

Spin Models on Thin Graphs

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We discuss the utility of analytical and numerical investigation of spin models, in particular spin glasses, on ordinary “thin” random graphs (in effect Feynman diagrams) using methods borrowed from the “fat” graphs of two dimensional gravity. We highlight the similarity with Bethe lattice calculations and the advantages of the thin graph approach both analytically and numerically for investigating mean field results.

1. INTRODUCTION

The analytical investigation of spin glasses on random graphs of various sorts has a long and honourable history [1,2], though there has been little in the way of numerical simulations. Random graphs with a fixed or fixed average connectivity have a locally tree like structure, which means that loops in the graph are predominantly large, so Bethe-lattice-like [3] (ie mean field) critical behaviour is expected for spin models on such lattices. Given this, the analytical solution for a spin model or, in particular, a spin glass on a Bethe lattice [4,5] can be translated across to the appropriate fixed connectivity random lattice. Alternatively, a replica calculation can be carried out directly in some cases for spin glasses on various sorts of random lattices.

A rather different way of looking at the problem of spin models on random graphs was put forward in [6], where it was observed that the requisite ensemble of random graphs could be generated by considering the Feynman diagram expansion for the partition function of the model. For an Ising ferromagnet with Hamiltonian

$$H = \beta \sum_{\langle ij \rangle} \sigma_i \sigma_j, \quad (1)$$

where the sum is over nearest neighbours on three-regular random graphs (ie ϕ^3 Feynman di-

agrams), the partition function is given by

$$Z_n(\beta) N_n = \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda^{2n+1}} \int \frac{d\phi_+ d\phi_-}{2\pi \sqrt{\det K}} \exp(-S)$$

where N_n is the number of undecorated graphs with $2n$ points, K is defined by

$$K_{ab} = \begin{pmatrix} \sqrt{g} & \frac{1}{\sqrt{g}} \\ \frac{1}{\sqrt{g}} & \sqrt{g} \end{pmatrix} \quad (2)$$

and the action itself is

$$S = \frac{1}{2} \sum_{a,b} \phi_a K_{ab}^{-1} \phi_b - \frac{1}{3} (\phi_+^3 + \phi_-^3). \quad (3)$$

where the sum runs over \pm indices. The coupling in the above is $g = \exp(2\beta J)$ where $J = 1$ for the ferromagnet and the ϕ_+ field can be thought of as representing “up” spins with the ϕ_- field representing “down” spins. An ensemble of z -regular random graphs would simply require replacing the ϕ^3 terms with ϕ^z and a fixed average connectivity could also be implemented with the appropriate choice of potential.

This approach was inspired by the considerable amount of work that has been done in recent years on $N \times N$ matrix¹ versions of such integrals which generate “fat” or ribbon graphs with sufficient structure to carry out a topological expansion [7] because of the matrix index structure.

¹ N , the size of the matrix is not to be confused with n , the number of vertices in the graph!

The natural interpretation of such fat graphs as the duals of triangulations, quadrangulations etc. of surfaces has led to much interesting work in string theory and particle physics [8]. The partition function here is a poor, “thin” (no indices, so no ribbons), scalar cousin of these, lacking the structure to give a surface interpretation to the graph. Such scalar integrals have been used in the past to extract the large n behaviour of various field theories [9] again essentially as a means of generating the appropriate Feynman diagrams, so a lot is known about handling their quirks.

2. (ANTI)FERROMAGNETS AND SPIN GLASSES

For the Ising ferromagnet on three-regular (ϕ^3) graphs, solving the saddle point equations at large n

$$\begin{aligned}\phi_+ &= \sqrt{g}\phi_+^2 + \frac{1}{\sqrt{g}}\phi_-^2 \\ \phi_- &= \sqrt{g}\phi_-^2 + \frac{1}{\sqrt{g}}\phi_+^2\end{aligned}\quad (4)$$

shows that the critical behaviour appears as an exchange of dominant saddle point solutions to the saddle point equations. The high and low temperature solutions respectively are

$$\begin{aligned}\phi_+, \phi_- &= \frac{\sqrt{g}}{g+1} \\ \phi_+, \phi_- &= \frac{\sqrt{g}}{2(g-1)} \left(1 \pm \sqrt{\frac{g-3}{g+1}}\right)\end{aligned}\quad (5)$$

which give a low temperature magnetized phase. The critical exponents for the transition can also be calculated in this formalism and, as expected, are mean field. In general a mean field transition appears at

$$\exp(2\beta_{FM}) = z/(z-2) \quad (6)$$

on ϕ^z graphs, which is the value predicted by the standard approaches. Simulations nicely confirm this mean field picture for the ferromagnet [10]. Analysis of the Binder’s cumulant for the magnetization also shows that the critical temperatures are identical to the corresponding Bethe lattices (ie $g = 3$ for ϕ^3 graphs). The specific heat is shown in Figure.1 for various sizes of ϕ^3 graphs.

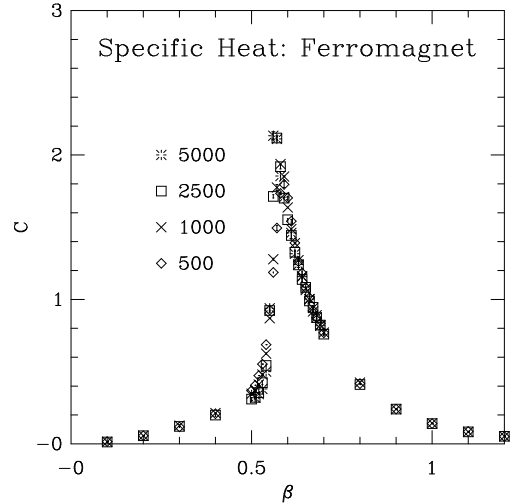


Figure 1. The specific Heat for the Ising ferromagnet on ϕ^3 graphs of various size.

There are various possibilities for addressing spin glass order in the Feynman diagram approach. In [6] the entropy per spin was calculated for the Ising *anti*-ferromagnet on ϕ^3 graphs and it was found to become negative for sufficiently negative β , which is often indicative of a spin glass transition. Simulations again confirm the picture. Taking a quenched distribution of couplings of the form

$$P(J) = p \delta(J-1) + (1-p) \delta(J+1), \quad (7)$$

which gives the antiferromagnet for $p = 0$, produces results for the spin glass order parameter, the overlap, that are very similar to the infinite range mean field (Sherrington-Kirkpatrick) model. Defining the overlap as

$$q = \frac{1}{n} \sum_i \sigma_i \tau_i. \quad (8)$$

with two Ising replicas on each graph, σ_i, τ_i and histogramming

$$P_n(q) = \left[\delta(q - 1/n \sum \sigma_i \tau_i) \right], \quad (9)$$

where $[]$ denotes the quenched disorder average, we get the distribution shown in Figure.2 [10,11]

in the putative spin glass phase at low temperature. The long tail stretching down to $q = 0$ is characteristic of the mean-field spin glass picture of many inequivalent states.

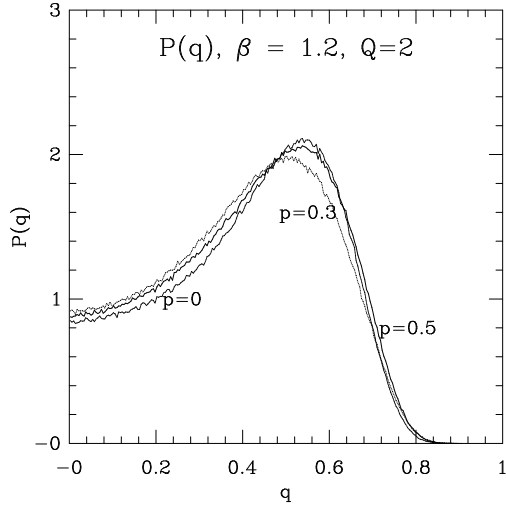


Figure 2. $P(q)$ at $\beta = 1.2$ for various p on ϕ^3 graphs of size 500.

It is possible to make some analytical inroads as well by looking at the solutions to the saddle point equations for k Ising replicas [11]

$$\vec{\phi} = \int P(J) \otimes^k K dJ \vec{\phi}^2 \quad (10)$$

where we have denoted the 2^k fields that now appear as $\vec{\phi}$. The Hessian for these equations is analytically calculable for *any* k

$$\prod_{m=0}^k \left[2 \int P(J) \tanh(\beta J)^m dJ - 1 \right]^{\binom{k}{m}} \quad (11)$$

and its zeroes show that the $k = 0$ transition temperature observed in simulations or calculated by using the analogy with the Bethe lattice is identical to the $k = 2$ transition temperature. This also occurs in the finite replica version of the Sherrington-Kirkpatrick model [12], so yet again the thin graph results are resolutely mean field.

For three or more replicas one does not see the continuous transition because a first order transition occurs at higher temperature to a replica-symmetric state. The situation appears to be rather similar for $Q > 2$ state Potts models where the saddle point calculation finds a continuous transition at one of the spinodal points and misses a first order transition occurring at higher temperature. The 3-state Potts model, for instance, with action

$$S = \frac{1}{2}(\phi_a^2 + \phi_b^2 + \phi_c^2) - c(\phi_a\phi_b + \phi_a\phi_c + \phi_b\phi_c) - \frac{1}{3}(\phi_a^3 + \phi_b^3 + \phi_c^3), \quad (12)$$

gives high and low temperature solutions

$$\begin{aligned} \phi_{a,b,c} &= 1 - 2c; \\ \phi_{a,b} &= \frac{1 + \sqrt{1 - 4c - 4c^2}}{2}, \\ \phi_c &= \frac{1 + 2c - \sqrt{1 - 4c - 4c^2}}{2} \end{aligned} \quad (13)$$

where $c = 1/(g + 1)$.

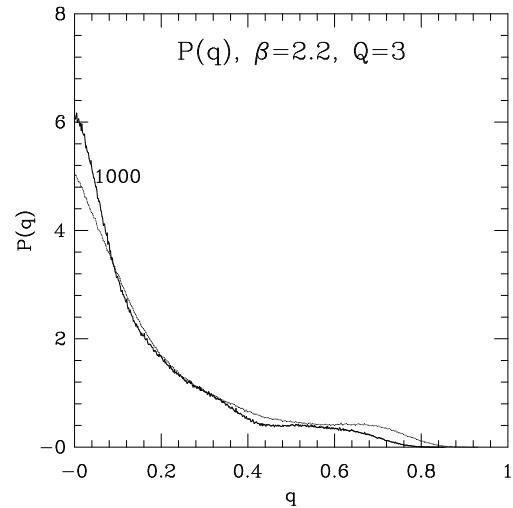


Figure 3. $P(q)$ at $\beta = 2.2$ for the 3-state Potts model for ϕ^3 graphs of size 1000 (labelled) and 500 (unlabelled).

Calculations and simulations for Potts glasses are just as easy as for the Ising spin glass. In Figure.3 overleaf we have plotted the distribution of overlaps for the three state Potts model at low temperature. The overlap for a Q state Potts model is now defined as

$$q = \frac{1}{n} \sum_{i=1}^n (Q \delta_{\sigma_i, \tau_i} - 1) \quad (14)$$

and the lack of an inversion symmetry in the spins gives a different pattern of replica symmetry breaking in mean-field theory to the Ising model. The numerical results are still consistent with a mean field picture.

3. Conclusions

In summary, spin models on thin graphs offer a promising arena for the application of ideas from matrix models, large- n calculations in field theory and bifurcation theory. In the spin glass case the tensor (or near-tensor) product structure of the inverse propagator allows some quite general expressions to be derived for the Hessian in the saddle point equations and offers a powerful line of attack on questions such as replica symmetry breaking. As a subject for numerical simulations they offer the great advantage of mean field results with no infinite range interactions and no boundary problems.

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REFERENCES

1. L. Viana and A. Bray, *J. Phys. C* **18** (1985) 3037.
2. M. Mezard and G. Parisi, *Europhys. Lett.* **3** (1987) 1067;
I. Kanter and H. Sompolinsky, *Phys. Rev. Lett.* **58** (1987) 164;
- K Wong and D. Sherrington, *J. Phys.* **A21** (1988) L459;
- C. de Dominicis and Y. Goldschmidt, *Phys. Rev.* **B41** (1990) 2184;
- P-Y Lai and Y. Goldschmidt, *J. Phys.* **A23** (1990) 399.
3. H. A. Bethe, *Proc. Roy. Soc. A* **150** (1935) 552;
T. P. Eggarter, *Phys. Rev.* **B9** (1974) 2989;
E. Muller-Hartmann and J. Zittartz, *Phys. Rev. Lett.* **33** (1974) 893.
4. S. Inawashiro and S. Katsura, *Physica* **100A** (1980) 24;
D. Thouless, *Phys. Rev. Lett.* **56** (1986) 1082;
P. Mottishaw, *Europhys. Lett.* **4** (1987) 333;
K Wong and D. Sherrington, *J. Phys.* **A20** (1987) L785.
5. Y. Goldschmidt, *Europhys. Lett.* **6** (1988) 7.
6. C. Bachas, C. de Calan and P. Petropoulos, *J. Phys.* **A27** (1994) 6121.
7. E. Brezin, C. Itzykson, G. Parisi and J.B. Zuber, *Commun. Math. Phys.* **59** (1978) 35;
M.L. Mehta, *Commun. Math. Phys.* **79** (1981) 327.
8. For a review see, J. Ambjorn, "Quantization of Geometry" Les Houches 1994, hep-th/9411179.
9. J. Le Guillou and J. Zinn-Justin (editors), "Large Order Behaviour of Perturbation Theory", Amsterdam: North Holland (1989).
10. C.F. Baillie, D. A. Johnston and J-P. Kownacki, *Nucl. Phys.* **B432** (1994) 551.
11. C.F. Baillie, W. Janke, D. A. Johnston and P. Plechac, "Spin Glasses on Thin Graphs", to appear in NPB[FS].
12. D. Sherrington, *J. Phys.* **A13** 637 (1980);
R. Penney, A. Coolen and D. Sherrington, *J. Phys.* **A26** 3681 (1993).