

A model universe with variable dimension: Expansion as decrumpling

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Abstract

We propose a model universe, in which the dimension of the space is a continuous variable, which can take any real positive number. The dynamics leads to a model in which the universe has no singularity. The difference between our model and the standard Friedman-Robertson-Walker models become effective for times much before the presently accepted age of the universe.

O Introduction

The standard model of cosmology is based on the following assumptions:

- 1 . Space-time is a differential manifold.
- 2 . Dimension of space-time is a fixed constant.
- 3 . Dimension of space-time is $3+1$.
- 4 . Space-time is homogenous and isotropic.
- 5 . Expansion of the universe is adiabatic.
- 6 . Dynamics is based on the Einstein field equations.

Up to now there has been suggestions in the literature to modify somehow the assumptions 3-6. Modification of the number of space-time to $D+1$, e.g., has been considered to account for inflation [1]. Inhomogeneous cosmological models has been considered to study the growth of inhomogeneities in the early universe or hoping to remove the big bang singularity [2]. In homogeneous cosmology and in quantum cosmology the assumption of isotropy may be abandoned [3,4]. But the most drastic change is the inflationary paradigm, which means assuming nonadiabatic expansion of the universe [5,6]. This paradigm claims to remove almost all the deficiencies of the standard model, except the singularity at the big bang. Lastly, some authors suggest to modify the underlying dynamics of the general relativity with very different motivations. Some like to remove the big bang singularity through using quadratic lagrangians [7]; Brans-Dicke theory is another common modification. Steady-State and Quasi-Steady-State theory through the assumption of continuous creation of matter brings in another drastic change in the common beliefs of the physicists [8]. It should be noted that none of these modifications tackle the problem of quantum gravity.

We want to bring in a completely new picture for the space-time, and abandon the first two assumptions of the standard model. Now, the observational evidence for differentiability of space-time is actually very poor. In fact the set of space-time events is not even continuous, and there are evidences that the matter

distribution in the universe, up to the present observed limits of $100 h^{-1}$ Mpc, is fractal or multifractal [9,10]. Such a fractal structure has been also observed in temperature fluctuations of the cosmic background radiation observed by COBE [11]. Fractal structure of space-time has also been used to interpret the quantum mechanics [12,13]. However, the dimension of space-time is always assumed to be a fix number, usually $3+1$. Authors using dynamical triangulation and Regge calculus in general relativity or quantum gravity [14] don't change this assumption either.

The picture we want to bring in cosmology is a generalization of polymeric or tethered surfaces, which are in turn simple generalizations of linear polymers to two-dimensionally connected networks [15,16]. Visualizing the universe as a piece of paper, then the crumpled paper will stand for the state of the early universe [17]. It should however be noted that the final formulation of our model in this paper could as well be interpreted as a generalization of fluid membranes [18]. In this case we can visualize the universe as a clay; it can be like a three dimensional ball, or like a two dimensional disc, or even like a one dimensional string. In each case the effective dimension of the universe is a continuous number between the dimension of the embedding space and some D which could be 3. To study the crumpling in the statistical physics one needs to define an embedding space, which does not exist in our case. Therefore, we assume an embedding space of arbitrary high dimension \mathcal{D} , which is allowed to be infinite. This is necessary, because the crumpling is highly dependent on the dimension of embedding space.

To simplify our picture, we introduce a cosmological model with just the space part having a continuously varying dimension. We call this space, with varying dimension, a D-space. Therefore we assume a homogeneous and isotropic universe, make a space-time decomposition, leaving the time coordinate unchanged. Now, we imagine our universe to be a D-space embedded in a space with arbitrary large, maybe infinite, dimension \mathcal{D} . This cosmic D-space consists of small cells with characteristic size of about the Planck length, denoted by δ . The cells, playing the role of the monomers in polymerized surfaces, are allowed to have as

many dimensions as the embedding space. Therefore, the cosmic space can have a dimension as large as the embedding space, like the polymers in crumpled phase. The radius of gyration of the crumpled cosmic space should play the role of the Friedman parameter of a FRW cosmology in $D + 1$ dimensional space-time, where D is the fractal dimension of the crumpled space in the embedding space and could be as high as \mathcal{D} . The expansion of space is understood now as decrumpling of cosmic space. In the course of decrumpling the fractal dimension of space changes from \mathcal{D} to D , where D is about three. To formulate the problem we write down the Hilbert-Einstein action for a FRW metric in D dimension. Now the Friedman parameter a , and the dimension D are both dynamical variables. The dynamical property of D could lead to difficulties if the model were not homogenous [19], and we had to consider a Lagrangian density in the action. The above mentioned cell structure of the universe brings in the next simplification which is a relation between a and D . Our model system becomes again a system with one degree of freedom, but the field equations are more complex. It turns out that these generalized field equation admits the FRW model as a limit.

I The action

We begin with a $D + 1$ dimensional space-time $M \times R$, where M is assumed to be homogeneous and isotropic.

The space-time metric is written as

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j. \quad (\text{I.1})$$

For simplicity, throughout this paper we take M to be flat. The Ricci scalar is then calculated to be

$$R = 2D \frac{d}{dt} \left(\frac{\dot{a}}{a} \right) + D(D + 1) \left(\frac{\dot{a}}{a} \right)^2. \quad (\text{I.2})$$

It is then easily seen that

$$R\sqrt{|g|} = -D(D - 1) \left(\frac{\dot{a}}{a} \right)^2 + \text{total time derivative}. \quad (\text{I.3})$$

Note that we are still considering D to be constant. The homogeneity of the metric allows one to integrate the gravitational Lagrangian density. Therefore, the gravitational Lagrangian becomes

$$L_G^{(0)} = -\frac{1}{2\kappa}D(D-1)\left(\frac{\dot{a}}{a}\right)^2 a^D, \quad (\text{I.4})$$

where a^D is the volume of M .

Coupling this Lagrangian to the source

$$L_M^{(0)} = \frac{1}{2}\theta^{\mu\nu}g_{\mu\nu}, \quad (\text{I.5})$$

gives the complete Lagrangian of the gravitational field. In (I.5),

$$T^{\mu\nu} := \frac{1}{\sqrt{|g|}}\theta^{\mu\nu}, \quad (\text{I.6})$$

and

$$T^{00} = \rho,$$

$$T^{0i} = 0, \quad (\text{I.7})$$

$$T^{ij} = g^{ij}p = a^{-2}p\delta_{ij},$$

where ρ and p are the energy density and pressure, respectively. (I.7) then leads to

$$\theta^{00} = \tilde{\rho} := \rho a^D, \quad (\text{I.8})$$

and

$$\theta^{ij} = \tilde{p}\delta^{ij} := pa^{D-2}\delta^{ij}. \quad (\text{I.9})$$

Note that, as in the 3 + 1 dimensional case, to get the correct field equations, in varying the action with respect to a , we have to take $\tilde{\rho}$ and \tilde{p} as constants [20]. The complete Lagrangian is then

$$L^{(0)} = -\frac{1}{2\kappa}D(D-1)\left(\frac{\dot{a}}{a}\right)^2 a^D + \left(-\frac{\tilde{\rho}}{2} + \frac{\tilde{p}Da^2}{2}\right). \quad (\text{I.10})$$

This Lagrangian suffers from the fact that its dimension is not constant. Note that, we have assumed that the dimension of κ is constant, that is $(\text{Length})^{D_0-1}$. To make a Lagrangian with a constant dimension,

we multiply the above Lagrangian by $a_0^{D_0-D}$, where a_0 is a quantity with the dimension of length, in fact the scale of the universe when the dimension is D_0 . Clearly, this factor brings no change in General Relativity, where $D = D_0 = 3$. Now, for our general case of variability of the space dimension, the constant part of this factor, $a_0^{D_0}$, can be omitted. Therefore, we arrive at the Lagrangian

$$\frac{L}{a_0^{D_0}} = -\frac{D(D-1)}{2\kappa} \left(\frac{\dot{a}}{a}\right)^2 \left(\frac{a}{a_0}\right)^D + \left(-\frac{\hat{\rho}}{2} + \frac{\hat{p}Da^2}{2}\right) =: \mathcal{L}, \quad (\text{I.11})$$

where,

$$\begin{aligned} \hat{\rho} &:= \rho \left(\frac{a}{a_0}\right)^D, \\ \hat{p} &:= pa^{-2} \left(\frac{a}{a_0}\right)^D, \end{aligned} \quad (\text{I.12})$$

We take this Lagrangian as our starting point. To check this Lagrangian, we derive the corresponding field equation for $D = D_0 = \text{const.}$; one expects to obtain the familiar Friedman equations [21], which are

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{2\kappa\rho}{D_0(D_0-1)}, \\ \frac{\ddot{a}}{a} &= -\frac{\kappa}{D_0(D_0-1)} [(D_0-1)\rho - D_0p]. \end{aligned} \quad (\text{I.13})$$

Now, the Lagrangian (I.11) leads to the following field equation

$$-(D_0-1) \left[\frac{\ddot{a}}{a} + \frac{D_0-2}{2} \left(\frac{\dot{a}}{a}\right)^2 \right] = \kappa p, \quad (\text{I.14})$$

which is just a combination of the two equations (I.13). The other equation comes from the continuity relation

$$\frac{d}{dt}(\rho a^{D_0}) + p \frac{d}{dt}(a^{D_0}) = 0. \quad (\text{I.15})$$

It is then seen that the Lagrangian (I.11), together with the continuity relation (I.15) completes the usual picture of Friedman-Robertson-Walker cosmology.

To implement the idea of variability of space dimension, we assume a cellular structure for space: the universe consists of N D_0 -dimensional cells. In a fictive embedding space, as far as N is finite, there is no D -dimensional arrangement of the cells with non-zero volume. So we assume the cells to have an arbitrary

number of extra dimensions, each of which has a length scale δ . Then the following relation holds between the D - and D_0 -dimensional volume of the cells.

$$\text{vol}_D(\text{cell}) = \text{vol}_{D_0}(\text{cell})\delta^{D-D_0}. \quad (\text{I.16})$$

Our D -dimensional universe will have an effective length scale, corresponding to the radius of gyration of a crumpled surface [22], equal to a . Hence we may write

$$\begin{aligned} a^D &= N\text{vol}_D(\text{cell}) \\ &= N\text{vol}_{D_0}(\text{cell})\delta^{D-D_0} \\ &= a_0^{D_0}\delta^{D-D_0}, \end{aligned} \quad (\text{I.17})$$

or

$$\left(\frac{a}{\delta}\right)^D = e^C, \quad (\text{I.18})$$

where C is a constant. This is an important constraint, which relates the length scale of the universe to its dimension: as D grows up to infinity, a decreases down to δ (but not less than it), and as a grows, D decreases. In other words, the expansion of the universe is through reduction of its dimension.

Using this constraint, the equation of motion becomes

$$(D-1)\left\{\frac{\ddot{a}}{a} + \left[\frac{D^2}{2D_0} - 1 - \frac{D(2D-1)}{2C(D-1)}\right]\left(\frac{\dot{a}}{a}\right)^2\right\} + \kappa p\left(1 - \frac{D}{2C}\right) = 0. \quad (\text{I.19})$$

Using (I.18), one gets

$$a\frac{dD}{da} = -\frac{D^2}{C}. \quad (\text{I.20})$$

Once again, we can check whether the equations (I.19) and (I.20) are consistent with the Einstein field equation (I.14). From (I.20), it is seen that in the limit $C \rightarrow \infty$, $D = D_0 = \text{const}$. In this case, the equation (I.19) becomes the same as (I.14). Hence, the standard Friedman cosmology is the $C \rightarrow \infty$ limit of our model. Moreover, assuming that $C \gg 1$, it is seen that in the vicinity of D_0 , our model universe behaves

like a Friedman model. Had we generalized the Lagrangian (I.10) by multiplying it by the other possibility (δ^{D_0-D}) , we would have gotten another field equation, which would not lead to the familiar Friedman equation. Therefore, the generalization (I.11) is unique.

The field equation (I.19) is not sufficient to obtain a . A continuity equation, and an equation of state, are also needed. A dimensional reasoning leads to the following generalization of the continuity equation (I.15):

$$\frac{d}{dt}(\rho a^D a_0^{D_0-D}) + p \frac{d}{dt}(a^D a_0^{D_0-D}) = 0, \quad (\text{I.21})$$

or

$$\frac{d}{dt} \left[\rho \left(\frac{a}{a_0} \right)^D \right] + p \frac{d}{dt} \left(\frac{a}{a_0} \right)^D = 0. \quad (\text{I.22})$$

Adding an equation of state to (I.18), (I.19), and (I.22), the formulation of the dynamics of our model universe is complete.

II Qualitative behavior of the system

Our system is defined through (I.18), (I.19), (I.22), and an equation of state. This system is more difficult to be solved analytically. To understand its qualitative behavior, we find a first integral of motion and study its properties.

From the Lagrangian (I.11), one can define a Hamiltonian:

$$\begin{aligned} \mathcal{H} &:= \dot{a} \frac{\partial \mathcal{L}}{\partial \dot{a}} - \mathcal{L}, \\ &= \dot{u} \frac{\partial \mathcal{L}}{\partial \dot{u}} - \mathcal{L}, \\ &= -\frac{D(D-1)}{2\kappa} e^C \left(\frac{\delta}{a_0} \right)^D \dot{u}^2 - \left(\frac{\hat{p}}{2} D \delta^2 e^{2u} - \frac{\hat{\rho}}{2} \right). \end{aligned} \quad (\text{II.1})$$

where u is defined through

$$u := \ln \frac{a}{\delta} = \frac{C}{D}. \quad (\text{II.2})$$

Using

$$\frac{d\mathcal{H}}{dt} = -\frac{\partial\mathcal{L}}{\partial t}, \quad (\text{II.3})$$

we have

$$\frac{d}{dt} \left[-\frac{D(D-1)}{2\kappa} e^C \left(\frac{\delta}{a_0}\right)^D \dot{u}^2 - \frac{\hat{p}}{2} D \delta^2 e^{2u} \right] = -\frac{D}{2} \delta^2 e^{2u} \frac{d\hat{p}}{dt}, \quad (\text{II.4})$$

or

$$\frac{d}{du} \left[\frac{D(D-1)}{2\kappa} \left(\frac{\delta}{a_0}\right)^D \dot{u}^2 \right] + p \left(\frac{\delta}{a_0}\right)^D \left(D - \frac{D^2}{2C}\right) = 0. \quad (\text{II.5})$$

Note that $\hat{\rho}$ and \hat{p} are considered as sources. The equation (II.5) is equivalent to

$$\frac{C^2}{2\kappa} \frac{d}{dD} \left(\frac{D-1}{D^3} e^{-\frac{CD}{B_0}} \dot{D}^2 \right) + p e^{-\frac{CD}{B_0}} \left(\frac{1}{2} - \frac{C}{D} \right) = 0. \quad (\text{II.6})$$

This means that the function

$$\mathcal{U}(D) := \int^D dD' p(D') e^{-\frac{CD'}{B_0}} \left(\frac{1}{2} - \frac{C}{D'} \right), \quad (\text{II.7})$$

serves as a potential for the kinetic energy

$$\mathcal{T} := \frac{C^2}{2\kappa} \left(\frac{D-1}{D^3} e^{-\frac{CD}{B_0}} \dot{D}^2 \right). \quad (\text{II.8})$$

Now, we will show that the system described by these, has two turning points. To do so, we must consider the behavior of the pressure. For large values of D , it is natural to use a radiation equation of state:

$$p = \frac{1}{D} \rho. \quad (\text{II.9})$$

From this, and the equation of continuity (I.22), one can calculate the pressure as

$$p = p_0 e^{C\left(\frac{D}{B_0}-1\right)} \left(\frac{D}{D_0}\right)^{\frac{C}{B_0}-1}. \quad (\text{II.10})$$

It is then easily seen that at large D 's,

$$\mathcal{U}(D) \sim \frac{D_0}{2C} e^{-C} \left(\frac{D}{D_0}\right)^{\frac{C}{B_0}}. \quad (\text{II.11})$$

The point $D = 2C$ is the point where the potential attains its minimum. For D near zero, assuming that the pressure remains finite (nonzero), it is seen that

$$\mathcal{U}(D) \sim -C \ln D, \quad (\text{II.12})$$

that is, \mathcal{U} grows unboundedly to infinity at $D \rightarrow \infty$, as well as $D \rightarrow 0$. This means that there are two turning points, one above $D = 2C$, the other below it.

However, the kinetic term (II.8) changes sign at $D = 1$. The above discussion is valid, provided $\mathcal{T} \geq 0$. So, to have two turning points, the constant $\mathcal{E} := \mathcal{U} + \mathcal{T}$ must be sufficiently low to make the lower turning point greater than 1. By the way, dimension of the universe less than 1 means that it is a disconnected set of cells, which we are not going to consider it.

III Behavior of the model near the lower turning point of the dimension

Taking D_0 as the lower turning point of the potential, we have, from (II.6),

$$\frac{C^2}{2\kappa} \frac{D-1}{D^3} e^{-\frac{CD}{D_0}} \dot{D}^2 + \int_{D_0}^D dD' p(D') e^{-\frac{CD'}{D_0}} \left(\frac{1}{2} - \frac{C}{D'} \right) = 0. \quad (\text{III.1})$$

Now, defining

$$\epsilon := D - D_0, \quad (\text{III.2})$$

and using (II.10), we have, to lowest orders in ϵ and $\dot{\epsilon}$,

$$\dot{\epsilon}^2 = 4A\epsilon, \quad (\text{III.3})$$

where

$$\bar{A} := \frac{\kappa D_0^2 p_0}{2C(D_0 - 1)}. \quad (\text{III.4})$$

This approximation holds wherever

$$\frac{C\epsilon}{D_0} \ll 1. \quad (\text{III.5})$$

It is now easy to solve this equation to obtain

$$\epsilon = A(\tau - t)^2. \quad (\text{III.6})$$

We use this expression for $t \leq \tau$, where τ is the lower-turning-point time. Here, all the times are measured from the big bang point of the standard cosmology. Now, we want to show that, even down to Planck's time the variations in D are negligible, and one can essentially take $D = D_0$. We have

$$\begin{aligned} \frac{\epsilon(t_i)}{\epsilon(t_f)} &=: \frac{\epsilon_i}{\epsilon_f} = \left(\frac{\tau - t_i}{\tau - t_f} \right)^2 \\ &= (1 + A^{1/2} T \epsilon_f^{-1/2})^2 \\ &=: (1 + \alpha)^2, \end{aligned} \quad (\text{III.7})$$

where

$$T := t_f - t_i, \quad (\text{III.8})$$

t_f is the present time, and t_i is some initial time, when we want to calculate the value of ϵ . To estimate the second term of the parentheses, we use equations of standard cosmology, during the radiation era:

$$\begin{aligned} p &= \frac{1}{3}\rho \\ &= \frac{1}{3}\rho_P \left(\frac{t_P}{t_f} \right)^2. \end{aligned} \quad (\text{III.9})$$

So we have

$$\begin{aligned} \alpha &= \left(\frac{9\kappa p_0}{4C} \right)^{1/2} T \epsilon_f^{-1/2}, \\ &= \left(\frac{3\kappa\rho_P}{4C} \right)^{1/2} \frac{t_P}{t_f} T \epsilon_f^{-1/2}. \end{aligned} \quad (\text{III.10})$$

Now using $\delta \sim t_P$, $a_0 \sim t_f$, $t_f \sim 10^{17}\text{s}$ (the age of the universe), $\rho_P \sim 10^{93}\text{kg m}^{-3}$ [21], $m_P^4 \sim 10^{97}\text{kg m}^{-3}$,

and $D_0 = 3$, we obtain

$$\begin{aligned} C &= \frac{1}{D_0} \ln \frac{t_f}{t_P} \\ &\sim 600, \end{aligned} \quad (\text{III.11})$$

and

$$\alpha \sim 10^{-3} \frac{T}{t_f} \epsilon_f^{-1/2}. \quad (\text{III.12})$$

If ϵ_f is small (less than 10^{-6}), α will be greater than one. So

$$\begin{aligned}\epsilon_i &\sim \epsilon_f \alpha^2 \\ &\sim 10^{-6} \left(\frac{T}{t_f}\right)^2.\end{aligned}\tag{III.13}$$

Comparing this with the criterion (III .5), we see that our approximation holds, and ϵ_i is small, for

$$T \ll 100t,\tag{III.14}$$

that is, the dimension of the cosmos has been constant from at least 10 times the "standard age of universe" before the "Big Bang".

IV Conclusions

The model we are proposing is qualitatively different from hitherto considered theories with extra dimensions, such as Kaluza-Klein theories [23], supergravity theories [23], and superstring theories [24]. There the 'external' and 'internal' dimensions are fixed, and the internal space being compactified, is of the size of Planck length. Therefore any change in dynamics comes from the change in the Lagrangian and not because of the variability of the dimension. We, in contrast, take the dimension as a dynamical variable. The picture we are using could be that of a decrumpling 3-dimensional space-membrane. This picture has led us to a model universe with a dynamics wich depends on the dimension of D-space. As we go back in time more and more, the dependence on the dimension becomes more effective. However, there is no beginning of time, and no Big Bang. Therefore, we consider the time of Big Bang in the standard model as a relative zero point of time! The higher turning point, where the dimension of D-space is more than 1000, could be considered as a beginning of the decrumpling or the expansion of the universe, but we should be aware that this point should not be considered as the 'creation'-time in the sence of standard model. In fact, our model does not have any real starting time, because it is an oscillating model. As our model doesn't have any starting time, the traditional horizon problem in standard cosmology does not show up in our model.

The most exciting feature of our model seems to be the absence of any singularity. Even, contrary to our initial expectations, the dimension of universe remains finite.

It should be noted that despite resolving the problems of the standard model, we could still have inflation within the decrumpling universe model. The impact of our model on the structure formation, nucleosynthesis, flatness problem, and dark matter remains to be considered. We are currently studying the other problems of the standard model and will turn to them in future publications.

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