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# $O(4)$ symmetric singular solutions and multiparticle cross sections in $\phi^4$ theory at tree level

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**Abstract**

We solve the classical euclidean boundary value problem for tree-level multiparticle production in $\phi^4$ theory at arbitrary energies in the case of $O(4)$ symmetric field configurations. We reproduce known low-energy results and obtain a lower bound on the tree cross sections at arbitrary energies.

**1.** In recent years, remarkable progress has been achieved in calculating the amplitudes of multiparticle production in scalar theories like $\frac{\lambda}{4}\phi^4$ (for a review see ref.[1]). At the tree level, the amplitude of the process $1 \to n$ has been calculated exactly at $n$-particle threshold [2],

$$A_{1\to n} = n! \left(\frac{\lambda}{8}\right)^{\frac{n-1}{2}}$$

It has been understood [3] that this amplitude is related to spatially homogeneous euclidean solution to the field equations which is singular at $\tau = 0$ ($\tau$ is euclidean time). The fact that singular euclidean solutions play a role in calculating multiparticle amplitudes is not too surprising: similar singular solutions are known to be relevant for calculating semiclassical matrix elements in quantum mechanics by the Landau technique [4]. Furthermore, perturbative calculations about the Brown solution [5, 6] strongly suggest that to all orders of the perturbation theory the multiparticle cross section in the limit

$$\lambda \to 0, \quad \lambda n = \text{fixed}, \quad \epsilon = (E - nm)/n = \text{fixed} \tag{1}$$

($E$ is the total center–of–mass energy) has the exponential form,

$$\sigma_{1\to n}(n, E) \sim \exp\left[\frac{1}{\lambda}F(\lambda n, \epsilon)\right]$$

and that the leading exponent is independent of the initial state provided the latter contains $O(1)$ particles. These features are again similar to the semiclassical matrix elements in quantum mechanics calculable by the Landau technique.

There have been several attempts to apply the Landau technique for calculating multiparticle cross sections in field theories [7, 8, 9, 10, 11]. In particular, Son [11] formulated an appropriate classical boundary value problem for evaluation of the exponent $F(\lambda n, \epsilon)$ to all loops. Generaly, this approach requires contours in comlex time plane, but at small $\lambda n$, the leading term in $F$ can be calculated by studying purely euclidean s ingular solutions of the field equation. In ordinary perturbation theory this leading term comes from tree graphs, so the dependence of $F$ on $\lambda$ is known explicitly,

$$F_{\text{tree}}(\lambda n, E) = \lambda n \ln(\frac{\lambda n}{16}) - \lambda n + \lambda n f(\epsilon)$$

and the euclidean technique of Son [11] enables one, at least in principle, to calculate the only unknown function, $f(\epsilon)$, of the average kinetic energy of the outgoing particles, $\epsilon$. The loop corrections add terms of order $(\lambda n)^2$ or higher into $F(\lambda n, \epsilon)$.

Even in the simplest case of small $\lambda n$, the calculation of the exponent $F_{\text{tree}}$ at all energies $\epsilon$ is a complicated problem. The corresponding solution has singularities on a three-dimentional hypersurface in four-dimentional euclidean space (in this paper we consider four-dimentional theories) [11]. This hypersurface depends on $\epsilon$ and has to be found in the process of calculation. In this paper we apply the Rayleigh-Ritz procedure and consider $O(4)$ symmetric field configurations. Since $F_{\text{tree}}$ can be obtained by maximization over all hypersurfaces [11], our estimate of the exponent for the tree cross section is in fact the lower bound on $F_{\text{tree}}$ at given $\epsilon$.

**2.** Let us consider the tree cross section of producing $n$ scalar particles with total energy $E$ by a few virtual ones in a model with the lagrangian

$$L = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\phi^2 - \frac{\lambda}{4}\phi^4$$

where the mass of the boson is set equal to 1. As outlined above, in the limit (1) the tree cross section, with exponential accuracy, does not depend on the initial state and has the form

$$\sigma_{\text{tree}}(E, n) \sim \exp\left[W_{\text{tree}}(E, n)\right]$$

where

$$W_{\text{tree}}(E, n) = \frac{1}{\lambda} F_{\text{tree}}(\lambda n, \epsilon) = n \ln \frac{\lambda n}{16} - n + n f(\epsilon) \qquad (2)$$

The prescription of ref.[11] for calculating $W_{\text{tree}}(E, n)$ is as follows. One first introduces two extra parametrs, $T$ and $\Theta$, which are Legendre conjugate to $E$ and $n$. Then one considers hypersurfaces in euclidean space,

$$\tau = \tau(\boldsymbol{x}) \tag{3}$$

with the condition that

$$\tau(\boldsymbol{x} = 0) = 0 \tag{4}$$

The field configurations of interest are the solutions to the euclidean field equations which are singular at the surfaces (3) and vanich at $\tau \to +\infty$. For a given surface of singularities the solution to this boundary value problem is argued to be unique; its spatial Fourier components exponentially decay at $\tau \to \infty$,

$$\phi(\mathbf{k}, \tau) = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} b_{\mathbf{k}} \exp\left(-\omega_{\mathbf{k}} \tau\right)$$

where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + 1}$. For a given surface of singularities one defines the object

$$S(T, \Theta) = \int d^3k b_{\mathbf{k}}^* b_{\mathbf{k}} \exp\left(\omega_{\mathbf{k}} T - \Theta\right) \tag{5}$$

and its Legendre transform with respect to $T$ and $(-\Theta)$,

$$W(E, n) = ET - n\Theta - S(T, \Theta) \tag{6}$$

$$\frac{\partial S}{\partial T} = -E$$
$$\frac{\partial S}{\partial \Theta} = n \tag{7}$$

Note that Eq.(7) can be written as

$$S = -n \tag{8}$$

The function $W(E, n)$ still depends on the form of the singularity surface. To obtain $W_{\mathrm{tree}}(E, n)$, one should maximize (6) over all singularity surfaces subject to the condition (4). Note that the specific dependence of $S(T, \Theta)$ on $\Theta$,

$$S(T, \Theta) \propto \mathrm{e}^{-\Theta}$$

ensures that $W(E,n)$ (and, correspondingly, $W_{\rm tree}(E,n)$) is indeed parametrized by the only function $f(\epsilon)$, as indicated in Eq.(2).

This program has been carried out analytically for small $\epsilon$ [11] and earlier perturbative results [5] have been reproduced,

$$f(\epsilon) = \frac{3}{2}\log\frac{\epsilon}{3\pi} + \frac{3}{2} - \frac{17}{12}\epsilon + O(\epsilon^2) \tag{9}$$

In the opposite limit, $\epsilon \gg 1$, the mass term in the field equation can be neglected and the exponent of the cross section approaches the limit $F(\lambda n, \infty)$ determined by the corresponding solution of the massless field equations. Provided the correct saddle point is the maximum over the singularity curves, one can obtain a lower bound on the tree cross section in the ultra-high energy regime from known [8] massless singular $O(4)$-symmetric solution,

$$\phi(\tau, \boldsymbol{x}) = \sqrt{\frac{8}{\lambda}}\,\frac{\rho}{\boldsymbol{x}^2 + (\tau+\rho)^2 - \rho^2} \tag{10}$$

($\rho$ is the radius of the 3-dimensional sphere where the solution is singular), from which one obtains

$$f(\infty) \geq \log(2/\pi^2)$$

Entirely different way of obtaining lower bound on the tree cross section is to estimate tree diagrams directly [12]. We will compare our results with this bound in what follows.

**3.** To estimate $f(\epsilon)$ entering Eq.(2) for all $0 \leq \epsilon < \infty$ we apply the Rayleigh-Ritz procedure and consider $O(4)$ symmetric fields. This will actually provide a lower bound for $\sigma_{\rm tree}(E,n)$ at given $E$ and $n$. The singularity surfaces are then three-spheres in euclidean space, and the only variational parameter is the radius of a sphere, $\rho$. Since the singularity surface should touch the origin (see Eq.(4)), the singularity sphere should be centered at

$$x_0 = (-\rho, \mathbf{0})$$

i.e., the $O(4)$ symmetric solution has the general form

$$\phi = \phi(r)$$

$$r = \sqrt{\boldsymbol{x}^2 + (\tau + \rho)^2}$$

At large $r$, the solution tends to the exponentially falling solution to the free field equation,

$$\phi \propto \frac{K_1(r)}{r} \propto \frac{\mathrm{e}^{-r}}{r^{3/2}}$$

i.e., at $\tau \to \infty$ one has

$$\phi = A(\rho) \frac{\exp(-\sqrt{\boldsymbol{x}^2 + (\tau + \rho)^2})}{(\boldsymbol{x}^2 + (\tau + \rho)^2)^{3/4}}$$

where the coefficient function $A(\rho)$ is to be determined by solving the field equations under the condition that it has a singularity at $r = \rho$. From this asymptotics one finds

$$b_{\mathbf{k}} = 2A(\rho) \frac{\mathrm{e}^{-\omega_{\mathbf{k}} \rho}}{\sqrt{2\omega_{\mathbf{k}}}}$$

The "action" (5) is then expressed through $A(\rho)$, so $W$ (Eq.(6)) reads

$$W = ET - n\Theta - 8\pi A^2(\rho) \mathrm{e}^{-\Theta} \frac{K_1(2\rho - T)}{2\rho - T} \tag{11}$$

Now we extremize Eq.(11) over $\rho$, $T$ and $\Theta$. This leads to the following equations which determine the saddle point values of these three parameters,

$$\frac{E}{n} = \frac{A'(\rho)}{A(\rho)} = \frac{K_2(2\rho - T)}{K_1(2\rho - T)} \tag{12}$$

$$\Theta = \ln \left\{ \frac{8\pi A^2(\rho)}{n} \frac{K_1(2\rho - T)}{2\rho - T} \right\} \tag{13}$$

So, we look for the classical solutions which are singular at the spheres $r^2 = \rho^2$ and from their asymptotics obtain $A(\rho)$, then express saddle point values of $\rho$, $T$ and

$\Theta$ through $E$ and $n$ (by making use of the Eqs. (12), (13)) and finally obtain the estimate for the exponent for the tree cross section (see Eqs.(6) and (8))

$$W_{\rm tree} = ET - n\Theta - n$$

It is straightforward to perform this calculation numerically for all $\epsilon$.

**4.** The result of numerical solution of the field equation with singularities at different $\rho$, the function $A(\rho)$, is presented in Fig.1. The resulting radius of the singularity sphere, $\rho(\epsilon)$, is shown in Fig.2. The exponent for the cross section indeed has the form of Eq.(2) with the function $f(\epsilon)$ plotted in Fig.3.

At low energies our result matches the perturbative results [5], Eq.(9). The fact that our variational approach leads to the exact results for $W_{\rm tree}$ at small $\epsilon$ can be understood as follows. At small $\epsilon$, the curvature of the singularity surface is always large, and only this curvature is relevant for the evaluation of $f(\epsilon)$ [11]. In other words, the surface of singularities has the form

$$\tau(\boldsymbol{x}) = \alpha \boldsymbol{x}^2 + O(\boldsymbol{x}^4) + \cdots$$

and only the leading term is important at small $\epsilon$. Clearly, this leading term can be reproduced exactly in our $O(4)$ symmetric ansatz, and our result is exact at small $\epsilon$.

At very high energies (small values of $\rho$) our field configuration tends to the solution of the massless equations, Eq.(10). So, at $\epsilon \gg 1$ it approaches the lower bound derived from this solution.

The alternative lower bound on $f(\epsilon)$ can be easily read out from ref. [12] and it is also shown in Fig.3. This bound has been obtained by direct analysis of diagrams. As one can see from Fig.3, our new bound is stronger than that of ref. [12].

To summarize, we have solved numerically the boundary value problem for tree multiparticle cross-sections at arbitrary energy for spherically symmetric field configurations. At low energies our solution reproduces the exact tree results, while for general energy it gives the lower bound on the tree cross section. To improve this estimate, one should consider more general field configurations whose symmetry is at most $O(3)$.

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