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Anisotropy beta functions

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The flow of couplings under anisotropic scaling of momenta is computed in ϕ^3 theory in 6 dimensions. It is shown that the coupling decreases as momenta of two of the particles become large, keeping the third momentum fixed, but at a slower rate than the decrease of the coupling if all three momenta become large simultaneously. This effect serves as a simple test of effective theories of high energy scattering, since such theories should reproduce these deviations from the usual logarithmic scale dependence.

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A novel approach to high energy scattering[1] in gauge theories was recently suggested by Verlinde and Verlinde[2]: Their idea was to scale the fields and coordinates in the classical action, to identify which pieces of the scaled action controlled the behaviour of high energy scattering with fixed (but larger than the confinement scale) momentum transfer. Their pretty result suggested that the theory essentially became a two-dimensional topological theory in this limit, making contact with earlier ideas of Lipatov[3]. Now, massless quantum field theories are *not* scale invariant in general, so one does not really expect this trivial scaling to be correct quantum mechanically. The question is, how does the renormalized vertex function depend on anisotropic scale changes? The literature on high energy scattering is enormous, but I have not found a first-principles account of this issue.

Ever since the pioneering work of Stueckelberg and Petermann[4], it has been understood that the renormalization group includes a great deal more information than just the overall scale dependence of Green functions. The invariance of physical quantities under changes of normalization conditions can be used to relate couplings constants at different momenta, for example. In the present note, I want to apply this textbook technique to the problem of computing the change in the coupling of three massless particles, if the three momenta are scaled anisotropically, subject to momentum conservation. Anisotropic renormalization group equations were considered by Robertson[5]—it is instructive to compare the two approaches. I shall do the calculation in the simplest possible, albeit non-perturbatively unstable, setting, that of asymptotically free ϕ^3 theory in 6 dimensions. The gauge theory case is somewhat more involved, since one has to work without the benefits of dimensional regularization with minimal subtraction, and will be treated elsewhere. Anisotropic gauge theory asymptotics on a lattice were considered in Ref. 6, following the approach of Karsch[7].

Green functions, or proper vertices, obtained by using different regulators or different normalization schemes are related by finite renormalizations. Thus the proper vertices obtained by using dimensional regularization with minimal subtraction, $\Gamma^{(n)}_{dm}$, are related to those obtained by using the R-operation with some normalization conditions (denoted collectively as Θ), $\Gamma^{(n)}_R$, by

$$\Gamma^{(n)}_{dm}(p_i;\mu,\hat{g}) = z^{-n/2}(\hat{g},\Theta)\Gamma^{(n)}_R(p_i;\mu\Theta,g),$$

where g is a function of $\hat{g}$ and Θ. It is assumed that both sides are computed in perturbation theory, so $\hat{g}$ and g are small. Since Θ does not appear on the left hand side of this equation, it follows that

$$\frac{\partial}{\partial\Theta} z^{-n/2}(\hat{g},\Theta)\Gamma^{(n)}_R(p_i;\mu\Theta,g) = 0.$$

z is computed by comparing $\Gamma^{(2)}$ in both schemes. The change in the coupling under changes of Θ can then be easily computed.

To be precise, the normalization conditions I will use are

$$\Gamma^{(2)}_R(p^2=0)=0$$
$$\frac{\mathrm{d}}{\mathrm{d}p^2}\Gamma^{(2)}_R(p^2=\mu^2)=1$$
$$\Gamma^{(3)}_R(p_i=\mu\theta_i)=g,$$

as appropriate for a massless theory. The vectors θ_i can be used to determine the change in g when momenta are varied, since physical correlation functions are independent of θ up to a field rescaling. Without any loss of generality (in Euclidean space), I assume $\theta_1^2=\theta_3^2=1$. It is of course unnecessary to use dimensional regularization with minimal subtraction as the 'reference' point. I do not see a way to do this calculation without specifying normalization conditions.

Actually, to the one-loop order I need, there is no need to compute z. For completeness I note that

$$\Gamma^{(2)}_{dm}(p^2)=p^2\left(1-\frac{g^2}{2(4\pi)^3}C_{dm}\right)\left[1-\frac{g^2}{12(4\pi)^3}\ln p^2/\mu^2\right],$$
$$\Gamma^{(2)}_R(p^2)=p^2\left(1-\frac{g^2}{12(4\pi)^3}\right)\left[1-\frac{g^2}{12(4\pi)^3}\ln p^2/\mu^2\right],$$

where C_{dm} is a constant. z is obviously needed at higher orders in the loop expansion. It remains therefore to evaluate

$$\Gamma^{(3)}_R=g+g^3\int\frac{\mathrm{d}^6q}{(2\pi)^6}\left[\frac{1}{q^2(q+p_1)^2(q+p_1+p_2)^2}-\frac{1}{q^2(q+\mu\theta_1)^2(q+\mu\theta_1+\mu\theta_2)^2}\right].$$

The integrals are standard, and I find that for any parameter τ parametrizing the normalization conditions,

$$\frac{\partial g}{\partial\tau}=\left(\frac{g}{4\pi}\right)^3\int_0^1\mathrm{d}t\left[-\frac{1}{2}\frac{\partial\ln w^2}{\partial\tau}+\frac{\partial b}{\partial\tau}\left\{1-(1+b)\ln\frac{1+b}{b}\right\}\right].$$

Here I have defined

$$w\equiv\mu\left(t\theta_1-(1-t)\theta_3\right)$$
$$b\equiv\frac{t(1-t)(\theta_1+\theta_3)^2}{(t\theta_1-(1-t)\theta_3)^2}.$$

As a check, note that if $\tau=\mu$, the only term contributing is the first term, giving

$$\mu\frac{\partial g}{\partial\mu}=-\left(\frac{g}{4\pi}\right)^3,$$

implying

$$\frac{1}{g^2}-\frac{1}{g_0^2}=\frac{1}{32\pi^3}\ln\frac{\mu}{\mu_0}.$$

Suppose now that $\theta_1 \to \tau\theta_1$. I now find

$$\frac{\partial g}{\partial\tau}=\left(\frac{g}{4\pi}\right)^3\mu\int_0^1 \mathrm{d}t\Bigg[-(1+2b)\frac{tw\cdot\theta_1}{w^2}$$
$$+2t(1-t)\frac{\theta_1\cdot(\tau\theta_1+\theta_3)}{w^2}-2(1+b)\ln\frac{1+b}{b}\left[t(1-t)\frac{\theta_1\cdot(\tau\theta_1+\theta_3)}{w^2}-b\frac{tw\cdot\theta_1}{w^2}\right]\Bigg].$$

This is still not very transparent, but setting $\theta_1\cdot\theta_3=0$ gives

$$\tau\frac{\partial g}{\partial\tau}=\left(\frac{g}{4\pi}\right)^3\int_0^1 \mathrm{d}t\left[2t-(3+2b)t^2-2\left(t(1-t)-bt^2\right)(1+b)\ln\frac{1+b}{b}\right]\frac{\tau^2\mu^2}{w^2}.$$

Taking $\tau\uparrow\infty$, this gives

$$\tau\frac{\partial g}{\partial\tau}(\tau=\infty)=-\left(\frac{g}{4\pi}\right)^3,$$

reflecting the fact that in this limit, θ_3 is set to zero, so changing τ is the same as an overall scale change of the normalization point.

It is also possible to do the integral (still with $\theta_1\cdot\theta_3=0$) explicitly at $\tau=1$,

$$\tau\frac{\partial g}{\partial\tau}(\tau=1)=-\frac{1}{2}\left(\frac{g}{4\pi}\right)^3,$$

which exhibits the fact that the anisotropy dependence is non-trivial. Write

$$\tau\frac{\partial g}{\partial\tau}(\tau)=-\left(\frac{g}{4\pi}\right)^3 f(\tau).$$

$f(\tau)$ is monotonic and rises from $\frac{1}{2}$ at $\tau=1$ to 1 at $\tau=\infty$, as shown in figure 1. Integrating this equation, we get

$$\frac{1}{g(\tau)^2}-\frac{1}{g(1)^2}=\frac{1}{32\pi^3}\int_1^\tau\frac{\mathrm{d}\varpi}{\varpi}f(\varpi)\equiv\frac{1}{32\pi^3}G(\tau).$$

The function G is shown in figure 2.

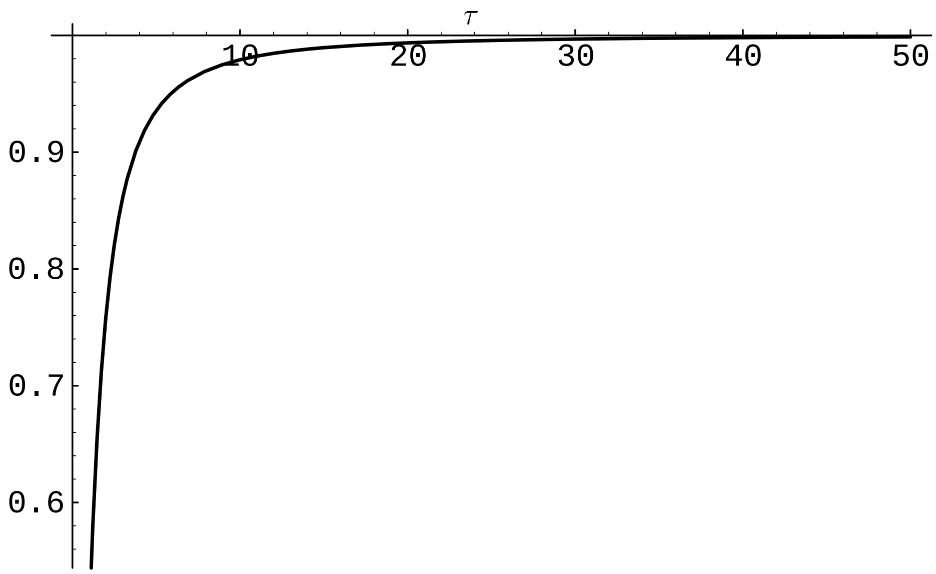

Fig. 1: $f(\tau)$

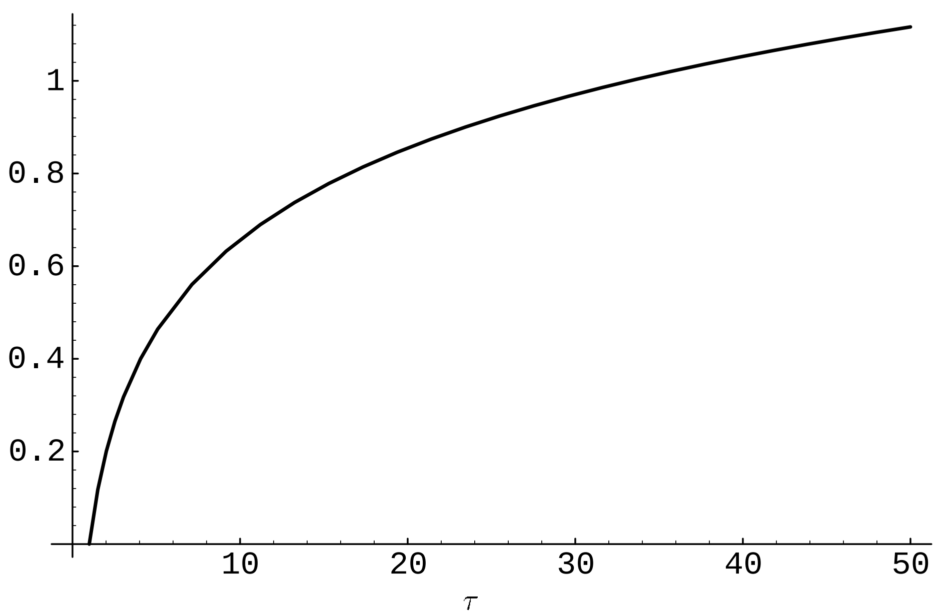

Fig. 2: $G(\tau)$

Consider now what happens if $\mu \to \mu_0 \tau^{-\alpha}$. The function $f(\tau)$ is now replaced by $f(\tau) - \alpha$, which is still positive for $\tau \geq 1$ if $\alpha < \frac{1}{2}$. In other words, in spite of the fact that the theory is asymptotically free, so one expects that the coupling will grow in the infrared, as long as the triangle formed by the three momenta has increasing area, the coupling of the three particles will decrease.

Since the correction from isotropic scaling is 50% at the point when the normalization point momenta are of equal magnitude, the predicted anisotropy dependence (when extended to gauge theories, of course) may be experimentally observable. The calculation in the present work used Euclidean momenta, so there is no direct comparison with high energy scattering data. The Verlinde scaling made no reference to the signature of the spacetime metric, hence should be testable within the present framework.

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