Building scars for integrable systems

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Abstract

It is shown, by means of a simple specific example, that for integrable systems it is possible to build up approximate eigenfunctions, called *asymptotic eigenfunctions*, which are concentrated as much as one wants to a classical trajectory and have a lifetime as long as one wants. These states are directly related to the presence of shell structures in the quantal spectrum of the system. It is argued that the result can be extended to classically chaotic system, at least in the asymptotic regime.

1. Introduction

The problem of quantising classically chaotic systems has attracted much attention in the last few years, especially in view of the subtle connection between classical and quantum mechanics in the semi-classical limit, generally referred as the $\hbar \to 0$ limit. Besides the well studied spectral properties of quantal systems, whose classical counterpart is chaotic, one of the most interesting and intriguing phenomenon is the appearance of the so called "scars", i.e. eigenfunctions which display a strong concentration of probability along a classical closed trajectory. This phenomenon is not yet well understood, and it has been observed in billiards as well as in smooth hamiltonian systems. It has been argued that the scars phenomenon can be directly related to the presence of a peak structure in the spectral density [1],[2], which seems to favour a particular stability of a wave-packet moving along a classically unstable trajectory. This problem has been considered in ref.[3] where it has been suggested that scars structures could be revealed through an energy averaging, in the framework of the Gutzwiller[4] formulation of the semi-classical limit. In order to shade some light on this problem, we consider the problem of establishing to what extent, for an *integrable* system, it is possible to construct an (approximate) eigenfuction which is close as much as possible to a classical trajectory. In the short wavelength limit, Erhenferst theorem assures that the motion of a wave packet can be as close as one wants to a classical trajectory, but, of course, an eigenfunction has to be time independent. Therefore, the motion of the wave packet in the semi-classical limit can only eventually suggest the particular superposition of wave packets which is (almost) time independent and still concentrated around the trajectory. However, to estimate the rate of spreading of a wave packet in the presence of a potential or a barrier is not an easy task, and can be performed only numerically, even for an integrable system. In the case of an integrable

system, one can try to build up directly a superposition of exact eigenstates in order to construct a wave packet which is concentrated along some classical trajectory and at the same time survives as long as one can. The essential point of the problem is to recognize that the lifetime of such a wave packet must be much longer than the characteristic time of the classical motion, namely the period of the trajectory, otherwise the wave packet cannot be considered an approximate eigenstate to any respect. This condition is *not* fulfilled in the treatment of ref.[3] for chaotic systems. We will show that in the case of integrable systems this condition can be fulfilled to any degree of precision, by building up a suitable superposition of eigenstates which are approximately degenerate, namely they belong to the same "shell" of the spectrum. It is hoped that this result could be of some help of solving the scars problem also in the case of chaotic systems, where the appearance of a peak in the density of state can be viewed as a sort of an "accidental" (approximate) shell. Actually, the wave functions belonging to this class, being strongly localized, are sensitive only to the local shape of the billiard. Therefore, their connection with classical orbits is not affected if the billiard shape is distorted along the contour where the wave function is essentially zero. This can include the chaotic cases, at least for asymptotically large quantum numbers.

2. Method

Let us consider a very simple integrable system with two degrees of freedom, the circular billiard. The system is trivially solvable, the eigenfunctions are cylindrical Bessel functions $J_l(\rho_{nl} r/R) \exp(il\phi)$, and the ρ_{nl} 's are the zeroes of the Bessel fuctions (BF). Here R is the billiard radius, r the radial coordinate and ϕ the angular coordinate. The quantum numbers n, l correspond to the quantisation of the radial and angular motion of the particle

respectively. This can be seen in the semi-classical limit, namely for large values of the quantum numbers, for which the asymptotic form of the BF[5] gives the semi-classical (Bhor-Sommerfeld) quantisation condition

$$\sqrt{k_{nl}^2 R^2 - l^2} - l\beta_0 = (2n+1)\pi/2 + \pi/4 \tag{1}$$

and it is readly verified that right hand side is just the action integral along the classical radial motion. For future considerations, it is convinient to keep in mind that the (constant) angular distance $2\beta_0$ between two successive hits of the particle at the billiard wall is given by $\cos(\beta_0) = l/(k_{nl}R)$. Shell structures are associated with closed orbits [6]. In fact, the condition of k_{nl} being stationary for variations Δn and Δl of the quantum numbers, from eq. (1), reads

$$\frac{\beta_0}{\pi} = l_0/(\rho_{n_0 l_0}) = \frac{\Delta n}{\Delta l} = \frac{p}{q} \tag{2}$$

being p and $q > p$ two integer numbers with no common prime factor. This implies that the corresponding classical trajectory closes after $m = pq$ hits at the wall, and the quantum numbers n and l are linearly related. Eq. (2) imposes also a condition for the quantum numbers l_0 and n_0, which asymptotically can be satisfied with arbitrary precision. The states which have quantum numbers $l = l_0 + \Delta l$ and $n = n_0 + \Delta n$ satisfying the linear condition of eq. (2) are approximately degenerate, and therefore form an energy shell, which, in turn, implies the appearance of a sharp peak in the density of states. We want to show now, by elementary considerations, that for large enough quantum numbers n_0 and l_0 one can construct linear superpositions of the eigenstates belonging to the same shell, which satisfy *at the same time* the two conditions, a) they are concentrated with arbitrary precision along a classical (closed) trajectory b) they have a lifetime arbitrarly longer than the classical characteristic time $T = M/\rho_{nl}\hbar$, being M the mass of the particle. Because

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of the simple symmetry of the system, each eigenfunction has a rotationally invariant probability density. A wave packet with a narrow angular spread $\Delta\phi = 1/\Delta_l$ can be written

$$\Psi_{l_0}(r,\phi) = \sum_{nl} \exp\left(\frac{(l-l_0)^2}{2\Delta_l^2}\right)\exp(il\phi)J_l(k_{nl}r) \tag{3}$$

In order to minimize the energy spread, we restrict the summation to the eigenstates belonging to the same shell. Asymptotically, for large quantum numbers, the summation can be performed by expanding the phase of the BF around the chosen value l_0 of the angular momentum and taking into account the condition of stationary value of k_{nl} and the corresponding linear relation between the quantum numbers n and l. The result reads

$$\Psi_{l_0}(r,\phi) \sim \exp\left(\frac{(\phi-\beta(r))^2}{2\Delta\phi^2}\right) \tag{4}$$

where $\cos(\beta(r)) = l_0/(k_{n_0 l_0}r)$, and $k_{n_0 l_0} = \rho_{n_0 l_0}/R$ is the eigenmomentum. The wave packet of eq. (3) is clearly concentrated around the classical trajectory, which is a polygon or a star with m sites, with a spatial spread $\Delta s \sim R\Delta\phi \sim R/\Delta_l$. The energy spread ΔE can be estimated in terms of the second derivative k'' of k_{nl} at $n_0 l_0$ along the direction defined by eq. (2), $\Delta E/E = (k''/k_{nl})\Delta_l^2$. After some algebra, one gets $\Delta E/E = g(\Delta_l/\rho_{n_0 l_0})^2$, where g is a costant factor of order unity, and it can be checked that the higher order terms are vanishing small for asymptotic quantum numbers. The ratio between the quantal lifetime $\tau_q = \hbar/\Delta E$ and the classical characteristic time T turns out

$$\frac{\tau_q}{T} \sim \left(\frac{\Delta s}{R}\right)^2 l_0 \tag{5}$$

This ratio, for a fixed value of the localization Δs, can be made arbitrarly large by increasing the values of the quantum numbers. Had we chosen a different linear combination, on the contrary this ratio would have been of order unity. In other words, the uncertainity

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ΔE, with the contraint of eq. (2), is asymptotically of the same order of the mean level spacing $n(E) = 2\pi MR^2/\hbar^2$. An example, corresponding to $p = 1$ and $q = 3$ is depicted

Table 1

l_0	n_0	ρ
111	30	241.87
114	29	242.00
117	28	242.09
120	27	242.14
123	26	242.13
126	25	242.07
129	24	241.96

in Fig. 1. The wave function is calculated with the expression of eq. (4) and with the full expansion of eq. (3) in part a and b respectively. Table 1 reports the quantum numbers and energies used in the calculation. One can notice that a high degree of localization can be obtained with only few terms and not too large quantum numbers. In the example $\tau_q/T \sim 15$. Higher degree of localization and longer lifetime can be obtained following the above described procedure.

3. Discussion and conclusions

The wave function depicted in Fig. 1 has a striking similarity with some scars reported in ref. [7] for a chaotic billiard. The authors show that this type of scars "live" in thin invariant tori embedded in a chaotic region. In the procedure of the present paper the integrable billiard can be deformed in regions of the contour where the wave function is vanishing small, and the billiard could become of the chaotic type leaving the scars essentially untouched. This could be a mechanism of generating scars in a chaotic system. Of course, the procedure does not exhaust all the possibilities, and indeed in ref. [7] many examples are shown where the scars "live" in the classically chaotic region (in the Wigner tranform sense). Another possibility of generalizing the procedure to chaotic billiard is the case of "local integrability", described in details in ref. [8]. In this case the presence of an adiabatic barrier allows to expand the hamiltonian, around a closed trajectory, in the longitudinal and transverse actions, which are approximate constants of the motion in the vicinity of the trajectory. In this case the present procedure can be repeated step by step, substituting the angular momentum with the transverse action and the angular coordinate with the transverse coordinate.

In conclusion we have presented a procedure to build up scars for integrable systems. Generalizing the method to chaotic systems appears possible, at least when thin invariant tori exist or local integrability is present. The extension of the method to more generic cases is under study and the results will be reported elsewhere.

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Figure captions

- 1.a Contour plot of the probability density associated to the wave function (3). Fixing an angular spread $\Delta\varphi = 0.25$, and considering the contribution of only 6 values of l centered around $l_0 = 120$ (cfr. table 1), we have obtained a ratio $\frac{\tau_q}{T} = 15$

- 1.b Contour plot of the probability density associated to the wave function (4) which is obtained from (3) expanding the phase of the BF around l_0 taking into account the condition of stationary value of $k_n l$ and replacing the summation in (3) by an integral over l.