ; if we define $\phi_n^{(k)}(-2\tilde{z})$ to be the k-th derivative of the function ϕ_n evaluated at the point $-2\tilde{z}$, we may turn Eq. (96) into the following approximate relationship,

$$
\frac{\tilde{z}}{2\beta} - d + \frac{\zeta}{2\beta} = 2\left[R_n(\zeta) + dQ_n(\zeta) \right] + (d-2) \sum_{k=0}^{n} \frac{(-\zeta)^k}{k!} \phi_n^{(k)}(-2\tilde{z}) \ , \tag{D4}
$$

which in turn decomposes into $n + 1$ equations in the $n + 1$ unknowns $q_i, \tilde{z}$,

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$$\frac{\tilde{z}}{2\beta} - d = (d-2)\phi_n(-2\tilde{z}) \,,$$

$$\frac{1}{2\beta} = 2(dq_1 - p_1) - (d-2)\phi_n^{(1)}(-2\tilde{z}) \,,$$

$$0 = 2(dq_k - p_k) + (d-2)\frac{(-1)^k}{k!}\phi_n^{(k)}(-2\tilde{z}) \qquad 1 < k \le n \,. \tag{D5}$$

These equations may be solved numerically for arbitrary values of β and d and offer better and better approximations to the true eigenvalue distribution (by applying Eq. (109) with increasing values of n). Numerical experiments have shown that when $\beta \gg 1/d$ even extremely small values of n give quite accurate predictions, while around the criticality, corresponding in this language to the condition

$$\phi(-\tilde{z}) = -1 \,, \tag{D6}$$

which determines β_c, but the accuracy is definitely weaker. This fact did not prevent us from determining the location of criticality, by the use of $n \le 8$, in the cases $d = 3, 4, 5$, with a precision of about 1%.

In the context of this discussion it is important to observe that (even approximate) knowledge of $\phi(\zeta)$ implies an (approximate) knowledge of the moments of the eigenvalue distribution, which may be obtained from the Laurent series expansion via the relationship

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \int_a^b dz' \, z'^m \rho(z') = \phi(z - \tilde{z}) \simeq \sum_{j=1}^{\infty} p_{-j}(z - \tilde{z})^{-j} \,, \tag{D7}$$

which in particular implies that

$$\int_a^b dz' \, z'^2 \rho(z') = \tilde{z}^2 + 2\tilde{z}p_{-2} + p_{-3} \,, \tag{D8}$$

and in turn we may extract the internal energy via the relationship

$$d(d-1)U = \frac{1}{4\beta^2}\left(\tilde{z}^2 + 2\tilde{z}p_{-2} + p_{-3}\right) - d - \frac{1}{\beta} \,. \tag{D9}$$

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FIGURES

FIG. 1. dC versus $d\beta$ for the $d = 2, 3, 4, \infty$ simplicial models.

FIG. 2. a_c and b_c versus $1/d$. The crosses mark the points corresponding to $d = 4$.

FIG. 3. Log-log plot of a_c and $\pi - b_c$ versus $d - 4$.

FIG. 4. Specific heat versus β for the $L = 2, 3, 4, \infty$ chiral chain models.