How to Measure a Beable

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Abstract

A brief discussion is given of measurement within the context of a theory of "beables", e.g. theories of de Broglie, Bohm, Bell, Vink, and also "modal" theories. It is shown that even in an ideal von Neumann measurement of a beable, the measured value may not agree with the value which the beable had prior to the measurement.

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1 Introduction

In standard quantum theory, the state of a system is described by a state vector $|\psi\rangle$, an element of the Hilbert space of possible states of the system. There have been several suggestions for "completing" this description by supposing that, unlike in standard quantum theory, a system could possess definite values of certain quantities even if the state vector was not an eigenstate of the operators associated with those quantities. In the theory of de Broglie [1] and Bohm [2], particles are taken to have, at each time, definite values of position. Bell [3] has proposed a theory in which it is the fermion number at (discretized) positions that has definite values. Vink [4] has shown how the formulation given by Bell can apply to any discrete quantity, and that in an appropriate limit this formulation reproduces the causal theory of Bohm. In the "modal" interpretations of van Fraassen [5], Kochen [6], Healey [7], Dieks [8], and Bub [9], the identity of the quantities whose values are definite can depend on the state vector, and so can be different at different times. In all of these theories [1-9] that we are considering, the time dependence of the state vector $|\psi\rangle$ is given by the Schrödinger equation; there is no "collapse" of the state vector.

Bell [3] has used the term "beable" to refer to a quantity whose value can be said to actually exist, as opposed to an "observable", which takes a value only when it is measured. Bell's intention was to have a theory in which the notion of measurement would not be fundamental; nevertheless, it is important to understand how and in what sense one could measure the value of a beable, and in this paper we will attempt to contribute to that understanding. In particular, we will discuss the question of whether the result of measuring a beable will necessarily correspond to the value that the beable had before the measurement.

First, let us define the notation we will use. We will denote by B a quantity which is a beable, and by $\hat{B}$ the corresponding Hilbert-space operator, and let us expand the state vector upon the eigenvectors of $\hat{B}$:

$$|\psi\rangle = \sum_i c_i |\psi_i\rangle \tag{1}$$

where $\hat{B}|\psi_i\rangle = b_i|\psi_i\rangle$. More generally, we can understand B to represent a collection of beables; if we are considering the theory of Bohm, we should understand the summation in eq. 1 to represent an integral, and the states $|\psi_i\rangle$ to be (unnormalized) position eigenstates. We are assuming that the system whose state vector is written in eq. 1 does possess a definite value of

B, which must equal b_i for some i; in the case in which the spectrum of $\hat{B}$ is not degenerate, this corresponds to a single term in the summation in eq. 1. We will denote by $|\psi\rangle^v$ the specification both of the state vector and of the value of B, and we will indicate the value of B by placing a bar above the appropriate term in the expansion of $|\psi\rangle$ upon the eigenvectors of $\hat{B}$. Thus for example we would write

$$|\psi\rangle^v = c_1\overline{|\psi_1\rangle} + c_2|\psi_2\rangle \tag{2}$$

in the case in which the state vector was $c_1|\psi_1\rangle + c_2|\psi_2\rangle$, and in which the value of B was b_1. It might be tempting to say that the bar in eq. 2 indicates that the first term on the right-hand side is the "correct" one, but more precisely eq. 2 indicates that the state vector contains *both* terms, and that the correct value of B corresponds to the first term. We can use the notation introduced in eq. 2 when discussing modal interpretations also, if we understand that B represents a quantity which is picked out by the state vector at the time at which eq. 2 is supposed to apply.

To make a measurement upon the system, let us couple it to a second system, which we shall call "the apparatus", and let the apparatus initially have state vector $|A_0\rangle$. We will represent the interaction between the system and the apparatus by an arrow, and assume, as usual, that for each i

$$|\psi_i\rangle|A_0\rangle \Longrightarrow |\psi_i\rangle|A_i\rangle \tag{3}$$

with $\langle A_i|A_j\rangle = \delta_{ij}$. Pauli has called such a measurement, which leaves the system in the eigenstate corresponding to the measured value, a "measurement of the first kind". Instead, we will use the term "von Neumann measurement" for any interaction between system and apparatus that satisfies eq. 3. It of course follows from eq. 3 that

$$\sum_i c_i|\psi_i\rangle|A_0\rangle \Longrightarrow \sum_i c_i|\psi_i\rangle|A_i\rangle \tag{4}$$

Since in the theories we are considering the state vector never collapses, the "result" of the measurement is reflected not in the final state vector, but instead in the final values of the beables. We will only consider cases in which each $|A_i\rangle$ is an eigenstate of beable operators for the apparatus; then, in the notation we have introduced above, we could indicate the result of the measurement by putting a bar above the appropriate term on the right-hand side of eq. 4.

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We will not be discussing any measurement of a quantity which is not a beable (e.g., a measurement of momentum for Bohm); we will consider only measurements of quantities which have definite values before the measurement. We will call a "faithful measurement" one in which the measured value of the beable is the same as the value before the measurement. For example,

$$[c_1\overline{|\psi_1\rangle} + c_2|\psi_2\rangle]\overline{|A_0\rangle} \Longrightarrow c_1\overline{|\psi_1\rangle|A_1\rangle} + c_2|\psi_2\rangle|A_2\rangle \tag{5}$$

represents a faithful measurement, but

$$[c_1\overline{|\psi_1\rangle} + c_2|\psi_2\rangle]\overline{|A_0\rangle} \Longrightarrow c_1|\psi_1\rangle|A_1\rangle + c_2\overline{|\psi_2\rangle|A_2\rangle} \tag{6}$$

represents a non-faithful measurement, in which before the measurement the value of B was b_1 but the result of the measurement (as reflected in the final value of the apparatus beables) was b_2.

Not every interaction that is a von Neumann measurement is faithful; whether or not an interaction is faithful can depend on the details of the Hamiltonian responsible for the interaction, even if we consider only those Hamiltonians which yield von Neumann measurements. We will show this in the next section by displaying two examples of Hamiltonians, both of which yield von Neumann measurements (as defined by eq. 3); one of these Hamiltonians yields measurements which are always faithful (as in eq. 5), but the other Hamiltonian yields measurements which are sometimes unfaithful (as in eq. 6). These examples assume the beable dynamics suggested by Bell [3] and elaborated upon by Vink [4], but in the third section we argue that with *any* beable dynamics von Neumann measurements will not necessarily be faithful.

2   Faithful and unfaithful examples

In the beable dynamics suggested by Bell [3], and elaborated upon by Vink [4], the probability that the value of the beable jumps from b_i to b_j in time dt is denoted by $T_{ij}dt$, and is given by

$$T_{ij} = \max[-2\operatorname{Im}(\tfrac{c_i}{c_j}H_{ji})\,,\ 0\,], \tag{7}$$

where H_{ji} is the matrix element of the Hamiltonian and c_i is defined in eq. 1. It follows from eq. 7 that the value of the beable B will be constant if H commutes with $\hat{B}$. This dynamics is of course stochastic; however,

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the examples we consider below will be determinate. To avoid confusion, we should note that we are not now considering the alternatives to eq. 7 suggested by Vink (eqs. 15 and 16 of ref. 4), nor will we be concerned with the further suggestion made by Vink that (in spite of the theorem of Kochen and Specker [10]) *all* quantities could be taken to be beables. We will only be interested in those beables which are, in fact, measured.

We want to consider a system (for which we use a superscript s) which is interacting with an apparatus (superscript a). We model each of them as a spin-$\frac{1}{2}$ particle, and take the beables to be the z components of the spin of each. Thus the basis of the combined system which corresponds to the $|\psi_i\rangle$ in eq. 1 will be the product basis, consisting of the four elements $|+\rangle^s|+\rangle^a$, $|+\rangle^s|-\rangle^a$, $|-\rangle^s|+\rangle^a$ and $|-\rangle^s|-\rangle^a$. Let $|\Psi\rangle$ denote the state vector for the combined system, and say that before the measurement we have

$$|\Psi\rangle_{t=0} = [c_+|+\rangle^s + c_-|-\rangle^s]|+\rangle^a. \tag{8}$$

<u>First example: a faithful measurement.</u> Let the interaction between s and a take place between times $t = 0$ and $t = \tau$, and take the Hamiltonian to be, during the interaction,

$$H = \frac{\pi}{4\tau}(I - \sigma_z)^s \otimes \sigma_y^a. \tag{9}$$

Note that we are taking this H to be the *full* Hamiltonian for the entire duration of the interaction. Then

$$
\begin{aligned}
U(\tau, 0) &= exp(-iH\tau) \\
&= \tfrac{1}{2}((I + \sigma_z)^s \otimes I^a - i(I - \sigma_z)^s \otimes \sigma_y^a).
\end{aligned}
\tag{10}
$$

From eqs. 8 and 10, we see that after the interaction we have

$$|\Psi\rangle_{t=\tau} = c_+|+\rangle^s|+\rangle^a + c_-|-\rangle^s|-\rangle^a. \tag{11}$$

Then if, in eqs. 8 and 11, we take alternatively $c_+ = 1$, $c_- = 0$ and $c_+ = 0$, $c_- = 1$, we can see that this interaction is indeed a von Neumann measurement (as in eq. 3), with the identifications $|A_0\rangle = |A_+\rangle = |+\rangle^a$ and $|A_-\rangle = |-\rangle^a$.

Since H commutes with σ_z^s, the value of the beable S_z^s will not change during the interaction; this means that we have a faithful measurement. In the notation introduced in eq. 2, suppose that before the interaction the bar belongs over the first term on the right-hand side of eq. 8; that is, suppose

$$|\Psi\rangle_{t=0}^v = c_+\overline{|+\rangle^s|+\rangle^a} + c_-|-\rangle^s|+\rangle^a. \tag{12}$$

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Then, since after the interaction the bar must still be over the state $|+\rangle^s$, and since that state only appears once on the right-hand side of eq. 11, we know that we have

$$|\Psi\rangle_{t=\tau}^v = c_+\overline{|+\rangle^s|+\rangle^a} + c_-|-\rangle^s|-\rangle^a. \tag{13}$$

Eqs. 12 and 13 (together with the analogous equations with the bar starting over the second term in eq. 8) tell us that we have a faithful measurement. In fact it is an obvious generalization of this example that, given the beable dynamics defined by eq. 7, any Hamiltonian which commutes with the measured beable, and which yields a von Neumann measurement, will yield a faithful measurement. This will be the case with the usual prescription, which is to take the Hamiltonian during the interaction to be proportional to the operator representing the measured quantity. However, this prescription is not included in the requirement (eq. 3) which we have used to define a von Neumann measurement.

 Second example: an unfaithful measurement. This time let the interaction occur between $t = 0$ and $t = 4\tau$, during which time the Hamiltonian is (with $\hbar = 1$)

$$H = \begin{cases} -\frac{\pi}{4\tau}\sigma_z^s \otimes \sigma_y^a & \text{for } 0 < t < \tau \\ -\frac{\pi}{2\tau}\sigma_x^s \otimes I^a & \text{for } \tau < t < 2\tau \\ +\frac{\pi}{4\tau}I^s \otimes \sigma_y^a & \text{for } 2\tau < t < 3\tau \\ +\frac{\pi}{2\tau}\sigma_x^s \otimes I^a & \text{for } 3\tau < t < 4\tau \end{cases} \tag{14}$$

Then

$$U(\tau,0) = \frac{1}{\sqrt{2}}(I^s \otimes I^a + i\sigma_z^s \otimes \sigma_y^a) \tag{15}$$

$$U(2\tau,\tau) = i\sigma_x^s \otimes I^a \tag{16}$$

$$U(3\tau,2\tau) = \frac{1}{\sqrt{2}}I^s \otimes (I - i\sigma_y)^a \tag{17}$$

$$U(4\tau,3\tau) = -i\sigma_x^s \otimes I^a, \tag{18}$$

and then

$$\begin{aligned} U(4\tau,0) &= U(4\tau,3\tau)U(3\tau,2\tau)U(2\tau,\tau)U(\tau,0) \\ &= \tfrac{1}{2}((I + \sigma_z)^s \otimes I^a - i(I - \sigma_z)^s \otimes \sigma_y^a). \end{aligned} \tag{19}$$

By comparing eqs. 10 and 19 , we can see that this second example has given us the same von Neumann measurement as did the first.

 Now let us take the initial state to be given by eq. 8 with the specification $c_+ = c_- = \frac{1}{\sqrt{2}}$. Let us also take the initial value of the beable S_z^s to be $+$

(the initial value of S_z^a is necessarily $+$); then we can write

$$|\Psi\rangle_{t=0}^v = \tfrac{1}{\sqrt{2}}\overline{|+\rangle^s|+\rangle^a} + \tfrac{1}{\sqrt{2}}|-\rangle^s|+\rangle^a. \tag{20}$$

Now from eq. 15 we see that

$$|\Psi\rangle_{t=\tau} = \tfrac{1}{2}|+\rangle^s|+\rangle^a - \tfrac{1}{2}|+\rangle^s|-\rangle^a + \tfrac{1}{2}|-\rangle^s|+\rangle^a + \tfrac{1}{2}|-\rangle^s|-\rangle^a. \tag{21}$$

Since for $0 < t < \tau$, H commutes with σ_z^s, the value of S_z^s is still $+$ at $t = \tau$, and so in eq. 21 the bar must go over one of the first two terms on the right-hand side. Because of the stochastic nature of the dynamics, we cannot say with certainty which one (in fact these two alternatives have equal probabilities); this ambiguity will be irrelevant when we get to the end of the calculation. So let's choose to put the bar over the first term; this gives

$$|\Psi\rangle_{t=\tau}^v = \tfrac{1}{2}\overline{|+\rangle^s|+\rangle^a} - \tfrac{1}{2}|+\rangle^s|-\rangle^a + \tfrac{1}{2}|-\rangle^s|+\rangle^a + \tfrac{1}{2}|-\rangle^s|-\rangle^a. \tag{22}$$

For $\tau < t < 2\tau$, eq. 7 implies that the values of the beables will not change. To see this, note that the solution of the Schrödinger equation in this time interval is

$$\begin{aligned}
|\Psi\rangle_t &= \tfrac{1}{2}[|+\rangle^s + |-\rangle^s]|+\rangle^a \exp[i\pi(t-\tau)/(2\tau)] \\
&\quad - \tfrac{1}{2}[|+\rangle^s - |-\rangle^s]|-\rangle^a \exp[-i\pi(t-\tau)/(2\tau)].
\end{aligned} \tag{23}$$

All matrix elements of H are real, and from eq. 23, the ratio c_i/c_j that appears in eq. 7 has the value one in all cases in which $H_{ji} \neq 0$. Thus $(c_i/c_j)H_{ji}$ is in all cases real, and so eq. 7 implies that all T_{ij} vanish. So the values of S_z^s and S_z^a at $t = 2\tau$ are the same as they were at $t = \tau$, namely both are $+$. Then using eqs. 16 and 21 we have

$$|\Psi\rangle_{t=2\tau}^v = \tfrac{i}{2}|-\rangle^s|+\rangle^a - \tfrac{i}{2}|-\rangle^s|-\rangle^a + \tfrac{i}{2}\overline{|+\rangle^s|+\rangle^a} + \tfrac{i}{2}|+\rangle^s|-\rangle^a. \tag{24}$$

For $2\tau < t < 3\tau$, H commutes with σ_z^s, so the value of S_z^s remains $+$; then from eq. 17 we have

$$|\Psi\rangle_{t=3\tau}^v = \tfrac{i}{\sqrt{2}}|-\rangle^s|+\rangle^a + \tfrac{i}{\sqrt{2}}\overline{|+\rangle^s|-\rangle^a}. \tag{25}$$

If in eq. 22 we had chosen to put the bar above the second term instead of the first, we would have obtained eq. 25 exactly as above. Finally, for

$3\tau < t < 4\tau$, H commutes with σ_z^a, so the value of S_z^a has the same value at $t = 4\tau$ as it did at $t = 3\tau$, namely $-$; then from eq. 18 we see

$$|\Psi\rangle_{t=4\tau}^v = \tfrac{1}{\sqrt{2}}|+\rangle^s|+\rangle^a + \tfrac{1}{\sqrt{2}}\overline{|-\rangle^s|-\rangle^a}. \tag{26}$$

By comparing eqs. 20 and 26, we see that this measurement is not faithful.

What is most surprising about this example is not that the final and initial values of S_z^s are different, but rather that the measured value of S_z^s as *read from the apparatus* differs from the initial value. In fact, the measured value (namely $-$) differs from the value that S_z^s had *for the entire time when s and a were actually interacting*. To see this, note that, from eq. 7, s and a evolve independently for $t > \tau$, and that S_z^s has the value $+$ for $0 < t < \tau$. In the terminology we are using, the "measurement" is not completed until $t = 4\tau$, when the state vector has the form given in eq. 4, with each $|A_i\rangle$ an eigenstate of apparatus beables.

We have seen that the Hamiltonians in our two examples, although they yield identical von Neumann measurements, treat the values of the beables differently. The Hamiltonian in the first example yields a faithful measurement, for any values of c_+ and c_- in eq. 8. The Hamiltonian in the second example would yield a faithful measurement for $c_+ = 1$, $c_- = 0$ or for $c_+ = 0$, $c_- = 1$ (as is required for any von Neumann measurement), but for $c_+ = c_- = \frac{1}{\sqrt{2}}$ it yields an unfaithful measurement.

3 Discussion

The second example of the preceding section shows that, assuming Bell-Vink dynamics for discrete beables (eq. 7), not every von Neumann measurement is faithful. In that example, the measured value of S_z^s differs from the value that S_z^s had *for the entire time when s and a were actually interacting*. Could this example be somehow exceptional? In the example, we took c_+ and c_- to be precisely equal. However, this choice was made to get a simple example in which, in spite of the stochastic nature of the dynamics, the measurement would always give the "wrong" answer. With other choices of c_+ and c_-, the answer would be wrong some of the time. It is complete faithfulness, not occasional unfaithfulness, which is the exceptional case.

Since Vink [4] has shown that, in an appropriate limit, this dynamics reproduces the causal theory of Bohm, one might therefore suspect that a von Neumann measurement of position in Bohm's theory might not correspond to the particle's position before the measurement. That this suspicion is

correct can be seen from the work of Englert, Scully, Sussman and Walther [11]. The example discussed in ref. 11, although not presented in quite the terms we are using here, can easily be modified to be an example of an unfaithful von Neumann measurement of position in Bohm's theory. (In fact, the second example presented in this paper could be considered as a discretized version of the example discussed in ref. 11.) Of course if one does make the usual choice (see for example p. 109 of ref. 12) of a Hamiltonian which is proportional to the position operator of the measured particle, one will be assured of obtaining a faithful measurement of position in Bohm's theory. In the terminology we are using here, the measurement is not over until the states $|A_i\rangle$ are eigenstates of apparatus beables. In the example discussed in ref. 11 (see also Dewdney, Hardy, and Squires [13]) this does not happen until the measured particle has moved away from the apparatus, and so the (full) Hamiltonian is *not* proportional to the position operator of the particle for the entire duration of the "measurement".

Vink [4] has also suggested alternative dynamics to the one suggested by Bell [3]. However, it is very difficult to see how *any* dynamics for discrete beables can avoid all examples of the type we have considered above, without putting restrictions on the Hamiltonian. The difficulty can be seen even without considering a measurement situation. So let's temporarily forget the apparatus, and say that the system s evolves for some time with a Hamiltonian $H = \frac{\pi}{4\tau}\sigma_y^s$. If at $t = 0$ the state vector is $|\psi\rangle_{t=0} = \frac{1}{\sqrt{2}}(|+\rangle^s + |-\rangle^s)$, at $t = \tau$ it will be $|\psi\rangle_{t=\tau} = |-\rangle^s$. Thus at $t = \tau$ the value of S_z^s will surely be $-$ (since there is no other term in $|\psi\rangle$), irrespective of its value at $t = 0$. So at $t = \tau$ the system has "forgotten" the value which S_z^s had at $t = 0$. Unless the beable dynamics were non-Markovian, there could not therefore be any correlation between the values of S_z^s at $t = 0$ and $t = 8\tau$, even though with this Hamiltonian we have $U(8\tau, 0) = I$. And if this kind of forgetfulness were to happen to our system in the middle of a measurement, there would be no relationship between the final (i.e., the measured) and the initial values of the beable.

We have been assuming that we are dealing with beables whose identity is independent of time, but that is not the case in modal theories [5-9]. Therefore it is not ruled out that there could be modal dynamics which would guarantee that any von Neumann measurement be faithful. However, this does not seem likely, since one would have thought that the necessity of taking different bases in eq. 1 at different times would make it harder, not easier, to guarantee faithfulness.

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What we have been calling a von Neumann measurement (eq. 3) is certainly not the most general interaction that should be considered to be a measurement; a "measurement of the second kind" can change the state of the system even if the system started out in an eigenstate of the quantity being measured. On the other hand, from the point of view of this paper, a von Neumann measurement is also not sufficiently restrictive, since it does not ensure faithfulness. For a von Neumann measurement with Bell-Vink (or its limiting case Bohmian) dynamics, it is at least possible to formulate a restriction on the Hamiltonian which *does* ensure that the measurement be faithful, namely that the Hamiltonian commute with the beable to be measured. Obviously this particular restriction could not be applied to a measurement of the second kind.

Acknowledgements

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