

Integrable Field Theories derived from 4D Self-dual Gravity

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Abstract

We reformulate the self-dual Einstein equation as a trio of differential form equations for simple two-forms. Using them, we can quickly show the equivalence of the theory and 2D sigma models valued in infinite-dimensional group, which was shown by Park and Husain earlier. We also derive other field theories including the 2D Higgs bundle equation. This formulation elucidates the relation among those field theories.

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1 Introduction

The four-dimensional self-dual Einstein equation (SdE) has been given attention for a long time both in physics and mathematics, as well as the self-dual Yang-Mills equation. Among a number of works associated with the SdE [1], an interesting and important subject is to connect it to other (possibly simple) field equations. Well-known examples of it are Plebanski's heavenly forms [2], there the SdE is given in terms of one function of space-time coordinates. Q-Han Park [3] and Ward [4] have shown that the SdE is derived from several two-dimensional sigma models with the gauge group of area preserving diffeomorphisms, $\text{SDiff}(\mathcal{N}_2)$. Park also has clarified the correspondence between the sigma models and first and second heavenly forms. On the other hand, by Ashtekar's canonical formulation for general relativity [5], the SdE has been reformulated as the Nahm equation [6], and its covariant version is given in Ref. [7]. Through this formulation, Husain has arrived at one of sigma models, that is, the principal chiral model [8]. Also by several reduction methods, other interesting models, e.g. the $\text{SL}(\infty)$ (affine) Toda equation [9][3], the KP equation [10] etc., are obtained.

Although we have various examples connected to the SdE, their relation is rather unclear, since their derivations from the SdE are more or less complicated and separated. Such a link of the models, however, should be investigated in order to understand the SdE further and in particular to develop the quantization of self-dual gravity.

In this paper, we describe the self-dual Einstein space by a trio of differential form equations for simple two-forms and derive several integrable theories quickly. This formulation elucidates their relation and may indicate the possibility to find further large class of models connected to the SdE.

2 Self-dual Einstein equation

We start from the observation that the SdE is expressed as closed-ness conditions of basis of the space of anti-self-dual two-forms,

$$d(e^0 \wedge e^i - \frac{1}{2}\epsilon_{ijk}e^j \wedge e^k) = 0, \quad i = 1, 2, 3, \quad (1)$$

where $e^{0,i} = e_{\mu}^{0,i}dx^{\mu}$ are tetrad one-forms on four-manifold. This formulation was employed by Plebanski to reduce the SdE to the heavenly forms [2], there the indices i, j, k are replaced with spinor ones A, B by the Pauli matrices $(\sigma^i)_A{}^B$. The equations in (1) appear also in Ref.[11][12]. We rewrite (1) by defining a null basis,

$$\begin{aligned} \mathcal{Z} &= e^0 + ie^1, & \bar{\mathcal{Z}} &= e^0 - ie^1, \\ \chi &= e^2 - ie^3, & \bar{\chi} &= e^2 + ie^3. \end{aligned} \quad (2)$$

Then (1) becomes

$$d(\mathcal{Z} \wedge \chi) = 0, \quad d(\bar{\mathcal{Z}} \wedge \bar{\chi}) = 0, \quad d(\mathcal{Z} \wedge \bar{\mathcal{Z}} + \chi \wedge \bar{\chi}) = 0. \quad (3)$$

As for symmetry, adding to the diffeomorphism invariance, (3) is invariant under local $SL(2, C)$ (self-dual) transformation, in matrix form,

$$\begin{bmatrix} \mathcal{Z}' \\ \chi' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \mathcal{Z} \\ \chi \end{bmatrix}, \quad \begin{bmatrix} \bar{\mathcal{Z}}' \\ \bar{\chi}' \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \begin{bmatrix} \bar{\mathcal{Z}} \\ \bar{\chi} \end{bmatrix}, \quad ad - bc = 1, \quad (4)$$

and also invariant under global $SL(2, C)$ (anti-self-dual) transformation for the pairs $(\bar{\mathcal{Z}}, \chi), (\mathcal{Z}, \bar{\chi})$ with the same form as (4).

The two-form in the last equation in (3) is rank-four, while others are rank-two, but we can re-express the last equation by defining two one-forms $\mathcal{P} = \mathcal{Z} - \bar{\chi}$, $\mathcal{Q} = \chi + \bar{\mathcal{Z}}$. Using them, we obtain the following equations equivalent to (3),

$$d(\mathcal{Z} \wedge \chi) = 0, \quad d(\bar{\mathcal{Z}} \wedge \bar{\chi}) = 0, \quad d(\mathcal{P} \wedge \mathcal{Q}) = 0. \quad (5)$$

Since all two-forms in (5) are simple and closed, they can be written as, on a local coordinate system,

$$\mathcal{Z} \wedge \chi = dz \wedge dx, \quad \bar{\mathcal{Z}} \wedge \bar{\chi} = d\bar{z} \wedge d\bar{x}, \quad \mathcal{P} \wedge \mathcal{Q} = dp \wedge dq, \quad (6)$$

where $(z, x, \bar{z}, \bar{x}, p, q)$ are functions. Although we use the notation suitable for the real, Euclidean case, we generally consider the complex SdE, so the bars in (6) do not mean the complex conjugation in usual. From the definition of \mathcal{P} and \mathcal{Q} , two identities are obtained,

$$\begin{aligned}\tilde{\mathcal{Z}} \wedge \chi \wedge \mathcal{P} \wedge \mathcal{Q} &= dz \wedge dx \wedge dp \wedge dq = dz \wedge dx \wedge d\bar{z} \wedge d\bar{x} = \tilde{\mathcal{Z}} \wedge \chi \wedge \tilde{\mathcal{Z}} \wedge \bar{\chi}, \\ \tilde{\mathcal{Z}} \wedge \bar{\chi} \wedge \mathcal{P} \wedge \mathcal{Q} &= d\bar{z} \wedge d\bar{x} \wedge dp \wedge dq = dz \wedge dx \wedge d\bar{z} \wedge d\bar{x} = \tilde{\mathcal{Z}} \wedge \chi \wedge \tilde{\mathcal{Z}} \wedge \bar{\chi}.\end{aligned}\quad (7)$$

(5), (6) and (7) are key equations in our formulation. For later use, we define the notation $x^a = (z, \bar{z})$ and $x^k = (p, q)$.

3 Integrable theories derived from the SdE

(a) *The principal chiral model*

At first, let us choose (z, \bar{z}, p, q) as four coordinate variables and $(x, \bar{x}) = (A_z, A_{\bar{z}})$ as functions of them. Then (7) reads

$$dz \wedge dA_z \wedge dp \wedge dq = dz \wedge dA_z \wedge d\bar{z} \wedge dA_{\bar{z}}, \quad d\bar{z} \wedge dA_{\bar{z}} \wedge dp \wedge dq = dz \wedge dA_z \wedge d\bar{z} \wedge dA_{\bar{z}}, \quad (8)$$

from which we can quickly derive the following equations after expanding $dA_{z(\bar{z})} = \partial_a A_{z(\bar{z})} dx^a + \partial_k A_{z(\bar{z})} dx^k$ and rescaling $A_{z(\bar{z})}$ by -2 ,

$$\partial_z A_{\bar{z}} + \partial_{\bar{z}} A_z = 0, \quad F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + \{A_z, A_{\bar{z}}\} = 0, \quad (9)$$

where $\{A_z, A_{\bar{z}}\}$ is the Poisson bracket with respect to (p, q) . The second equation means that generators $\mathcal{A}_{z(\bar{z})} = \{, A_{z(\bar{z})}\}$ of $\text{sdiff}(\mathcal{N}_2)$ are pure-gauge, $\mathcal{A}_{z(\bar{z})} = g^{-1} \partial_{z(\bar{z})} g$, where g is an element of the group $\text{SDiff}(\mathcal{N}_2)$. Substituting them into the first equation, we obtain

$$\partial_{\bar{z}}(g^{-1} \partial_z g) + \partial_z(g^{-1} \partial_{\bar{z}} g) = 0. \quad (10)$$

This is precisely the chiral model equation on the (z, \bar{z}) space-time with (p, q) treated as coordinates on the internal surface \mathcal{N}_2 . Let us solve $A_{z(\bar{z})}$ for a single

scalar function Θ by the first equation in (9), that is, $A_z = 2\partial_z\Theta$ and $A_{\bar{z}} = -2\partial_{\bar{z}}\Theta$. Then the second equation becomes

$$\Theta_{z,\bar{z}} + \Theta_{z,p}\Theta_{\bar{z},q} - \Theta_{z,q}\Theta_{\bar{z},p} = 0 , \quad (11)$$

where $\Theta_{z,p} = \partial_z\partial_p\Theta$. By an adequate gauge condition for the local $SL(2, C)$ symmetry, tetrads can take the form,

$$\begin{aligned} \mathcal{Z} &= h^{\frac{1}{2}}dz , & \chi &= -h^{\frac{1}{2}}d\bar{z} - h^{-\frac{1}{2}}\Theta_{z,k}dx^k , \\ \bar{\mathcal{Z}} &= h^{\frac{1}{2}}d\bar{z} , & \bar{\chi} &= h^{\frac{1}{2}}dz + h^{-\frac{1}{2}}\Theta_{\bar{z},k}dx^k , \end{aligned} \quad h = \{\Theta_{\bar{z}}, \Theta_z\} , \quad (12)$$

and the line element is

$$ds^2 = \mathcal{Z} \otimes \bar{\mathcal{Z}} + \chi \otimes \bar{\chi} = -\Theta_{\alpha,k} dx^\alpha \otimes dx^k + \frac{1}{\{\Theta_z, \Theta_{\bar{z}}\}} \Theta_{z,k} \Theta_{\bar{z},l} dx^k \otimes dx^l . \quad (13)$$

It is obvious that all self-dual metrics are obtained from this sigma model. In the case of $\{A_z, A_{\bar{z}}\} = 0$, the volume form $\frac{1}{4}(\mathcal{Z} \wedge \bar{\mathcal{Z}} \wedge \chi \wedge \bar{\chi})$ vanishes, which corresponds to a degenerate space-time.

(b) *The topological model with the WZ term only*

Next we take (z, x, \bar{z}, \bar{x}) as our coordinates and $(p, q) = (B_0, B_1)$ as functions of them. After changing the notation $(z, x, \bar{z}, \bar{x}) = (z, \bar{z}, p, q)$, (7) gives

$$\{B_0, B_1\}_{(z,\bar{z})} = 1 , \quad \{B_0, B_1\} = 1 . \quad (14)$$

The bracket in the first equation is defined with respect to (z, \bar{z}) . These equations are rather unfamiliar, but if we define $\partial_k A_{z(\bar{z})} = \{B_0, B_1\}_{(z(\bar{z}), x^k)}$, we can easily check the integrability $\partial_{[k}\partial_{l]}A_{z(\bar{z})} = 0$ and

$$\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z = 0 , \quad \{A_z, A_{\bar{z}}\} = 1 . \quad (15)$$

Also in this case, generators $\mathcal{A}_{z(\bar{z})}$ are pure-gauge and from the first equation, we obtain

$$\partial_{\bar{z}}(g^{-1}\partial_z g) - \partial_z(g^{-1}\partial_{\bar{z}} g) = 0 . \quad (16)$$

This is the topological model derived from the lagrangian of the Wess-Zumino term only [3]. The potentials $A_{z(\bar{z})}$ are given in terms of one function Ω by the first

equation in (15). Then $A_{z(\bar{z})} = \partial_{z(\bar{z})}\Omega$ and the second equation becomes

$$\Omega_{z,p}\Omega_{\bar{z},q} - \Omega_{z,q}\Omega_{\bar{z},p} = 1 , \quad (17)$$

which is Plebanski's first heavenly form [2]. With a gauge condition for the local $SL(2, C)$ symmetry, tetrads are given by

$$\mathcal{Z} = dz , \quad \chi = d\bar{z} , \quad \bar{\mathcal{Z}} = \Omega_{z,k}dx^k , \quad \bar{\chi} = \Omega_{\bar{z},k}dx^k , \quad (18)$$

and the line element is $ds^2 = \Omega_{a,k}dx^a \otimes dx^k$.

(c) *The WZW model*

The equation of the Wess-Zumino-Witten model is obtained by dropping the term in the right-hand-side of the first equation in (8),

$$dz \wedge dA_z \wedge dp \wedge dq = 0 , \quad d\bar{z} \wedge dA_{\bar{z}} \wedge dp \wedge dq = dz \wedge dA_z \wedge d\bar{z} \wedge dA_{\bar{z}} . \quad (19)$$

After relabeling $(z, A_z, \bar{z}, A_{\bar{z}}, p, q)$ as $(A_z, z, p, q, \bar{z}, A_{\bar{z}})$, (19) becomes

$$\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z = 0 , \quad \{A_z, A_{\bar{z}}\} = 0 . \quad (20)$$

Solving the potentials as in the case (b), we have

$$\Omega_{z,p}\Omega_{\bar{z},q} - \Omega_{z,q}\Omega_{\bar{z},p} = 0 . \quad (21)$$

Comparing it with the first heavenly form (17), we see that the WZW model describes a fully degenerate space-time [3].

(d) *The Higgs Bundle equation*

Here, note that it is not necessary to choose all four coordinate variables from the set $(z, x, \bar{z}, \bar{x}, p, q)$. Instead, we can regard more than two variables in it as functions. This observation leads us to the possibility to obtain further large class of models connected to the SdE.

Let us examine the observation by taking (z, \bar{z}) as two coordinate variables and $(x, \bar{x}, p, q) = (\phi, \bar{\phi}, B_0, B_1)$ as functions. Then (7) reads

$$dz \wedge d\phi \wedge dB_0 \wedge dB_1 = dz \wedge d\phi \wedge d\bar{z} \wedge d\bar{\phi} , \quad d\bar{z} \wedge d\bar{\phi} \wedge dB_0 \wedge dB_1 = dz \wedge d\phi \wedge d\bar{z} \wedge d\bar{\phi} . \quad (22)$$

As like the case (b), we introduce $\partial_k A_{z(\bar{z})} = \{B_0, B_1\}_{(z(\bar{z}), x^k)}$, in which the coordinates (p, q) are defined as the condition $\{B_0, B_1\} = 1$ is satisfied. This condition ensures the integrability $\partial_{[k}\partial_{l]}A_{z(\bar{z})} = 0$. Using $\phi, \bar{\phi}$ and $A_{z(\bar{z})}$, (22) and another identity $\mathcal{P} \wedge \mathcal{Q} \wedge \mathcal{P} \wedge \mathcal{Q} = 0$ become

$$\partial_{\bar{z}}\phi + \{A_{\bar{z}}, \phi\} = -\{\phi, \bar{\phi}\}, \quad \partial_z\bar{\phi} + \{A_z, \bar{\phi}\} = \{\phi, \bar{\phi}\}, \quad F_{z\bar{z}}(A) = 0, \quad (23)$$

or, changing $A_z \rightarrow A_z + \phi$ and $A_{\bar{z}} \rightarrow A_{\bar{z}} + \bar{\phi}$,

$$\partial_{\bar{z}}\phi + \{A_{\bar{z}}, \phi\} = 0, \quad \partial_z\bar{\phi} + \{A_z, \bar{\phi}\} = 0, \quad F_{z\bar{z}} = -\{\phi, \bar{\phi}\}. \quad (24)$$

This is the two-dimensional Higgs bundle equation with the group $\text{SDiff}(\mathcal{N}_2)$, which is mentioned in Ref. [4]. A self-dual Einstein metric is given by a solution of the model through the tetrads,

$$\begin{aligned} \mathcal{Z} &= (\partial_p\phi + \frac{g}{h}\partial_p\bar{\phi})dz + h^{-1}\partial_p\bar{\phi}\partial_I\phi dx^I, \quad \chi = (\partial_q\phi + \frac{g}{h}\partial_q\bar{\phi})dz + h^{-1}\partial_q\bar{\phi}\partial_I\phi dx^I, \\ \bar{\mathcal{Z}} &= (\partial_q\bar{\phi} - \frac{\tilde{g}}{h}\partial_q\phi)d\bar{z} - h^{-1}\partial_q\phi\partial_J\bar{\phi} dx^J, \quad \bar{\chi} = -(\partial_p\bar{\phi} - \frac{\tilde{g}}{h}\partial_p\phi)d\bar{z} + h^{-1}\partial_p\phi\partial_J\bar{\phi} dx^J, \end{aligned} \quad (25)$$

where $h = \{\phi, \bar{\phi}\}$, $g = \{\phi, A_z\}$ and $\tilde{g} = \{\bar{\phi}, A_{\bar{z}}\}$, and $x^I = \{\bar{z}, p, q\}$, $x^J = \{z, p, q\}$. The Higgs bundle equation was originally derived from a dimensional reduction of the four-dimensional self-dual Yang-Mills theory [13].

(e) Let us consider another example by choosing (p, q) as two coordinates and $(z, x, \bar{z}, \bar{x}) = (C_0, C_1, D_0, D_1)$ as functions of (p, q) and other (z, \bar{z}) . Also in this case, we define $\partial_k A_{z(\bar{z})} = \{C_0, C_1\}_{(z(\bar{z}), x^k)}$, $\partial_k B_{z(\bar{z})} = \{D_0, D_1\}_{(z(\bar{z}), x^k)}$ and impose the condition $\{C_0, C_1\} = 1$, $\{D_0, D_1\} = 1$ to ensure the integrabilities of $\partial_k A_{z(\bar{z})}$ and $\partial_k B_{z(\bar{z})}$. Then we obtain

$$F_{z\bar{z}}(A) = F_{z\bar{z}}(B) = 0, \quad \epsilon^{ab}(\partial_a B_b + \{A_a, B_b\}) = 0, \quad \epsilon^{ab}(\partial_a A_b + \{B_a, A_b\}) = 0. \quad (26)$$

The first two equations result from $\mathcal{Z} \wedge \chi \wedge \bar{\mathcal{Z}} \wedge \bar{\chi} = \tilde{\mathcal{Z}} \wedge \bar{\chi} \wedge \tilde{\bar{\mathcal{Z}}} \wedge \bar{\chi} = 0$. The tetrads are, for example,

$$\begin{aligned} \mathcal{Z} &= h^{\frac{1}{2}}dz - h^{-\frac{1}{2}}\partial_k A_{\bar{z}} dx^k, \quad \chi = h^{\frac{1}{2}}d\bar{z} + h^{-\frac{1}{2}}\partial_k A_z dx^k, \quad h = \{A_z, A_{\bar{z}}\}, \\ \bar{\mathcal{Z}} &= -g^{\frac{1}{2}}d\bar{z} - g^{-\frac{1}{2}}\partial_k B_z dx^k, \quad \bar{\chi} = g^{\frac{1}{2}}dz - g^{-\frac{1}{2}}\partial_k B_{\bar{z}} dx^k, \quad g = \{B_z, B_{\bar{z}}\}. \end{aligned} \quad (27)$$

In this model, the flat potentials $A_{z(\bar{z})}$ and $B_{z(\bar{z})}$ interact with each other through the third and fourth equations in (26).

4 Conclusion

In this paper, we have shown a formulation of the self-dual Einstein space which leads to low-dimensional field theories quickly and clearly. Now the relation among those theories is rather clear. For example, if we obtain a solution of (a) the principal chiral model, then it is straightforward, at least formally, to derive the corresponding solution of (b) the topological model or the first heavenly form (17); solving the potentials $A_{z(\bar{z})}$ in (9) for (p, q) , relabeling $(z, A_z, \bar{z}, A_{\bar{z}}, p, q)$ as $(z, \bar{z}, p, q, B_0, B_1)$, and following the step in the case (b). Here, for the opposite direction from (b) to (a), we give a simple example. A solution Ω of (17) corresponding to the $k = 0$ (flat) Gibbons-Hawking metric [14][1] is given by

$$\Omega = 2\epsilon^{-1}\sqrt{\bar{z}q}\sinh z \sinh p + 2\epsilon\sqrt{\bar{z}q}\cosh z \cosh p, \quad (28)$$

where ϵ is an arbitrary constant. We can make Ω real by setting the complex conjugate condition $z^* = p$, $\bar{z}^* = q$. Through the relation $\partial_k A_{z(\bar{z})} = \{B_0, B_1\}_{(z(\bar{z}), x^k)}$, the pair (B_0, B_1) can take the form,

$$B_0 = \epsilon^{-1/2}\sqrt{2\bar{z}}\sinh z - \epsilon^{1/2}\sqrt{2q}\cosh p, \quad B_1 = \epsilon^{1/2}\sqrt{2\bar{z}}\cosh z + \epsilon^{-1/2}\sqrt{2q}\sinh p. \quad (29)$$

By changing the notation as noted above and solving for $(A_z, A_{\bar{z}})$, we obtain a solution of the principal chiral model (9),

$$\begin{aligned} A_z &= -\left[\frac{\epsilon^{-1/2}p \sinh \bar{z} + \epsilon^{1/2}q \cosh \bar{z}}{\epsilon^{-1} \sinh z \sinh \bar{z} + \epsilon \cosh z \cosh \bar{z}}\right]^2, \\ A_{\bar{z}} &= -\left[\frac{\epsilon^{1/2}p \cosh z - \epsilon^{-1/2}q \sinh z}{\epsilon^{-1} \sinh z \sinh \bar{z} + \epsilon \cosh z \cosh \bar{z}}\right]^2. \end{aligned} \quad (30)$$

Also it may be interesting to discuss various reduction procedures through this formulation. For example, in the case (b) with the above complex conjugate condition, suppose that $(A_z, A_{\bar{z}})$ are functions of \bar{z}, q and the imaginary part of (z, p) only. Then changing the imaginary part and iA_z , we obtain the three-dimensional Laplace equation, which is just the ‘translational’ Killing vector reduction by Boyer and Finley [9]. Also for the ‘rotational’ case, the $SL(\infty)$ Toda equation is derived from (7) quickly. It is intriguing to pursue further large class of models connected to

the SdE by arranging the functions $(z, \bar{z}, x, \bar{x}, p, q)$ suitably and also to investigate the relation among the models.

To check the coordinate variables which permit real, Euclidean metrics, let us write down the form of $\mathcal{P} \wedge \mathcal{Q}$ explicitly,

$$\mathcal{P} \wedge \mathcal{Q} = (\mathcal{Z} \wedge \chi + \bar{\mathcal{Z}} \wedge \bar{\chi}) + (\mathcal{Z} \wedge \bar{\mathcal{Z}} + \chi \wedge \bar{\chi}) . \quad (31)$$

If all tetrads are real, the real, Euclidean case, the first term in (31) is real, while the second term pure imaginary. In the case of (a) the principal chiral model, suppose (p, q) are real or complex conjugate to each other. According to it, $\mathcal{P} \wedge \mathcal{Q} = dp \wedge dq$ becomes real or pure-imaginary. But then either the first or the second term in (31) vanishes, which corresponds to a degenerate space-time, not interesting case. Therefore, with such (p, q) , the chiral model permits only complex metric or signature $(+, +, -, -)$ real metric in which two of four tetrad one-forms may be taken as pure imaginary. In fact (B_0, B_1) in (29), which are (p, q) in the case (a), are neither real nor complex conjugate to each other.

The infinite-dimensional group $\text{SDiff}(\mathcal{N}_2)$ is known to be realized as a large N limit of the $\text{SU}(N)$ group when the surface \mathcal{N}_2 is the sphere or the torus [15]. Hence an approach to the SdE is to start from the $\text{SU}(N)$ principal chiral model, there its explicit solutions can be determined by the Uhlenbeck's uniton construction [16]. Adding to the $\text{SDiff}(\mathcal{N}_2)$, several (hidden) symmetrical structures in the SdE have been studied by the sigma model approach [3][8][17]. Our formulation may be useful to investigate the structure since, from our key equations (6),(7), we can easily see fundamental symmetries of our models, e.g. the conformal invariance on the (z, \bar{z}) space and $\text{SDiff}(\mathcal{N}_2)$, before we derive their field equations explicitly.

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