TAUP-2285-95
August 3, 1995

Mesonic Spectra of Bosonized QCD_2 Models

A. Armoni and J. Sonnenschein [1]

School of Physics and Astronomy
Beverly and Raymond Sackler Faculty of Exact Sciences
Tel Aviv University, Ramat Aviv, 69978, Israel

Abstract

We discuss bosonized two-dimensional QCD with massless fermions in the adjoint and multi-flavor fundamental representations. We evaluate the massive mesonic spectra of several models by using the light-front quantization and diagonalizing the mass operator $M^2 = 2P^+P^-$. We recover previous results in the case of one flavor adjoint fermions and we find the exact massive spectrum of multi flavor QCD in the limit of large number of flavors.

[1] Work supported in part by the Israel Academy of Science, and the US-Israel Binational Science Foundation

1 Introduction

QCD in two space-time dimensions (QCD_2) is believed to be a useful laboratory to examine ideas about the hadronic physics of the real world. One such an idea, the expansion in large number of colors N_c, was applied in the pioneering work of 't Hooft [1]. A mesonic spectrum of a "Regge trajectory" type was discovered in that work for QCD_2 with Dirac fermions in the fundamental representation. The large N_c limit is taken while keeping $e^2 N_c$ fixed, where e is the gauge coupling. The "orthogonal" approach, the strong coupling limit, was also found to be a fruitful technique when combined with bosonizing the system. In that framework the low energy effective action of the theory can be derived exactly. Quantization of soliton solutions of that effective theory led to the low lying semi-classical baryonic spectrum[2].

The large N_c approach combined with a light-front quantization was applied to the study of QCD_2 with Majorana fermions in the adjoint representation. Unlike the case of fundamental fermions, for adjoint fermions pair creation is not suppressed in the large N_c limit. The bosonic spectrum includes an infinite number of approximately linear Regge trajectories which are associated with an exponential growing density of states at high-energy[3, 4]. Recently, a "universality" behavior of QCD_2 was found in the sense that the physics of massive states depends only on the gauge group and the afine Lie algebra level but not on the representation of the group[5].

In the present paper we combine the bosonization technique with that of a large N expansion and a light-front quantization in the analysis of the massive mesonic spectrum of several QCD_2 models. The massless sector which was discussed in ref.[5] is not addressed in the present work. The models include massless fermions in the adjoint representation and multi flavor fundamental representations. In the former case we expand in N_c- the number of colors whereas in the latter case we consider large N_f- number of flavors . In case of massless fermions the bosonization formalism is convenient since it separates the color, flavor and baryon number degrees of freedom[6]. Moreover the generalization from fermions coupled to YM fields to an arbitrary gauged afine Lie algebra system is natural in the bosonized picture. The basic difference between our approach and the one taken in refs. [3, 4] is the use of current quanta rather then those of quarks in constructing the mass operator M^2 and thus also the wave equation and the Hilbert space. Our results are in agreement with the results of refs.[3, 4] for the case of adjoint fermions. In the large N_f it is shown that the exact massive spectrum is a single particle with $M^2 = \frac{e^2 N_f}{\pi}$. This phenomenon is explained by the fact that this limit can be viewed as an "abelianization" of the model.

The paper is organized as follows. In section 2 we present the models. The actions are written down in their bosonized versions. We then derive explicit expressions for the momentum operators and the mass operator. The afine Lie algebra currents are expanded in terms of annihilation and creation operators with which a Fock space of physical states is built. In section 3 the model with adjoint fermions is analyzed in the large N_c limit. We introduce wave functions and write down an eigenvalue equation which generalizes 't Hooft equation[1]. The massive mesonic spectrum was then deduced. In section 4 we find that the exact spectrum of multi-flavor QCD with fundamental fermions in the regime of $N_f \gg N_c$ include only one single mesonic state. We discuss the relation of this spectrum to that of multi-flavor QED. The case of large level WZW models is also considered. Some conclusions and certain open problems are discussed in section 5.

2 The Models

Consider two dimensional $SU(N_c)$ Yang-Mills gauge fields coupled to (i) N_f massless Dirac fermions in the fundamental representation, or (ii) massless Majorana fermions in the adjoint representation. These theories are described by the following classical Lagrangian:

$$\mathcal{L} = -\frac{1}{4e^2} Tr[F_{\mu\nu}^2 + i\bar{\Psi}\, \not{D}\Psi] \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ and the trace is over the color and flavor indices. For the case (i) Ψ has the following group structure Ψ_{ia} where $i = 1, ..., N_c$ and $a = 1, ..., N_f$, $D_\mu = \partial_\mu - iA_\mu$ whereas for the case (ii) $\Psi \equiv \Psi^i_j$ and $D_\mu = \partial_\mu - i[A_\mu, \;]$. In both cases Ψ is a two spinor parametrized as follows $\Psi = \begin{pmatrix} \bar{\psi} \\ \psi \end{pmatrix}$.

It is useful to handle these models in the framework of the light-front quantization, namely, to use light-cone space-time coordinates and to choose the chiral gauge $A_- = 0$. In this scheme the quantum Lagrangian takes the form

$$\mathcal{L} = -\frac{1}{2e^2}(\partial_- A_+)^2 + i\psi^\dagger\partial_+\psi + i\bar{\psi}^\dagger\partial_-\bar{\psi} + A_+ J^+ \tag{2}$$

where color and flavor indices were omitted and J^+ denotes the $+$ component of the color current $J^+ \equiv \psi^\dagger\psi$.

By choosing x^+ to be the 'time' coordinate it is clear that A_+ and $\bar{\psi}$ are non-dynamical degrees of freedom. In fact, $\bar{\psi}$ are decoupled from the other fields so in order to extract the physics of the dynamical degree of freedom one has to functionally integrate over A_+. The result of this integration is the following simplified Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I = i\psi^\dagger\partial_+\psi + i\bar{\psi}^\dagger\partial_-\bar{\psi} - e^2 J^+ \frac{1}{\partial_-^2} J^+ \tag{3}$$

Since our basic idea is to solve the system in terms of the "quanta" of the colored currents, it is natural to introduce bosonization descriptions of the various models.

(i) The bosonized action of colored flavored Dirac fermions in the fundamental representation is expressed in terms of a WZW[7] action of a group element $u \in U(N_c \times N_f)$ with an additional mass term that couples the color, flavor and baryon number sectors[8]. In the massless case when the latter term is missing the action takes the following form

$$S_0^{fun} = S_{(N_f)}^{WZW}(g) + S_{(N_c)}^{WZW}(h) + \frac{1}{2}\int d^2x \partial_\mu\phi\partial^\mu\phi \tag{4}$$

where $g \in SU(N_c)$, $h \in SU(N_f)$ and $e^{i\sqrt{\frac{4\pi}{N_c N_f}}\phi} \in U_B(1)$ with $U_B(1)$ denoting the baryon number symmetry and

$$S_{(k)}^{WZW}(g) = \frac{k}{8\pi}\int_\Sigma d^2x \; tr(\partial_\mu g \partial^\mu g^{-1}) + \\ \frac{k}{12\pi}\int_B d^3y \epsilon^{ijk} \; tr(g^{-1}\partial_i g)(g^{-1}\partial_j g)(g^{-1}\partial_k g),$$

(ii) The current structure of free Majorana fermions in the adjoint representation can be recast in terms of a WZW action of level $k = N_c$, namely, $S_0^{adj} = S_{(N_c)}^{WZW}(g)$ where now g is in the adjoint representation of $SU(N_c)$, so that it carries a conformal dimensions of $\frac{1}{2}$[9].

Multi-flavor adjoint fermions can be described as $S^{WZW}_{N_f}(g) + S^{WZW}_{N_c^2-1}(h)$ where $g \in SO(N_c^2-1)$ and $h \in SO(N_f)$. In the present work we discuss only gauging of $SU(N_c)$ WZW so the latter model would not be considered.

Substituting now S_0^{fun} or S_0^{adj} for S_0 the action that corresponds to (3) is given by

$$S = S_0 - \frac{e^2}{2}\int d^2x J^+ \frac{1}{\partial_-^2} J^+ \tag{5}$$

where the current J^+ now reads $J^+ = i\frac{k}{2\pi} g\partial_- g^\dagger$, where the level $k = N_f$ and $k = N_c$ for the multi-flavor fundamental and adjoint cases respectively.

The light-front quantization scheme is very convenient because the corresponding momenta generators P^+ and P^- can be expressed only in terms of J^+. We would like to emphasize that this holds only for the massless case.

Using the Sugawara construction, the contribution of the colored currents to the momentum operator, P^+, takes the following simple form:

$$P^+ = \frac{1}{N+k}\int dx^- : J^i_j(x^-) J^j_i(x^-) : \tag{6}$$

where $J \equiv \sqrt{\pi} J^+$, N_c in the denominator is the second Casimir operator of the adjoint representation and the level k takes the values mentioned above. Note that for future purposes we have added the color indices $i, j = 1 \ldots N_c$ to the currents. In the absence of the interaction with the gauge fields the second momentum operator, P^- vanishes. For the various QCD_2 models it is given by

$$P^- = -\frac{e^2}{2\pi}\int dx^- : J^i_j(x^-) \frac{1}{\partial_-^2} J^j_i(x^-) : \tag{7}$$

In order to find the massive spectrum of the model we should diagonalize the mass operator $M^2 = 2P^+P^-$. Our task is therefore to solve the eigenvalue equation

$$2P^+P^- \; |\psi\rangle = M^2 \; |\psi\rangle. \tag{8}$$

We write P^+ and P^- in term of the Fourier transform of $J(x^-)$ defined by $J(p^+) = \int \frac{dx^-}{\sqrt{2\pi}} e^{-ip^+x^-} J(x^-)$. Normal ordering in the expression of P^+ and P^- are naturally with respect to p, where $p < 0$ denotes a creation operator. To simplify the notation we write from here on p instead of p^+. In terms of these variables the momenta generators are

$$P^+ = \tfrac{2}{N+k}\textstyle\int_0^\infty dp J^i_j(-p) J^j_i(p)$$

$$P^- = \frac{e^2}{\pi}\int_0^\infty dp \frac{1}{p^2} J^i_j(-p) J^j_i(p) \tag{9}$$

Recall that the light-cone currents $J^i_j(p)$ obey a level k, $SU(N_c)$ affine Lie algebra

$$[J^k_i(p), J^n_l(p')] = \frac{1}{2}kp(\delta^n_i\delta^k_l - \frac{1}{N}\delta^k_i\delta^n_l)\delta(p+p') + \frac{1}{\sqrt{2}}(J^n_i(p+p')\delta^k_l - J^k_l(p+p')\delta^n_i) \tag{10}$$

We can now construct the Hilbert space. The vacuum $|0, R\rangle$ is defined by the annihilation property:

$$\forall p > 0,\ J(p)\,|0, R\rangle = 0 \tag{11}$$

Where R is an "allowed" representations depending on the level[10]. Thus, a typical state in Hilbert space is
$Tr\ J(-p_1)\dots J(-p_n)\ |0, R\rangle$.

Diagonalizing M^2, is in general, a complicated task, hence we will examine in detail some special cases of the theory.

3 Large N_c Adjoint Fermions

The first case we analyze is QCD_2 with fermions in the adjoint representation. This model was investigated in the past [3] [4] and recently it was shown [5] that its massive spectrum is the same as the model of $N_f = N_c$ fundamental quarks.

Since an exact solution of the model is beyond our reach we employ a large $N = N_c$ approximation where some simplifications occur. As stated above the $N_c = N$ is also the corresponding level of the model. First, we write down the general symmetric singlet states:

$$|\psi\rangle = \sum_{n=2}^{\infty}\frac{1}{N^n}\int_0^P \dots \int_0^P dp_1\dots dp_n \delta(P - \sum_{i=1}^n p_i)\phi_n(p_1,\dots,p_n) \times$$
$$\times \frac{1}{n}\sum_\sigma J^{j_1}_{j_2}(-p_{\sigma(1)}) J^{j_2}_{j_3}(-p_{\sigma(2)}) \dots J^{j_n}_{j_1}(-p_{\sigma(n)})\ |0\rangle$$

$$|\psi\rangle' = \sum_{n=2}^{\infty}\frac{1}{N^n}\int_0^P \dots \int_0^P dp_1\dots dp_n \delta(P - \sum_{i=1}^n p_i)\chi_n(p_1,\dots,p_n) \times$$
$$\times \frac{1}{n}\sum_\sigma J^{j_1}_{j_2}(-p_{\sigma(1)}) J^{j_2}_{j_3}(-p_{\sigma(2)}) \dots J^{j_n}_{j_{n+1}}(-p_{\sigma(n)})\ |0\rangle^{j_{n+1}}_{j_1}$$

Here ϕ_n, χ_n are the Schrödinger wave functions, and the summation over σ is, as explained below, over the cyclic group Z_n. The difference between the two states is that in $|\psi\rangle$ the currents act on the identity vacuum, whereas in $|\psi\rangle'$ the currents act on the adjoint vacuum. The latter is a state that transforms in the adjoint representation of $SU(N)$ and obeys $J(p)\ |0\rangle^i_j = 0$ for all $p > 0$. The two states form two distinct equivalence classes, in a sense that one state can not be related to another one by acting with an operator which is built from a finite number of current operators. In principle, other vacuum representations are allowed, but is was proven [5] that in the large N limit it is sufficient to consider only the identity vacuum and the adjoint vacuum. Other representations lead to multi particle states which are suppressed in the large N limit. Note that $M^2\ |0\rangle = 0$, but $M^2\ |0\rangle^i_j = m_0^2\ |0\rangle^i_j \neq 0$. The reason is that the zero mode of the current $J^i_j(0)$ does not annihilate the adjoint vacuum and actually $J^k_l(0)J^l_k(0)\ |0\rangle^i_j = N\ |0\rangle^i_j$[10] .The factor N is the second Casimir of the adjoint representation. Thus $M^2\ |0\rangle^i_j = 2P^+P^-\ |0\rangle^i_j = 2 \times \frac{1}{2N} \times N \times \frac{e^2}{2\pi} \times N\ |0\rangle^i_j = \frac{e^2 N}{2\pi}\ |0\rangle^i_j$. This non-zero value of m_0^2 will lead to mass splitting between eigenvalues of $|\psi\rangle$ and $|\psi\rangle'$.

The summation of σ over all elements of the cyclic group Z_n was introduce to achieve the following symmetry property of the wave functions

$$\begin{aligned} \phi_n(p_1, p_2, \ldots, p_{n-1}, p_n) &= \phi_n(p_n, p_1, p_2, \ldots, p_{n-1}) \\ \chi_n(p_1, p_2, \ldots, p_{n-1}, p_n) &= \chi_n(p_n, p_1, p_2, \ldots, p_{n-1}) \end{aligned} \tag{12}$$

Another property of ϕ_n, χ_n is their boundary condition

$$\begin{aligned} \phi_n(0, p_2, \ldots, p_n) &= 0 \\ \chi_n(0, p_2, \ldots, p_n) &= 0 \end{aligned} \tag{13}$$

which is a consequence of hermiticity of the operators[1]. A general state whether of the type $|\psi\rangle$ or $|\psi\rangle'$ is an eigenstate of P^+ with eigenvalue P

$$P^+\ |\psi\rangle = P\ |\psi\rangle \tag{14}$$

$$P^+\ |\psi\rangle' = P\ |\psi\rangle' \tag{15}$$

due to the following commutation relation

$$[P^+, J^{j_1}_{j_2}(-p)] = pJ^{j_1}_{j_2}(-p). \tag{16}$$

In the more familiar CFT terminology this relation translates into $[L_0, J_n] = -nJ$. The calculation of the commutator $[P^-, J^{j_1}_{j_2}(-p)]$ is more complicated. In the procedure of evaluating the spectrum we find the eigen wave-function by solving an integral equation that generally mixes ϕ_n (or χ_n) with different n's, namely, it mixes the number of currents. Since the general equation is highly non-trivial we will use an approximation in which we take into account only singular terms and terms with the same number of current operators. Those terms have the dominant contribution to the wave equation[3]. A detailed calculation of the commutators which includes all the terms is written in the Appendix. Here we write only the most significant term of the commutator

$$\begin{aligned}
&[P^-, J^{j_1}_{j_2}(-p_1)] = \\
&\quad \frac{e^2}{\pi}\frac{1}{\sqrt{2}}\int_0^\infty dp\Big(\frac{1}{(p+p_1)^2} - \frac{1}{p^2}\Big)\big(J^i_{j_2}(-p-p_1)J^{j_1}_i(p) - J^{j_1}_i(-p-p_1)J^i_{j_2}(p)\big) \\
&\quad +other\ terms
\end{aligned} \tag{17}$$

The above term includes an annihilation operator $J(p)$, which has non-zero contribution when it commutes with another creation operator, say $J(-p')$. A non-negligible commutator occurs only with a 'neighbor' of $J^{j_1}_{j_2}(-p_1)$, namely, a commutator with a current which carries the same group indices. Other commutators are suppressed by factors of $\frac{1}{N}$. A tedious though straightforward calculation yields

$$\begin{aligned}
&\tfrac{1}{\sqrt{2}}\textstyle\int_0^\infty dp\big(\frac{1}{(p+p_1)^2} - \frac{1}{p^2}\big)\big[\big(J^i_{j_2}(-p-p_1)J^{j_1}_i(p) - J^{j_1}_i(-p-p_1)J^i_{j_2}(p), J^{j_2}_{j_3}(-p_2)\big] = \\
&\quad \tfrac{N}{2} \times \textstyle\int_0^{p_2} dp\big(\frac{1}{(p_1+p_2-p)^2} - \frac{1}{(p_2-p)^2}\big)J^{j_1}_i(-p_1-p_2+p)J^i_{j_3}(-p) + other\ terms
\end{aligned} \tag{18}$$

were 'other terms' are either terms that are suppressed in large N, or terms that change the number of currents. We comment on the validity of dropping the latter terms in section 5. The result of the last two commutators is that if one starts with two currents $J(-p_1)$ and $J(-p_2)$ he ends with two other currents which are multiplied by singular denominators. This result leads to the following eigenvalue equation

$$\begin{aligned}
M^2\phi_n(p_1,\ldots,p_n) = -\frac{e^2N}{\pi}\int dy\frac{1}{(p_1-y)^2}\phi_n(y, p_1+p_2-y, p_3,\ldots,p_n) + \\
cyclic\ permutations
\end{aligned} \tag{19}$$

This is a generalization of *'t Hooft's equation.* Obviously, the same equation holds also for χ_n.

Equation (19) can be solved analytically with the boundary conditions (12) and (13). The simplest solution is for ϕ_2:

$$\begin{aligned} \phi_2(x) &= \sin(\pi k x) \quad k \in 2Z+1 \\ M_k^2 &= e^2 N \pi k \end{aligned}$$

The general solution for ϕ_{2n} is quite involved but the eigenvalues are rather simple:

$$M^2_{k_1,k_2,\ldots,k_n} = e^2 N\pi(k_1+k_2+\ldots+k_n) \qquad k_1,k_2,\ldots,k_n \in 2Z+1 \tag{20}$$

The result for the $|\psi\rangle'$ sector is similar, with the small m_0^2 difference:

$$M^2_{k_1,k_2,\ldots,k_n} = e^2 N\pi(k_1+k_2+\ldots+k_n) + m_0^2 \qquad k_1,k_2,\ldots,k_n \in 2Z+1 \tag{21}$$

This expression of the eigenvalues indicates an exponential growth of the number of states, in accordance with previous results [3]. There is, however, a slight difference between our results and those of ref.[3]. We found that the values of the integers must be odd whereas in Kutasov's paper, they are even. The source of this difference is the symmetric boundary conditions that we have used (12).

The spectrum contains two blocks. The identity-block has integer dimension and hence can be interpreted as the boson-block,whereas the adjoint-block has half integer dimension and thus will be referred as the fermion-block. We have seen that the two blocks have similar structure since the two sectors obey the same wave function equation. Furthermore, mixture between the two sectors is avoided due to the fact that the hamiltonian creates and destroys even numbers of "quarks".

The"quark" content of the spectrum is determined by the relation between the currents and the quarks. This relation is given by

$$J^i_j(-p) = \int_{-\infty}^{\infty} dq \psi^i_k(q)\psi^k_j(-p-q) \tag{22}$$

which is the Fourier transform of $J(x) = \psi(x)\psi(x)$, and hence the "boson -block" can be written as

$$\sum_\sigma J^{j_1}_{j_2}(-p_{\sigma(1)})J^{j_2}_{j_3}(-p_{\sigma(2)})\ldots J^{j_n}_{j_1}(-p_{\sigma(n)})\,|0\rangle =$$

$$\sum_{\sigma}\int_{-\infty}^{\infty}dq_1\psi_{i_1}^{j_1}(q_1)\psi_{j_2}^{i_1}(-p_{\sigma(1)}-q_1)\int_{-\infty}^{\infty}dq_2\psi_{i_2}^{j_2}(q_2)\psi_{j_3}^{i_2}(-p_{\sigma(2)}-q_2)$$
$$\dots\int_{-\infty}^{\infty}dq_n\psi_{i_n}^{j_n}(q_n)\psi_{j_1}^{i_n}(-p_{\sigma(n)}-q_n)\ |0\rangle$$

and can be thought of as a mixture of $2n, 2n-2, \dots, 2$ quarks. In a similar way, a state in the fermion-block is a mixture of $2n+1, 2n-1, \dots, 3$ quarks.

4 Large N_f QCD

Massless QCD_2 with N_f flavors of quarks in the fundamental representation is described by the Lagrangian of (4). Setting aside the flavor and baryon number sectors, the left over system is that of a $k = N_f$ level $SU(N_c)$ WZW action with an additional non-local interaction term. We are thus led to analyze the spectrum of the model with level equal to N_f. In practical terms the latter means substituting k by N_f in the expression for P^+ eqn. (9) and in the Affine Lie algebra eqn. (10).

The idea is to invoke a large N_f approximation, namely, to consider models in which $k = N_f \gg N_c$. Models with number of colors and flavors which fall into this regime, are significantly simpler than models that don't obey this inequality. The basic reason for that is the simplification of the algebra eqn. (10). The commutator of P^- with J in the large N_f limit takes the form

$$[P^-, J_j^i(-p)] = \frac{e^2 N_f}{2\pi p} J_j^i(-p) + e^2 JJ\ term \tag{23}$$

which means that

$$M^2\ \frac{J_j^i(-p)}{N_f^{\frac{1}{2}}}\ |0,R\rangle = \frac{e^2 N_f}{\pi}\ \frac{J_j^i(-p)}{N_f^{\frac{1}{2}}}\ |0,R\rangle + e^2 N_f^{\frac{1}{2}} \frac{JJ}{N_f}\ |0,R\rangle \tag{24}$$

Upon neglecting the second term which is suppressed by a factor of $\frac{1}{\sqrt{N_f}}$ we get the following solution to the eigenvalue problem

$$M^2\ J_j^i(-p)\ |0\rangle_{ib}^{ja} = \frac{e^2 N_f}{\pi}\ J_j^i(-p)\ |0\rangle_{ib}^{ja}. \tag{25}$$

Notice that the current operator acts on the adjoint vacuum and not on the identity vacuum. The reason for that is obviously the requirement that states have to be color singlets. $|0\rangle^{ja}_{ib}$ stands for $|0\rangle^{j}_{i}\otimes\ |0\rangle^{a}_{b}$, namely, the tensor product of the color adjoint vacuum and its flavor counterpart. Recall that following eqn. (4) the vacuum state is an outer product of the vacua of the three independent Hilbert-spaces of the color, flavor and baryon number sectors. This "decoupling" of these spaces of states is an artifact of the massless limit of the bosonized picture. It is further discussed in section 5.

The above state is the only one-particle state of the theory. All other massive state are multi particle state which are built from this 'meson'.

The content of massless multi-flavor QCD_2 in the large N_f limit is thus very simple and is in fact closely related to multi-flavor massless QED[11]. In the latter case model the spectrum of single particles contains the following single state

$$M^2\ J(-p)\ |0\rangle = \frac{e^2 N_f}{\pi}\ J(-p)\ |0\rangle \tag{26}$$

The reason of this similarity is very clear. Neglecting the $O(N_f^{-\frac{1}{2}})$ term in eqn.(24) corresponds to dropping the structure constant term in the Affine-Lie algebra. In other words, in this limit we perform an Abelianization of the theory.

However, one may identify a difference in the substructure of the two mesons. The Abelian meson is built up from two quarks since current quanta is made out of two quarks, whereas the non Abelian meson is a more complex object. The difference can be seen by writing the state explicitly in terms of quarks operators. The (traceless) current is written as:

$$J^i_j(-p) = \int_{-\infty}^{\infty} dq\psi^{\dagger i}_c(q)\psi^c_j(-p-q) - \frac{\delta^i_j}{N_c}\int_{-\infty}^{\infty} dq\psi^{\dagger k}_c(q)\psi^c_k(-p-q) \tag{27}$$

The flavored adjoint vacuum is written as:

$$|0\rangle^{ja}_{ib} = \psi^{\dagger j}_b(0)\psi^a_i(0)\ |0\rangle \tag{28}$$

It is a state of two quarks with zero (light-cone) momentum acting on the identity vacuum. Thus the massive meson is

$$\begin{aligned}
J^i_j(-p)\ |0\rangle^{ja}_{ib} &= (\textstyle\int_{-\infty}^{\infty} dq[\psi^{\dagger i}_c(q)\psi^c_j(-p-q) - \frac{\delta^i_j}{N}\psi^{\dagger k}_c(q)\psi^c_k(-p-q)]) \times \psi^{\dagger j}_b(0)\psi^a_i(0)\ |0\rangle \\
&= \textstyle\int_0^p dq\psi^{\dagger i}_c(-q)\psi^c_j(-p+q)\psi^{\dagger j}_b(0)\psi^a_i(0)\ |0\rangle - \frac{1}{N}\int_0^p dq\psi^{\dagger i}_c(-q)\psi^c_i(-p+q)\psi^{\dagger j}_b(0)\psi^a_j(0)\ |0\rangle \\
&\quad + \tfrac{N^2-1}{N}\psi^{\dagger i}_b(-p)\psi^a_i(0)\ |0\rangle,
\end{aligned}$$

namely, a color singlet which is a mixture of four quarks and two quarks.

The basic feature used in this section has been the fact that $k \gg N_c$. This holds, in fact, not only for large number of fundamental representations but obviously also for any large level gauged WZW $SU(N)$ model.

5 summary

In this work we have calculated the mesonic spectra of several QCD_2 models by employing bosonization, light-front quantization and expansion in large number of colors or flavors.

The main results of the work are

a) An approximated spectrum of the "adjoint fermions" model.

b) The exact leading order in $\frac{1}{N_f}$ spectra of multi-flavor fundamental representation.

As for (a), our approximation is similar to the one used in the fermionic picture [3], namely, dropping the terms that mix wave-functions with different number of current creation operators. The physical meaning of such approximation is suppressing pair creation and pair annihilation. This approximation is not quite justified, but it gives us hint about the structure of the spectrum. Obviously, the most urgent task in this direction is to look for methods to solve the full wave equation.

The second result states that the spectrum of the large N_f fundamental fermions is built out of a single massive particle with $M^2 \sim e^2 N_f$. For comparison see ref.[12] where an analysis of similar cases is discussed in the Hamiltonian formalism. In fact, it is a universal behavior of any gauged $SU(N)$ WZW model with $k \gg N$.

The interesting question that naturally arise is what happens in the intermediate region, where $k \sim N$. This region can be realized for instance, in the case of multi-flavor QCD_2 with $N_f \sim N_c$. It is reasonable to expect that the mesonic spectrum in this region will lie between that of a single massive state and that with an exponential density growth. In the absence of exact analytical methods one may have to invoke numerical diagonalization of the bosonized M^2 operator in a similar way to that of the fermionic picture[4].

For the analysis of the baryonic spectrum in the bosonization approach it is essential to consider the case of massive quarks[2]. This maybe also the case for the mesonic spectrum since the mass term couples the colored, flavored and baryon number sectors. In fact, even for the massless case a better strategy is to solve the massive case and then go to the massless limit. The extraction of the mesonic spectrum in the massive case is much more evolved since the mass term can not be written in a simple fashion in terms of the currents. However, one can systematically expand the mass term in powers of $\frac{m_q}{e}$ where m_q is the quark mass. Solving the wave equations in the presence of these massive perturbation deserves a further future study.

Recently, the theories of YM_2 and QCD_2 were analyzed as "perturbed" topological coset models[13]. The spectrum in that approach which was deduced using a BRST procedure includes a peculiar massive state which was not detected in other approaches including the present. This discrepancy maybe related to the different approximation used in that approach. Clearing up this point as well as the implementation of that method to the case of adjoint fermions and other possible generalization deserves a further investigation. Other methods inherited from string theory and conformal field theory can be also be applied to the analysis of the models discussed[14].

QCD_2 models can be generalized to a much richer class of theories which are also gauge invariant and renormalizable[15]. The framework of this generalization is the formulation of the YM_2 functional integral in terms of an action which is linear in F and includes an additional auxiliary pseudoscalalr field. Ordinary QCD_2 has a quadratic term in the latter field while taking any arbitrary function f of this auxiliary field spans the space of generalized models. The analysis presented in the present paper can be applied also to those models. The momentum operator P^- rather then being quadratic in $\frac{1}{\partial_-}J$ it will take the general form of $f(\frac{1}{\partial_-}J)$. It will be interesting to compare the outcome of the methods used in the current work to those derived in ref.[15].

ACKNOWLEDGEMENTS

We thank for M. Engelhardt, Y. Frishman and D. Kutasov for stimulating conversations. We would specially like to thank S. Yankielowicz for numerous discussions in various part of the work.

6 Appendix

A detailed calculation of currents commutators. We would like to calculate $[P^-, J^{j_1}_{j_2}(-p_1)]$, where $P^- = \frac{e^2}{\pi}\int_0^\infty \frac{dp}{p^2} J^i_j(-p)J^j_i(p)$. Therefore we would calculate $[\int_0^\infty \frac{dp}{p^2} J^i_j(-p)J^j_i(p), J^{j_1}_{j_2}(-p_1)]$ by using the affine Lie algebra(10):

$$
\begin{aligned}
&[\int_0^\infty \frac{dp}{p^2} J^i_j(-p)J^j_i(p), J^{j_1}_{j_2}(-p_1)] = \\
&\quad \int_0^\infty \frac{dp}{p^2} J^i_j(-p)[J^j_i(p), J^{j_1}_{j_2}(-p_1)] + \int_0^\infty \frac{dp}{p^2}[J^i_j(-p), J^{j_1}_{j_2}(-p_1)]J^j_i(p) = \\
&\quad \int_0^\infty \frac{dp}{p^2} J^i_j(-p)\{p\frac{k}{2}(\delta^j_{j_2}\delta^{j_1}_i - \frac{1}{N}\delta^j_i\delta^{j_1}_{j_2})\delta(p-p_1) \\
&\quad + \frac{1}{\sqrt{2}}(J^{j_1}_i(p-p_1)\delta^j_{j_2} - J^j_{j_2}(p-p_1)\delta^{j_1}_i)\} \\
&\quad + \frac{1}{\sqrt{2}}\int_0^\infty \frac{dp}{p^2}(J^{j_1}_j(-p-p_1)\delta^i_{j_2} - J^i_{j_2}(-p-p_1)\delta^{j_1}_j)J^j_i(p) = \\
&\quad \frac{k}{2p_1}J^{j_1}_{j_2}(-p_1) \\
&\quad + \frac{1}{\sqrt{2}}\int_{-p_1}^\infty \frac{dp}{(p+p_1)^2} J^i_{j_2}(-p-p_1)J^{j_1}_i(p) - \frac{1}{\sqrt{2}}\int_{-p_1}^\infty \frac{dp}{(p+p_1)^2} J^{j_1}_j(-p-p_1)J^j_{j_2}(p) \\
&\quad + \frac{1}{\sqrt{2}}\int_0^\infty \frac{dp}{p^2} J^{j_1}_j(-p-p_1)J^j_{j_2}(p) - \frac{1}{\sqrt{2}}\int_0^\infty \frac{dp}{p^2} J^i_{j_2}(-p-p_1)J^{j_1}_i(p) =
\end{aligned}
$$

The above expression includes annihilation currents as well creation ones. Separating them one from the other we obtain:

$$
\begin{aligned}
&= \frac{k}{2p_1}J^{j_1}_{j_2}(-p_1) \\
&\quad + \frac{1}{\sqrt{2}}\int_0^\infty dp(\frac{1}{(p+p_1)^2} - \frac{1}{p^2})J^i_{j_2}(-p-p_1)J^{j_1}_i(p) \\
&\quad - \frac{1}{\sqrt{2}}\int_0^\infty dp(\frac{1}{(p+p_1)^2} - \frac{1}{p^2})J^{j_1}_j(-p-p_1)J^j_{j_2}(p) \\
&\quad + \frac{1}{\sqrt{2}}\int_{-p_1}^0 \frac{dp}{(p+p_1)^2} J^i_{j_2}(-p-p_1)J^{j_1}_i(p)
\end{aligned}
$$

$$-\frac{1}{\sqrt{2}}\int_{-p_1}^{0}\frac{dp}{(p+p_1)^2}J_j^{j_1}(-p-p_1)J_{j_2}^{j}(p) =$$

$$\frac{k}{2p_1}J_{j_2}^{j_1}(-p_1)$$
$$+\frac{1}{\sqrt{2}}NJ_{j_2}^{j_1}(-p_1)\int_{-p_1}^{0}\frac{dp}{(p+p_1)^2}$$
$$+\frac{1}{\sqrt{2}}\int_{-p_1}^{0}dp(\frac{1}{(p+p_1)^2}-\frac{1}{p^2})J_j^{j_1}(p)J_{j_2}^{j}(-p-p_1)$$
$$+\frac{1}{\sqrt{2}}\int_{0}^{\infty}dp(\frac{1}{(p+p_1)^2}-\frac{1}{p^2})\{J_{j_2}^{i}(-p-p_1)J_i^{j_1}(p)-J_i^{j_1}(-p-p_1)J_{j_2}^{i}(p)\}$$

In the above expression there are four terms:

The first one $\frac{k}{2p}J(-p)$, does not change the number of currents (it has only one current) and it is proportional to the level k. It will play a central role in the large k limit.

The second term is similar to the first one, but it is proportional to N and it diverges. The divergent part will be compensated by another divergent term (which arises from the fourth term).

The third term include two creation currents. Thus the interaction P^-, with the help of the algebra created a current. In our discussion we will ignore this part, for the sake of simplicity.

The last term includes annihilation currents, and therefore we should evaluate its commutator with other creation current.

$$[\frac{1}{\sqrt{2}}\int_{0}^{\infty}dp(\frac{1}{(p+p_1)^2}-\frac{1}{p^2})\{J_{j_2}^{i}(-p-p_1)J_i^{j_1}(p)-J_i^{j_1}(-p-p_1)J_{j_2}^{i}(p)\},J_{j_3}^{j_2}(-p_2)] =$$

$$\frac{1}{\sqrt{2}}\int_{0}^{\infty}dp(\frac{1}{(p+p_1)^2}-\frac{1}{p^2})\{J_{j_2}^{i}(-p-p_1)[J_i^{j_1}(p),J_{j_3}^{j_2}(-p_2)]$$
$$+[J_{j_2}^{i}(-p-p_1),J_{j_3}^{j_2}(-p_2)]J_i^{j_1}(p)-J_i^{j_1}(-p-p_1)[J_{j_2}^{i}(p),J_{j_3}^{j_2}(-p_2)]$$
$$-[J_i^{j_1}(-p-p_1),J_{j_3}^{j_2}(-p_2)]J_{j_2}^{i}(p)\} =$$

$$\frac{1}{\sqrt{2}}\int_{0}^{\infty}dp(\frac{1}{(p+p_1)^2}-\frac{1}{p^2})\{J_{j_2}^{i}(-p-p_1)p\frac{k}{2}(\delta_i^{j_2}\delta_{j_3}^{j_1}-\frac{1}{N}\delta_i^{j_1}\delta_{j_3}^{j_2})\delta(p-p_2)$$
$$\frac{1}{\sqrt{2}}J_{j_2}^{i}(-p-p_1)J_i^{j_2}(p-p_2)\delta_{j_3}^{j_1}-\frac{1}{\sqrt{2}}J_{j_2}^{i}(-p-p_1)J_{j_3}^{j_1}(p-p_2)\delta_i^{j_2}$$

$$-\frac{1}{\sqrt{2}}NJ^i_{j_3}(-p-p_1-p_2)J^{j_1}_i(p)$$
$$-J^{j_1}_i(-p-p_1)p\frac{k}{2}(\delta^i_{j_3}\delta^{j_2}_{j_2}-\frac{1}{N}\delta^i_{j_2}\delta^{j_2}_{j_3})\delta(p-p_2)$$
$$+\frac{1}{\sqrt{2}}NJ^{j_1}_i(-p-p_1)J^i_{j_3}(p-p_2)$$
$$-\frac{1}{\sqrt{2}}J^{j_2}_i(-p-p_1-p_2)J^i_{j_2}(p)\delta^{j_1}_{j_3}+\frac{1}{\sqrt{2}}J^{j_1}_{j_3}(-p-p_1-p_2)J^i_{j_2}(p)\delta^{j_2}_i\}=$$

This leads to the following expression:

$$\begin{aligned}
&=-\frac{1}{\sqrt{2}}p_2N\frac{k}{2}(\frac{1}{(p_1+p_2)^2}-\frac{1}{{p_2}^2})J^{j_1}_{j_3}(-p_2-p_1)\\
&+\frac{1}{2}\int_0^\infty dp(\frac{1}{(p+p_1)^2}-\frac{1}{p^2})\times\\
&\times\{J^i_j(-p-p_1)J^j_i(p-p_2)-J^j_i(-p-p_1-p_2)J^i_j(p)\}\delta^{j_1}_{j_3}\\
&+\frac{N}{2}\int_0^\infty dp(\frac{1}{(p+p_1)^2}-\frac{1}{p^2})\times\\
&\times\{J^{j_1}_i(-p-p_1)J^i_{j_3}(p-p_2)-J^i_{j_3}(-p-p_1-p_2)J^{j_1}_i(p)\}
\end{aligned}$$

A few remarks about the last expression:

The first term includes only creation currents, therefore there is no need to evaluate its commutators with other currents.

The second term contains creation and annihilation currents, but it is suppressed in the large N limit.

The last term is an important term. It is not suppressed at large N, and it contains the following expression in it:

$$\frac{N}{2}\int_0^{p_2} dp(\frac{1}{(p+p_1)^2}-\frac{1}{p^2})J^{j_1}_i(-p-p_1)J^i_{j_3}(p-p_2) \qquad (29)$$

Which may rewritten as:

$$\frac{N}{2}\int_0^{p_2} dp(\frac{1}{(p_1+p_2-p)^2}-\frac{1}{(p_2-p)^2})J^{j_1}_i(p-p_1-p_2)J^i_{j_3}(-p) \qquad (30)$$

This expression leads to the generalized ’t Hooft equation.

References

[1] G. 't Hooft, *"A Two-Dimensional Model For Mesons"*, *Nucl. Phys.* **B75** (1974) 461.

[2] G. D. Date, Y. Frishman, and J. Sonnenschein, *"The Spectrum of Multiflavor QCD in Two-Dimensions"*, *Nucl. Phys.* **B283** (1987) 365.

[3] D. Kutasov, *"Two Dimensional QCD coupled to Adjoint matter and String Theory"*, *Nucl. Phys.* **B414** (1994) 33 [hep-th/9306013].

[4] G. Bhanot,K. Demeterfi and I.R. Klebanov, *"1+1 Dimensional Large N QCD coupled to Adjoint Fermions"*, *Phys. Rev.* **D48** (1993) 4980 [hep-th/9307111].

[5] D. Kutasov and A. Schwimmer, *"Universality in Two Dimensional Gauge Theory"* [hep-th/9501024].

[6] Y. Frishman and J. Sonnenschein, *"Bosonization and QCD in Two Dimensions"*, *Physics Reports* **223 # 6** (1993) 309.

[7] E. Witten, *"Non-Abelian Bosonization in Two Dimensions"*, *Comm. Math. Phys.* **92** (1984) 455.

[8] Y. Frishman and J. Sonnenschein, *"Bosonization of colored flavored Fermions and QCD in Two-Dimensions"*, *Nucl. Phys.* **B294** (1987) 801.

[9] O. Aharony, O. Ganor, J. Sonnenschein and S. Yankielowicz, *"On the Twisted G/H Topological Models"*, *Nucl. Phys.* **B399** (1993) 560.

[10] D. Gepner and E. Witten, *"String Theory on Group manifolds"*, *Nucl. Phys.* **B278** (1986) 493.

[11] I.G. Halliday, E. Rabinovici, A. Schwimmer and M. Chanowitz, *"Quantization Of Anomalous Two-Dimensional Models"*, *Nucl. Phys.* **B268** (1986) 413.

[12] M. Engelhardt, *"QCD In (1+1)-Dimensions in The Limit of A Large Number of Colors and Flavors"*, *Nucl. Phys.* **B440** (1995) 543 [hep-ph/9412314].

[13] Y. Frishman, A. Hanany and J. Sonnenschein, *"Subtleties in QCD Theory in Two-Dimensions"*, *Nucl. Phys.* **B429** (1994) 75.

[14] For a review see E. Abdalla and M.C.B. Abdalla, *"Updating QCD_2"* [hep-th/9503002] and references therein.

[15] M. Douglas, K. Li and M. Staudacher, *"Generalized Two-Dimensional QCD"*, *Nucl. Phys.* **B420** (1994) 118.