Inflaton Potential Reconstruction and Generalized Equations of State

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Abstract

We extend a previous analysis concerning cosmological fluids with generalized equations of state in order to study inflationary scenarios. In the framework of the slow-roll approximation we find the expressions for the perturbation parameters ϵ, η and the density perturbation spectra in terms of the adiabatic index $\gamma(a)$ as a function of the universe scale factor. This connection allows to find straightforwardly $\gamma(a)$ corresponding, for example, to the simplest *chaotic* model and to the Harrison-Zeldovich potential and shows its capability to be applied to more complicate situations. Finally, we use this description to develop a new approach to the early universe dynamics, based on a $1/N$ expansion, where N is the e-fold number. To this aim, we introduce a set of suitable dimensionless variables and show that at the zero-th order in $1/N$, an improved slow-roll approximation is obtained.

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1 Introduction

The idea of deriving the inflaton potential by imposing particular theoretical or phenomenological requirements has been widely discussed in the recent literature. In [1], the potential $V(\phi)$ is deduced by explicitly assuming the time evolution of the universe scale factor $a = a(t)$, and substituting it into the Friedmann-Robertson-Walker equations. The resulting potential is given in a parametric form, using the time as the parameter. In [2] the hamiltonian approach was followed to postulate that the scalar field hamiltonian is some function of $\log(a)$, a being the average scale factor in Bianchi models I and V. Finally in [3] it is shown that, for any arbitrary equation of state, $V(\phi)$ can be related to the solution of a Riccati equation. More recently other methods have been proposed. In particular in [4] it is shown how a family of inflaton potentials can be reconstructed from the scalar density fluctuation spectrum. This approach is improved in [5] where it is shown how the potential can be uniquely determined from the knowledge of the tensor gravitational wave spectrum. These spectra may be obtained from the detection of fluctuations in the temperature distribution in the Cosmic Microwave Background Radiation (CMBR). These results, which are not general as they are obtained in the slow-roll approximation, are corrected up to second-order in slow roll parameters [6].

In this letter the scalar field potential reconstruction procedure contained in [7], is applied to inflationary scenarios. In particular, the properties of some relevant models are related to a suitable class of equations of state for the cosmological fluid. In section 2 the potential reconstruction procedure proposed in [7] is briefly reviewed. This formalism allows to relate in a very simple way the adiabatic index for the cosmological fluid considered to the slow-roll parameters ϵ and η [5]. In section 3, we obtain the expression of fluctuation amplitudes and spectra in terms of the adiabatic index $(\gamma(a))$, and for some relevant inflationary potentials, i.e. the most simple chaotic theory $V(\phi) = \mu^2 \phi^2$, and the Harrison Zeldovich model we determine $\gamma(a)$. Thus, in section 4, by using the approach outlined in section 2 we develop the $1/N$ expansion for some relevant quantities concerning the early universe dynamics, where N is the e-fold number of the inflationary stage. Remarkably, at the zero-th order in this parameter we get an improved slow-roll approximation which contains as a particular case the usual slow-roll regime. Finally in section 5 we give our conclusions.

2 Potential reconstruction from the equation of state

In a homogeneous and isotropic universe, whose metric can be cast in the standard form

$$ds^2 = dt^2 - a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right\} \quad , \tag{1}$$

the corresponding Friedmann-Robertson-Walker equations are

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3} \rho \quad , \tag{2}$$

and

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi G \; p \quad , \tag{3}$$

where k can be chosen to be $+1$, -1, or 0 for spaces of constant positive, negative, or zero spatial curvature respectively[1].

For a perfect fluid corresponding to a minimally coupled self-interacting scalar field the energy density ρ and the pressure p read

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad , \tag{4}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \quad , \tag{5}$$

where $V(\phi)$ is the potential of the self-interacting scalar field. By using (4) and (5) the Klein-Gordon equation

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{\delta V(\phi)}{\delta \phi} = 0 \quad , \tag{6}$$

can be derived from the conservation law

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p) \quad . \tag{7}$$

Hence from (7), dividing by $\dot{a}$, we can obtain the equation of state $p = p(\rho)$ for the fluid. In general due to the arbitrariness of the potential, we expect this equation to be non linear

$$p = \left(-\frac{1}{3}\frac{a}{\rho}\frac{d\rho}{da} - 1 \right)\rho \equiv (\gamma(\rho) - 1)\rho \quad . \tag{8}$$

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[1]in units $\hbar = c = 1$, $G = 1/m_{Pl}^2$.

We note that, by virtue of (7), to give $\gamma(\rho)$ is equivalent to assign $\gamma(a)$, by which we can describe the most general family of inflationary potentials. This procedure, in fact, allows us to find a parametric form for the scalar potential in terms of the equation of state in a very simple and intuitive way, where the expansion factor a is the parameter.

From the definition of γ contained in (8), we obtain

$$\rho(a) = \rho_0 \exp\left\{-3 \int_{a_i}^{a} \frac{\gamma(a')}{a'} da'\right\} \quad , \tag{9}$$

where a_i and $\rho_0 = \rho(a_i)$ are the initial values for the scale factor and the energy density respectively. Eqs. (7) and (8) together with (9) give,

$$\frac{1}{2}\dot{\phi}^2 - V(\phi) = (\gamma(a) - 1)\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \quad . \tag{10}$$

Thus, the expressions for the kinetic energy and for the potential follow

$$\frac{1}{2}\dot{\phi}^2 = \frac{\gamma(a)}{2}\rho(a) \quad , \tag{11}$$

$$V(a) = \frac{2 - \gamma(a)}{2} \rho(a) = \frac{2 - \gamma(a)}{2} \rho_0 \exp\left\{-3 \int_{a_i}^{a} \frac{\gamma(a')}{a'} da'\right\} \quad . \tag{12}$$

In Eq. (11), the factor $\gamma(a)/2$ can be seen as the weight of the kinetic term with respect to the total energy density. By virtue of (11) $\gamma(a)$ is bound to be positive, and it is exactly equal to zero for pure exponential inflation.

To obtain the evolution of ϕ with respect to a, let us consider the square root of (11)

$$\dot{\phi} = \pm\sqrt{\gamma(a)\rho(a)} \quad . \tag{13}$$

By comparing (13) with

$$\dot{\phi} = \frac{d\phi}{da}\dot{a} = a\frac{d\phi}{da}\sqrt{\frac{8\pi G}{3}\rho(a)} \tag{14}$$

(we are assuming a spatially flat expanding universe), we get

$$\phi(a) = \phi(a_i) \pm \sqrt{\frac{3}{8\pi G}} \int_{a_i}^{a} \frac{\sqrt{\gamma(a')}}{a'} da' \quad . \tag{15}$$

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Note that both (12) and (15) are completely general, and determine $V(\phi)$ once we assign ρ_0, and the function $\gamma(a)$. Thus the above description provides a natural classification scheme for single-fluid inflationary models, given in terms of the $\gamma(a)$ function. Note moreover that by construction (12) and (15) formally solve (6), and that the inflationary phase is characterized by $\ddot{a} > 0$ [8], [9], and therefore, whenever $\gamma(a) < 2/3$.

The last step in reconstructing the potential is the determination of the parameter ρ_0. We can see that this problem is related to the *efficiency* (how fast the inflation occurs) of the inflationary process devised here. The function $\gamma(a)$ contains the information of the amount of growth of the universe during inflation, and also allows to obtain explicitly the inflationary time interval Δt in terms of ρ_0

$$\Delta t = \sqrt{\frac{3}{8\pi G \rho_0}} \int_{a_i}^{a_f} \frac{da}{a} \exp\left[\frac{3}{2} \int_{a_i}^{a} \frac{\gamma(a')}{a'} da'\right] \quad . \tag{16}$$

From $\gamma(a)$, immediately follows the value of the e-fold number $N = \log(a_f/a_i)$. Finally, we notice that this reconstruction procedure can also be reversed in order to obtain the properties of the cosmological fluid (equation of state) once that the inflationary potential, or the dynamics of the scalar fields are known. We will show in the following one example of this inverse treatment.

To end this section it is worth pointing out the connection existing between the slow-roll approximation and this description given in terms of $\gamma(a)$. As it is well-known, the slow-roll regime can be conveniently characterized in terms of two parameters ϵ and η, defined in [5], by the conditions ϵ, $|\eta| << 1$. These two quantities are simply connected to deviation from the flat Harrison-Zeldovich scalar perturbation spectrum, and according to the standard approach, this connection allows to reconstruct perturbatively the inflaton potential. By definition the expressions of ϵ and η in terms of $\gamma(a)$ result to be

$$\epsilon = \frac{3}{2}\left(\frac{p}{\rho} + 1\right) = \frac{3}{2}\gamma(a) \quad , \tag{17}$$

$$\eta = \frac{3}{2}\left(\frac{\partial p}{\partial \rho} + 1\right) = \frac{3}{2}\gamma(a) - \frac{1}{2}\frac{d\log(\gamma(a))}{d\log(a/a_i)} \quad . \tag{18}$$

Interestingly, from (17) and (18) ϵ represents the adiabatic index, whereas η is linearly related to the squared value of the generalized sound speed.

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3 Density perturbations and relevant examples of equations of state.

A typical feature of inflationary models is to provide a general scheme for the production of primordial fluctuations. This aspect is of particular importance, since the experimental information on the CMBR temperature fluctuations, and large scale galactic structure can constraint, to some extent, the form of the inflationary potential. In this section we will stress how, starting from the adiabatic index $\gamma(a)$, perturbation spectra are easily reconstructed. As in [5] we introduce the perturbation amplitudes A_S and A_G corresponding, respectively, to quantum fluctuation of the inflaton field (scalar perturbations) and metric (tensorial perturbations)

$$A_S = \sqrt{\frac{2}{\pi}} G \, H^2(\phi) \left| \frac{\delta H(\phi)}{\delta \phi} \right|^{-1} \quad , \tag{19}$$

$$A_G = \sqrt{\frac{G}{2\pi^2}} \, H(\phi) \quad , \tag{20}$$

where $H(\phi) = \sqrt{8\pi G\rho/3}$ is the Hubble parameter.

In particular, from (17) the ratio A_G/A_S is related to $\gamma(a)$ in a simple way

$$\frac{A_G}{A_S} = \sqrt{\epsilon} = \sqrt{\frac{3}{2} \, \gamma(a)} \quad , \tag{21}$$

and therefore it is small in the slow-roll regime. In the same limit the scale dependence of the spectra can be expressed in terms of $\gamma(a)$

$$1 - n_S \equiv \frac{d \log \left[A_S^2(\lambda) \right]}{d \log(\lambda/\lambda_0)} = 4\epsilon(a) - 2\eta(a) = 3\gamma(a) + \frac{d \log(\gamma(a))}{d \log(a/a_i)} \quad , \tag{22}$$

$$n_G \equiv \frac{d \log \left[A_G^2(\lambda) \right]}{d \log(\lambda/\lambda_0)} = 2\epsilon(a) = 3\gamma(a) \quad , \tag{23}$$

where the parameters $\epsilon(a)$, $\eta(a)$ have to be evaluated at the value for a when the scale λ goes outside the Hubble radius during the inflationary era. To this end we recall that the physical scale $\lambda(\phi)$, decoupled when the value of the scalar field was ϕ, grew between the time of horizon crossing and today by a factor [5]

$$\lambda(\phi) = H^{-1}(\phi) \frac{a_0}{a(\phi)} \quad , \tag{24}$$

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where $H(\phi)$ and $a(\phi)$ represent, respectively, the value of the Hubble radius and of the scale factor at decoupling, and a_0 is the present value of the scale factor. Since

$$a(\phi) = a_f \, \exp[-N(\phi)] \quad , \tag{25}$$

with $N(\phi)$ the number of e-fold between the value ϕ and the the end of inflation, from (24) and (25) we get

$$\lambda(\phi) = \frac{\exp[N(\phi)]}{H(\phi)} \frac{a_0}{a_f} \quad . \tag{26}$$

The function $N(\phi)$ can be obtained using the definition of the Hubble parameter and the equation of motion. Differentiating in fact H^2 with respect to ϕ

$$\frac{\delta H^2}{\delta \phi} = 2\dot{H}H\dot{\phi}^{-1} = \frac{8\pi G}{3}\left(\ddot{\phi} + \frac{\delta V(\phi)}{\delta \phi}\right) \quad , \tag{27}$$

which, by virtue of (6), gives

$$\frac{\delta H}{\delta \phi} = -4\pi G \dot{\phi} \quad . \tag{28}$$

Therefore, if t_f is the value of time at the end of the inflationary phase, using (28)

$$N(\phi) = \int_{t(\phi)}^{t_f} H(t')dt' = \int_{\phi}^{\phi_f} \frac{H(\phi')}{\dot{\phi}'}d\phi' = -4\pi G \int_{\phi}^{\phi_f} H(\phi')\left(\frac{\delta H(\phi')}{\delta \phi'}\right)^{-1} d\phi' \quad , \tag{29}$$

If we now differentiate (26) with respect to a we find, using (15) and (29)

$$\begin{aligned}
\frac{dlog(\lambda/\lambda_0)}{d\log(a/a_i)} &= \left[4\pi GH\left(\frac{\delta H}{\delta \phi}\right)^{-1} - H^{-1}\frac{\delta H}{\delta \phi}\right]\frac{d\phi}{d\log(a/a_i)} \\
&= \left[4\pi GH\left(\frac{\delta H}{\delta \phi}\right)^{-1} - H^{-1}\frac{\delta H}{\delta \phi}\right]\sqrt{\frac{3\gamma(a)}{8\pi G}}\mathrm{sign}\left(\frac{d\phi}{da}\right) \quad , \tag{30}
\end{aligned}$$

which, from the definition of A_S and A_G in (19) and (20), can also be cast in the form

$$\frac{d\log(\lambda/\lambda_0)}{d\log(a/a_i)} = \mathrm{sign}\left(\frac{\delta H(\phi)}{\delta \phi}\right)\,\mathrm{sign}\left(\frac{d\phi}{da}\right)\left[\frac{A_S}{A_G} - \frac{A_G}{A_S}\right]\sqrt{\frac{3\gamma(a)}{2}} \quad . \tag{31}$$

Finally, from (21) we get the simple result

$$\frac{d\log(\lambda/\lambda_0)}{d\log(a/a_i)} = -\mathrm{sign}\left(\frac{dH(a)}{da}\right)\left(\frac{3}{2}\gamma(a) - 1\right) = \frac{3}{2}\gamma(a) - 1 \quad , \tag{32}$$

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since combining (2) and (3), for $\gamma(a) > 0$

$$\dot{H} = -4\pi G \gamma \rho < 0 \quad , \tag{33}$$

and during the expansion $\dot{a} > 0$, implying that $dH/da < 0$.

Equation (32) allows us to express directly the perturbation spectra as functions of a: using (22) and (23)

$$\frac{d\log\left[A_S^2(a)\right]}{d\log(a/a_i)} = \left[3\gamma(a) + \frac{d\log(\gamma(a))}{d\log(a/a_i)}\right]\left(\frac{3}{2}\gamma(a) - 1\right) \quad , \tag{34}$$

$$\frac{d\log\left[A_G^2(a)\right]}{d\log(a/a_i)} = 3\gamma(a)\left(\frac{3}{2}\gamma(a) - 1\right) \quad . \tag{35}$$

To end this section with two relevant examples, we will deduce the expressions for the adiabatic index $\gamma(a)$, for the most simple "chaotic" potential, $V(\phi) = \mu^2\phi^2$, and for the potential leading to the flat Harrison-Zeldovich spectrum for scalar perturbations.

In the slow-roll limit the case $V(\phi) = \mu^2\phi^2$ may be easily solved, considering the evolution of the scalar field once it has reached *limit velocity* condition, and neglecting the kinetic energy contribution to the Hubble parameter. From (2) and (6) one obtains

$$\phi(t) = \phi_0 - \frac{\mu}{\sqrt{6\pi G}}t \quad , \tag{36}$$

$$a(t) = a_0 \exp\left\{\frac{\sqrt{8\pi G}\mu}{\sqrt{3}}\left[\phi_0 t - \frac{\mu}{2\sqrt{6\pi G}}t^2\right]\right\} \quad . \tag{37}$$

The end-point of the inflationary phase is determined by the condition $\ddot{a}(t) = 0$, giving the final value of the inflaton field ϕ_f and the value of N

$$\phi_f = (4\pi G)^{-1/2} \quad , \tag{38}$$

$$N = 2\pi G\phi_0^2 - \frac{1}{2} \quad . \tag{39}$$

Using the previous relations we obtain the adiabatic index

$$\gamma(a) = \frac{2}{3}\left[2N + 1 - 2\log\left(\frac{a}{a_i}\right)\right]^{-1} \quad , \tag{40}$$

which shows that the inflaton field starts its evolution with condition which differs from *perfect slow-roll* for term $1/N$ ($\gamma(a_i) \approx 1/3N$), increases its velocity and finally exit

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from the inflationary phase $\gamma(a_f) = 2/3$).

Starting from (40) we can now easily obtain the perturbation amplitudes. We first notice that in the present approximation $\eta = 0$, so $1 - n_S = 2 n_G$. From this by using (32)-(35) we obtain at the lower order the well-known expressions [5]

$$A_S(\lambda) = A_S^0 \left[1 + \frac{2}{2N+1} \log\left(\frac{\lambda}{\lambda_0}\right) \right] \quad , \tag{41}$$

$$A_G(\lambda) = A_G^0 \left[1 + \frac{2}{2N+1} \log\left(\frac{\lambda}{\lambda_0}\right) \right]^{1/2} \quad , \tag{42}$$

where, from relations (19) and (21), since λ_0 is defined as the scale which decouples at ϕ_0

$$A_S^0 = \frac{4G^{3/2}\mu\phi_0^2}{\sqrt{3}} \quad , \tag{43}$$

$$A_G^0 = \frac{A_S^0}{\sqrt{2N+1}} \quad . \tag{44}$$

To reconstruct the Harrison-Zeldovich potential we remind that in order to have a scale independent fluctuation amplitude, it is necessary that $2\epsilon(a) = \eta(a)$, or equivalently that the r.h.s. of (22) identically vanishes. Hence, from this condition we straightforwardly get the differential equation for $\gamma(a)$

$$\frac{d\gamma(a)}{d\log(a/a_i)} = -3\,\gamma^2(a) \quad , \tag{45}$$

whose solution is

$$\gamma(a) = \frac{2}{3} \left[\frac{2}{3\gamma_0} + 2\log\left(\frac{a}{a_i}\right) \right]^{-1} \quad , \tag{46}$$

with γ_0 the initial value of $\gamma(a)$. In particular if $\gamma_0 = 0$ the adiabatic index remains constant and it is easily seen from (50) that the potential reduces to a constant, and the amplitude of scalar perturbations tends to infinity. Notice that $\gamma(a)$ remains always under the inflationary threshold of 2/3, leading, in absence of other fields to eternal inflation.

Evaluating the expression of $V(a)$ and $\phi(a)$ we find for $\gamma_0 \neq 0$:

$$V(\phi) = \frac{\rho_0}{3(1-\hat{\gamma}_0)} \left\{ 3\frac{\bar{\phi}^2}{\phi^2} - \frac{\bar{\phi}^4}{\phi^4} \right\} \quad , \tag{47}$$

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with $\bar{\phi} = (4\pi G)^{-1/2}$. Eq. (47), obtained by suitably choosing $\phi(0)$, recovers the well-known form of the only potential leading to the Harrison-Zeldovich spectrum for scalar perturbations [5].

4   1/e-fold number expansion

As well-known an inflationary phase in the universe evolution is able to solve a number of important problems occurring in the standard cosmology provided that the final e-fold number, $N = \log(a_f/a_i)$, is large enough. A reasonable estimate for this quantity would be $N \approx 100$. The value of N is completely fixed once the equations of motion and the initial conditions are assigned. By using the approach described in the previous sections, we will adopt here a different point of view, namely we will formally look at N as a quantity in terms of which the potential and the field dynamics can be parameterized. Since, as already mentioned, its value is very large it is quite natural to expand in $1/N$ all physical quantities, building in this way a perturbative approach to inflaton dynamics. Notice that N is a *global* quantity, being related to the total growth of the scale factor during the inflationary phase, and the $1/N$ expansion is completely independent on whether slow-roll conditions are satisfied or not: this is quite different from the usual perturbative approach in terms of the slow-roll parameters ϵ and η [5], which allows for an accurate description only in the slow-roll regime.

In order to simplify our analysis let us introduce the dimensionless quantities y and $\hat{\gamma}(y)$ defined by

$$y \equiv \frac{1}{N} \log\left(\frac{a}{a_i}\right) \quad , \tag{48}$$

and

$$\hat{\gamma}(y) \equiv 1 - \frac{3}{2}\gamma(a(y)) \quad . \tag{49}$$

By definition, the y variable during the inflationary epoch is bound to be $0 \leq y \leq 1$, and the starting and ending points of inflation correspond to $y = 0$ and $y = 1$ respectively. Actually, in this interval $0 \leq \hat{\gamma}(y) \leq 1$. The definition of the variable y is quite natural by observing that it would simply correspond to Ht for a pure exponential expansion. Eqs. (12) and (15), when expressed in terms of y, suggest to introduce two dimensionless quantities, obtained by properly rescaling potential and field configuration

$$\widehat{V}(y) \equiv \left[\frac{V(y)}{\rho_0}\right]^{1/N} = \left[\frac{1}{3}(2 + \hat{\gamma}(y))\right]^{1/N} \exp\left\{-2\int_0^y [1 - \hat{\gamma}(y')] \, dy'\right\} \quad , \tag{50}$$

and

$$\hat{\phi}(y) \equiv \sqrt{4\pi G} \left[\frac{\phi(y) - \phi(0)}{N} \right] = \pm \int_0^y \sqrt{1 - \hat{\gamma}(y')} \, dy' \quad . \tag{51}$$

We first note that $\widehat{V}(y)$ results to be an analytical function of $1/N$ provided that $\hat{\gamma}(y)$ is analytical as well: in particular, expanding (50) in $1/N$ up to the first order

$$\widehat{V}(y) \simeq \exp \left\{ -2 \int_0^y [1 - \hat{\gamma}(y')] \, dy' \right\} \left[1 + \frac{1}{N} \log \left(\frac{2 + \hat{\gamma}(y)}{3} \right) \right] \quad , \tag{52}$$

$\widehat{V}(y)$ gets at zero-th order a very simple functional dependence on $\hat{\gamma}(y)$. In terms of the quantities (49), (50) and (51) we can also rewrite the equation of motion (6) in the following form

$$\frac{1}{N} \hat{\phi}''(y) + [2 + \hat{\gamma}(y)] \hat{\phi}'(y) + \frac{1}{2} [2 + \hat{\gamma}(y)] \frac{1}{\widehat{V}(y)} \frac{\delta \widehat{V}(y)}{\delta \hat{\phi}} = 0 \quad , \tag{53}$$

where prime denotes differentiation with respect to y. In the large N limit it reduces simply to

$$\hat{\phi}'(y) + \frac{1}{2} \frac{1}{\widehat{V}(y)} \frac{\delta \widehat{V}(y)}{\delta \hat{\phi}} = 0 \quad , \tag{54}$$

which corresponds to the zero-th term of the $1/N$ expansion for the $\hat{\phi}$-field equation of motion; in particular note that, consistently with the expansion, (51) and (52) at the lowest order in $1/N$, exactly solve (54). The first order differential equation (54) provides an *improved* slow-roll approximation for the scalar field dynamics. It is in fact the analogous of the slow-roll limit for the classical equation of motion for ϕ, but it does also contain a contribution coming from the second-time derivative of the field. Had we neglected the second time derivative term, as in the usual slow-roll approximation, we would have rather obtained

$$\hat{\phi}'(y) + \frac{2 + \hat{\gamma}(y)}{6} \frac{1}{\widehat{V}(y)} \frac{\delta \widehat{V}(y)}{\delta \hat{\phi}} = 0 \quad , \tag{55}$$

which reduces to (54) only in the case of a perfectly flat potential ($\hat{\gamma}(y) = 1$). Consequently, we point out that slow-roll approximation implies large N, but, viceversa, there are efficient inflationary models, for which $1/N$ expansion is still reliable, which do not satisfy slow-roll conditions. This conclusion can be qualitatively got from the

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expression of the e-fold number N: we first remind that, using (28), ϵ and η can also be written in the form[2]

$$\epsilon = \frac{1}{4\pi G}\left(\frac{1}{H}\frac{\delta H}{\delta \phi}\right)^2 \tag{56}$$

$$\eta = \frac{1}{4\pi G}\frac{1}{H}\frac{\delta^2 H}{\delta \phi^2} \tag{57}$$

so from (29)

$$N = -4\pi G\int_{\phi_0}^{\phi_f}\frac{d\phi}{\sqrt{\epsilon(\phi)}} \tag{58}$$

If ϵ is small one expects the integral of $(\sqrt{\epsilon})^{-1}$, as a function of ϕ, to be large, though this is only a sufficient condition: it is in fact conceivable to still have $N \gg 1$ even if ϵ is close to 1 somewhere in the interval $[min\{\phi_0, \phi_f\}, max\{\phi_0, \phi_f\}]$.
The relationship between the slow-roll and the $1/N$ expansions can be however formalized by introducing a new parameter

$$\nu = \frac{\hat{\phi}''}{N(2 + \hat{\gamma}(y))\hat{\phi}'} \tag{59}$$

The condition $\nu \ll 1$ corresponds to the possibility of neglecting the $1/N$ term in the dimensionless equation of motion (53), which therefore reduces to (54): we will hereafter refer to the latter as the $1/N$ approximation and to the expansion of the dynamical quantities in terms of the two parameters $1/N$ and ν as the $1/N$ expansion. Using (51), (17) and (18) one easily gets

$$\nu = \frac{\epsilon - \eta}{3 - \epsilon} \tag{60}$$

Remarkably ν contains all power terms in ϵ. This relation shows that if ϵ and η are small, namely if the slow-roll approximation holds, then also ν is small and the $1/N$ approximation is legitimate; viceversa, if ν is small, this only implies that the difference $\epsilon - \eta$ is small, while both ϵ and η can be quite large: in this case the $1/N$ approximation is still valid, while the slow-roll one breaks down. Summarizing, slow-roll $\Rightarrow 1/N$, but $1/N \not\Rightarrow$ slow-roll, so, as we wanted to show, the $1/N$ expansion seems to be more widely applicable to inflationary dynamics since it grasps its main feature (large e-fold).

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[2]This is actually the form in which they were introduced in reference [5].

It is interesting at this point to look for the potential satisfying the condition $\nu = 0$, i.e. for which the $1/N$ approximation already gives an exact solution: from (17) and (18) it follows that γ is a constant, $\gamma(a) = \gamma_0 = 2\epsilon_0/3 > 0$, so, using (50) and (51), it is easy to obtain the expression of $V(\phi)$

$$V(\phi) = \rho_0 \left(1 - \frac{\epsilon_0}{3}\right) \exp\left[-\sqrt{4\pi G \epsilon_0} \, |\phi - \phi_0|\right] \tag{61}$$

Let us now finally consider the slow-roll regime in further details. It is well-known that the consistency of this approximation for the scalar field dynamics requires [10]

$$\left|\frac{1}{V(\phi)}\frac{\delta V(\phi)}{\delta\phi}\right| \ll \sqrt{48\pi G} \quad, \tag{62}$$

$$\left|\frac{1}{V(\phi)}\frac{\delta^2 V(\phi)}{\delta\phi^2}\right| \ll 24\pi G \quad. \tag{63}$$

Also in this case the rescaled quantities (49), (50) and (51) allow to rewrite (62) and (63) in a more suitable form

$$\left|\frac{1}{\widehat{V}(\hat{\phi})}\frac{\delta\widehat{V}(\hat{\phi})}{\delta\hat{\phi}}\right| \ll 2\sqrt{3} \quad, \tag{64}$$

and

$$\left|\frac{1}{\widehat{V}^2(\hat{\phi})}\left(\frac{\delta\widehat{V}(\hat{\phi})}{\delta\hat{\phi}}\right)^2 + \frac{1}{N}\frac{1}{\widehat{V}^2(\hat{\phi})}\left[\widehat{V}(\hat{\phi})\frac{\delta^2\widehat{V}(\hat{\phi})}{\delta\hat{\phi}^2} - \left(\frac{\delta\widehat{V}(\hat{\phi})}{\delta\hat{\phi}}\right)^2\right]\right| \ll 6 \quad. \tag{65}$$

Remarkably, the transformation introduced in (48)-(51), besides allowing us to perform the $1/N$ expansion of the physical quantities, has the nice property to leave (62) formally unchanged. Furthermore, we notice that at the zero-th order in $1/N$ expansion, (64) and (65) are essentially equivalent and are both satisfied provided that

$$\left|\frac{1}{\widehat{V}(\hat{\phi})}\frac{\delta\widehat{V}(\hat{\phi})}{\delta\hat{\phi}}\right| \ll \sqrt{6} \quad. \tag{66}$$

5 Conclusions

Following the reconstruction procedure of the inflaton potential based on the knowledge of the adiabatic index for the cosmological fluid as a function of the universe scale

factor a [7], we have studied the slow-roll regime and in particular the connection existing between $\gamma(a)$ and the perturbation parameters ϵ and η [5]. In this framework we have also computed the scalar and tensorial perturbation spectra as a function of $\gamma(a)$, and as an example we have analyzed two simple cases, namely the most elementary chaotic model and the Harrison Zeldovich potential; for both this cases the adiabatic index $\gamma(a)$ is found to be a simple rational function of $\log(a/a_i)$. Interestingly, the simple connection occurring between the adiabatic index and the slow-roll parameters ϵ and η provides once that scalar and tensorial perturbation amplitudes are known, to completely characterize the cosmological fluid. Finally, by applying the reconstruction procedure we have developed a formalism in which it is natural to perform a $1/N$ expansion (N is the e-fold number) of the most relevant physical quantities characterizing the inflationary dynamics. This expansion seems to be more general than the one given in terms of the slow-roll parameters, being applicable also in cases when even though the e-fold number is large, slow-roll conditions are not satisfied. In particular this approach is based on the idea of using two new parameters, $1/N$ and ν, in describing inflationary dynamics, instead of the slow-roll ones ϵ and η. At zero-th order in this expansion an improved slow-roll approximation is straightforwardly obtained.

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