ELECTROMAGNETIC FIELD IN SOME ANISOTROPIC STIFF FLUID UNIVERSES

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The electromagnetic field is studied in a family of exact solutions of the Einstein equations whose material content is a perfect fluid with stiff equation of state (p = ϵ). The field equations are solved exactly for several members of the family.

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I. INTRODUCTION

Recently Lotze [1], using a method developed by Sagnotti and Zwiebach [2], has considered the Maxwell equations in anisotropic universes with a diagonal Bianchi type I metric and found exact solutions for two particular expansions laws in the axisymmetric case. Electromagnetic fields in space-times with local rotational symmetry, using the Debye-formalism were also considered by Dhurandhar et al. [3]; they obtained exact solutions for Kantowski-Sachs universes, Taub space and for Bianchi type I with dust as material content. Here I consider the same problem in a family of Bianchi type I models that are solutions of Einstein equation with either a free massless scalar field or with a perfect fluid with stiff equation of state. The metric was obtained by Jacobs [4] and is also given by Vajk and Eltgroth [5] and it is a particular case of the metrics studied by Thorne [6], more recently it was rediscovered by Iyer and Vishveshwara [7] while looking for exact solutions in which Dirac equation separates. Recently, the production of scalar particles in these model was considered [9]. We write the metric in the following form,

$$ds^2 = -dt^2 + t^{2q}(dx^2 + dy^2) + t^{2(1-2q)}dz^2, \tag{1}$$

This metric is a one parameter family of solutions to Einstein equations with a perfect stiff fluid. The parameter q is related to the Lagrangian of the scalar field or to the energy density of the perfect fluid by the relation

$$L = (1/2)\phi^{;a}\phi_{;a} = \frac{q(2-3q)}{t^2}. \tag{2}$$

or

$$\epsilon = p = \frac{q(2-3q)}{t^2}. \tag{3}$$

This metric is also the solution to Einstein equations with a massless minimally coupled scalar field. The qualitative features of the expansion depend on q in the following way: for $1/2 < q$, the universe expands from a "cigar" singularity; for $q = 1/2$, the universe expands purely transversely from an initial "barrel" singularity; for $0 < q < 1/2$, the initial singularity is "point"-like; if $q \leq 0$ we have a "pancake" singularity. The case $q = 1/3$ is the isotropic universe with a stiff fluid; the case p=q is the Minkowski spacetime. This family of metrics is "Kasner-like" in the sense that the sum of the exponents is equal to one but the sum of the squares is not equal to one except in the two cases when $q = 0$ and $q = 2/3$ when we have vacuum. The symmetries of these spacetimes can be described by four spacelike Killing vector fields,

$$\xi_1 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad \xi_2 = \frac{\partial}{\partial x}, \quad \xi_3 = \frac{\partial}{\partial y}, \quad \xi_4 = \frac{\partial}{\partial z}, \tag{4}$$

The first vector corresponds to the rotational simmetry in the plane xy and the other three to the translational simmetry along the x, y and z axis. The non-vanishing commutators are

$$[\xi_1, \xi_2] = \xi_3, [\xi_3, \xi_1] = -\xi_2. \tag{5}$$

In the next section we review the formalism used by Sagnotti and Zwiebach [2], that write the metric in the following form,

$$ds^2 = -C^2 d\tau^2 + C_1^2(dx^2 + dy^2) + C_3^2 dz^2, \tag{6}$$

comparing with Eq.(1) we see that in our particular case we have

$$\tau = t; C = 1, \quad C_1 = t^q, \quad C_3 = C^{1-2q}. \tag{7}$$

Notations and conventions. $c = 1$, metric signature (-+++); greek indices run from 0 to 3 , latin indices from 1 to 3. The derivative with respect to the time τ (or t) is denoted by an overdot.

II. FIELD EQUATIONS

The Maxwell equations will be written in the following the method developed by Sagnotti and Zwiebach [2] . The field strength tensor $F^{\mu\nu}$ is written as

$$F^{\mu\nu} \propto \int d^3 b f k f^{\mu\nu}(\mathbf{k};\tau)\exp(i\mathbf{kx}). \tag{8}$$

The Maxwell equation are written in terms of the quantities

$$F^{m(\sigma)} = \sqrt{-g}(f^{0m} + i\sigma\ {}^*f^{m0}), \quad \sigma = \pm, \tag{9}$$

using spherical coordinates in $\mathbf{k}$ space, only the θ and ϕ and componets of $F^{m(\sigma)}$ are nonvanishing and will be denoted by $F^{(\sigma)}$ and $G^{(\sigma)}$, respectively, satisfying the equations

$$\dot{F}^{(\sigma)} = -\sigma k[\alpha F^{(\sigma)} + \beta G^{(\sigma)}] \tag{10}$$

and

$$\dot{G}^{(\sigma)} == \sigma k[\gamma F^{(\sigma)} + \alpha G^{(\sigma)}] \tag{11}$$

with

$$\alpha = \frac{C^2}{\sqrt{-g}}\frac{k_1 k_2 k_3}{k k_\perp{}^2}(C_2^2 - C_1^2), \tag{12}$$

$$\beta = \frac{C^2}{\sqrt{-g}}\frac{1}{k_\perp{}^2}(C_1^2 k_2^2 + C_2^2 k_1^2) \tag{13}$$

The parameter γ is given by $\beta\gamma - \alpha^2 = C^2\Omega^2/k^2$, where

$$\Omega^2 = \sum (k_i/C_i)^2. \tag{14}$$

After the elimination of $G^{(\sigma)})$ the relevant field equation is

$$\ddot{F}^{(\sigma)} - \frac{\dot{\beta}}{\beta}\dot{F}^{(\sigma)} + [C^2\Omega^2 + \sigma k\beta(\frac{\alpha}{\beta})^{\cdot}\ ]F^{(\sigma)} = 0. \tag{15}$$

In the present case we have $\alpha = 0$ and

$$\beta = (b\tau)^{3q-1}, \qquad \Omega^2 = k_\perp^2(b\tau)^{-3q} + k_3^2(b\tau)^{6q-3} \tag{16}$$

$$\ddot{F}^{(\sigma)} + \frac{(1-3q)}{\tau}\dot{F}^{(\sigma)} + [k_\perp^2(b\tau)^{1-3q} + k_3^2(b\tau)^{6q-2}]F^{(\sigma)} = 0. \tag{17}$$

Because we are considering axisymmetric case the solutions to the field equation become independent of the polarisation σ. There are several cases where the above equation can be solved exactly and are considered in the following section.

III. EXACT SOLUTIONS

In this section we consider those values of q for which it is possible to solve equation (10) for arbitrary values of k_3 and $k_\perp$.

A. q=0.

In this case the field equation is

$$\ddot{F}^{(\sigma)} + \frac{\dot{F}^{(\sigma)}}{\tau} + [k_\perp^2(b\tau) + \frac{k_3^2}{(b\tau)^2}]F^{(\sigma)} = 0, \tag{18}$$

with the solution

$$F^{(\sigma)} = c_1 H_\nu^{(1)}(|k_\perp|(b\tau)^{3/2}) + c_2 H_\nu^{(2)}(|k_\perp|(b\tau)^{3/2}), \tag{19}$$

where $\nu = i|k_3|/2$ and $H_\nu^{(i)}$ is a Hankel function of order ν and c_i are integration constants.

B. q=1/5.

For this value of q the field equation is

$$\ddot{F}^{(\sigma)} + \frac{2}{5}\frac{\dot{F}^{(\sigma)}}{\tau} + [k_\perp^2(b\tau)^{2/5} + k_3^2(b\tau)^{-4/5}]F^{(\sigma)} = 0 \tag{20}$$

and the solution is given by

$$F^{(\sigma)} = c_1 D_a(\eta) + c_2 D_{-(a+1)}(i\eta), \tag{21}$$

here D_a is the parabolic function of order a with

$$\eta = \pm(1+i)\sqrt{5k_\perp/2}(b\tau)^{3/5}, \tag{22}$$

and

$$a = -\frac{1}{2} - \frac{5k_3^2}{4k_\perp}. \tag{23}$$

C. q=1/4.

Now the field equation is

$$\ddot{F}^{(\sigma)} + \frac{1}{4\tau}\dot{F}^{(\sigma)} + [k_\perp^2(b\tau)^{\frac{1}{4}} + k_3^2(b\tau)^{\frac{-1}{2}}]F^{(\sigma)} = 0, \tag{24}$$

and the solution is as follows

$$F^{(\sigma)} = \sqrt{\eta}[c_1 H_{1/3}^{(1)}(\frac{2}{3}\eta^{3/2}) + c_2 H_{1/3}^{(2)}(\frac{2}{3}\eta^{3/2})], \tag{25}$$

here $H_\nu^{(i)}$ is a Hankel function of order ν with

$$\eta = \frac{\kappa(b\tau)^{3/4} + \lambda}{\kappa^{2/3}}, \tag{26}$$

and

$$\kappa = 4k_\perp^2, \qquad \lambda = 4k_3^2. \tag{27}$$

D. q=1/3.

This case is the isotropic Robertson-Walker with a stiff fluid and a $t^{1/3}$ expansion law. The field equation is

$$\ddot{F}^{(\sigma)} + k^2 F^{(\sigma)} = 0, \tag{28}$$

with

$$k^2 = k_\perp^2 + k_3^2, \tag{29}$$

and the solutions is,

$$F^{(\sigma)} = c_1 \exp(ik\tau) + c_2 \exp(-ik\tau). \tag{30}$$

E. q=1/2.

$$\ddot{F}^{(\sigma)} - \frac{\dot{F}^{(\sigma)}}{2\tau} + [k_3^2(b\tau) + \frac{k_\perp^2}{\sqrt{(b\tau)}}]F^{(\sigma)} = 0, \tag{31}$$

with the solution

$$F^{(\sigma)} = c_1 F_0(\eta,\rho) + c_2 G_0(\eta,\rho), \tag{32}$$

where $F_0(\eta,\rho)$and $G_0(\eta,\rho)$ are the regular and the irregular (logaritmic) Coulomb wave functions [7] with null angular momentum and

$$\eta = -(\frac{k_\perp^2}{2k_3^2}) \qquad \text{and} \qquad \rho = (b\tau^{2/3}). \tag{33}$$

F. q=1.

The equation (10) is in this case

$$\ddot{F}^{(\sigma)} - 2\frac{\dot{F}^{(\sigma)}}{\tau} + [k_\perp^2(b\tau)^{-2} + k_3^2(b\tau)^4]F^{(\sigma)} = 0, \tag{34}$$

and the solutions is

$$F^{(\sigma)} = \tau^{3/2}[c_1 H_\nu^{(1)}(\frac{|k_3|b^2(\tau)^6}{3}) + c_2 H_\nu^{(2)}(\frac{|k_3|b^2(\tau)^6}{3})], \tag{35}$$

where

$$\nu = \frac{\sqrt{(3/2)^2 - (k_\perp/b)^2}}{3}, \tag{36}$$

and $H_\nu^{(i)}$ is a Hankel function of order ν.

IV. RESTRICTED SOLUTIONS

In the previous section we considered those values of q for which it is possible to solve equation (10) for arbitrary values of k_3 and $k_\perp$, on the other hand it is possible to solve the field equation for arbitrary values of q but the particular case where k_3 or $k_\perp$ or both are zero.

V. $K_3 = K_\perp = 0$

The field equation is in this case

$$\ddot{F}^{(\sigma)} + \frac{1-3q}{\tau}\dot{F}^{(\sigma)} = 0, \tag{37}$$

The solutions are

$$F^\sigma = \begin{cases} c_1 + c_2\tau^{3q}, & q \neq 0 \\ c_1 + c_2\ln(\tau), & q = 0 \end{cases} \tag{38}$$

VI. $K_3 = 0, K_\perp \neq 0$.

The field equation and the solutions are

$$\ddot{F}^{(\sigma)} - 2\frac{\dot{F}^{(\sigma)}}{\tau} + [\frac{k_\perp^2}{(b\tau)^2}]F^{(\sigma)} = 0, \tag{39}$$

$$F^\sigma = \begin{cases} \tau^{3q/2}[c_1 Z_{\frac{q}{1-q}}(\frac{2|k_\perp b^{\frac{1-3q}{2}}|\tau^{\frac{3(1-q)}{2}}}{3(1-q)}) + c_2 Z_{\frac{-q}{1-q}}(\frac{2|k_\perp b^{\frac{1-3q}{2}}|\tau^{\frac{3(1-q)}{2}}}{3(1-q)})]; & q \neq 1; \\ c_1\tau^\alpha + c_2\tau^\beta, & q = 1, b^2 \neq 4k_\perp^2/9 \\ \tau^\alpha(c_1 + c_2\log\tau), & q = 1, b^2 = 4k_\perp^2/9 \end{cases} \tag{40}$$

where

$$\alpha = \frac{3 \pm \sqrt{9 - 4k_\perp^2/b^2}}{2}, \beta = \frac{3 \mp \sqrt{9 - 4k_\perp^2/b^2}}{2}. \tag{41}$$

and Z_ν is a solution of Bessel equation of order ν.

VII. $K_3 \neq 0, K_\perp = 0$.

Now, Eq. (10) and its solutions are

$$\ddot{F}^{(\sigma)} + (1-3q)\frac{\dot{F}^{(\sigma)}}{\tau} + [k_3^2(b\tau)^{6q-2}]F^{(\sigma)} = 0, \tag{42}$$

$$F^\sigma = \begin{cases} \tau^{3q/2}[c_1 Z_{\frac{|q|}{2q}}(\frac{|k_3 b^{3q-1}|\tau^{3q}}{3q}) + c_2 Z_{\frac{|q|}{2q}}(\frac{|k_3 b^{3q-1}|\tau^{3q}}{3q})]; & q \neq 0; \\ c_1\tau^\alpha + c_2\tau^\beta, & q = 0, b^2 \neq 4k_\perp^2 \\ \tau^\alpha(c_1 + c_2\log\tau), & q = 0, b^2 = 4k_\perp^2 \end{cases} \tag{43}$$

where

$$\alpha = \frac{1 \pm \sqrt{1 - 4k_\perp^2/b^2}}{2}, \beta = \frac{1 \mp \sqrt{1 - 4k_\perp^2/b^2}}{2}. \tag{44}$$

and Z_ν is a solution of Bessel equation of order ν.

In this paper we have found several exact solutions to the Maxwell equations in some anisotropic axisymmetric Bianchi type I cosmological models. The possibility of having a self-consistent model for the Einstein-Maxwell-Klein-Gordon equations, as well as the second quantization and particle production is under consideration and will be reported in a forthcoming paper.

VIII. ACKNOWLEDGMENTS

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