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L^AT_EX-ed on August 2, 1995**VORTEX PRODUCTION AT PHASE TRANSITIONS
IN NONRELATIVISTIC AND RELATIVISTIC MEDIA**

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ABSTRACT

We examine string (vortex) formation at a quench for a weakly-coupled global U(1) theory when the excitation spectrum is non-relativistic. It is so similar to vortex production in the corresponding relativistic plasma as to reinforce arguments for the similarity of vortex production in the early universe and in low-temperature many-body physics.

1. Introduction

For many years it has been argued by Kibble¹ and others² that the large-scale structure of the universe can be attributed to cosmic strings formed at phase transitions in the Grand Unification era. Unfortunately, given the unlikely event of observing a cosmic string (vortex) directly it is difficult to make the case compelling.

However, the formation of topological defects like vortices during symmetry-breaking phase transitions is not unique to the early universe but generic to many physical systems. In particular, recent experiments on the production of vortices in superfluid ⁴He³ and ³He⁴ have excited considerable interest. A priori, we might expect the early universe to have little in common with many-body systems. Its highly relativistic regime is characterised by initial temperatures (or energies) $T \gg m$ for all particle masses m (in units $k_B = c = 1$), whereas low temperature non-relativistic media are characterised by $T \ll m$, to freeze out antiparticles, but with $T \simeq \mu_{\text{nr}}$, the non-relativistic chemical potential. Nonetheless, the proposition has been made by the experimental groups concerned that these, and other experiments⁵, may provide an insight on the early universe. In this they have been championed by some theoretical astroparticle physicists, most notably Zurek⁶, one of whose most recent articles⁷ was, indeed, titled *Cosmological Experiments in Superfluids and Superconductors*.

In this talk I shall indicate how comparisons between vortex production in these very different environments can be more than superficial analogies by sketching out some preliminary quantitative steps. The talk has three components.

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Firstly, I shall remind you of the generalities of vortex production and of the way a network of simple Z -vortices is characterised by field configuration probabilities. The next step is to show how, by varying the chemical potential, it is possible to interpolate between a relativistic scalar theory (which, in curved space-time, could be of the early universe), and a non-relativistic field theory of condensed bosons. Finally, these ideas will be given a concrete realisation in a model of Z -vortex formation by unstable long-wavelength Gaussian fluctuations.

The reader seeking greater detail will find it in recent work by myself and my collaborators, Tim Evans and Alasdair Gill^{8,9,10,11}, from which this talk is drawn.

2. Vortex Production and Vortex Distributions

What encourages us in the hope that vortex production in the early universe and the laboratory has close parallels is that the most plausible production mechanism refers to neither. We recapitulate it now for the simplest theory permitting vortices, that of a complex scalar field $\phi(\mathbf{x}, t)$. The complex order parameter of the theory is $\langle \phi \rangle = \eta e^{i\alpha}$ and the theory possesses a global $O(2)$ symmetry that we take to be spontaneously broken at a phase transition. The field ϕ could be either a complex non-relativistic order field appropriate to a superfluid or a relativistic field in the early universe.

Initially, we take the system to be in the symmetry-unbroken (disordered) phase. We have no reason to choose any particular initial field configuration, beyond the requirement that the field is distributed about $\phi = 0$ with zero mean. The simplest assumption is that, beginning at some time $t = t_0$, the $O(2)$ symmetry of the ground-state (vacuum) is broken by a rapid change in the environment inducing an explicit time-dependence in the field parameters. Once this quench is completed the ϕ -field potential $V(\phi) = -a|\phi|^2 + b|\phi|^4$ is taken to have the symmetry-broken form $a > 0$, $b > 0$ of the familiar ‘wine-bottle’ bottom. The ground-state manifold (the circle S^1 , labelled by the phase α of $\langle \phi \rangle$) is infinitely connected and the theory possesses *global* strings or vortices, labelled by a winding number $n \in \mathbb{Z}$.

These strings cost considerable energy to produce. That they should appear at all follows from a general argument, due to Kibble¹, which goes as follows. During the transition, the complex scalar field begins to fall from the false ground-state into the true ground-state, choosing a point on the ground-state manifold at each point in space. For continuous transitions, for which this collapse to the true ground-state occurs by phase separation, the resulting field configuration is expected to be one of domains within each of which the scalar field has relaxed to a constant ground-state value.

If this is so, then the requirements of continuity and single valuedness will sometimes force the field to remain in the false ground-state between some of the domains. For example, the phase of the field may change by an integer multiple of 2π on going round a loop in space. This requires at least one *zero* of the field within the loop, each of which has topological stability and characterises a vortex passing

through the loop. The density of strings is then closely linked to the number of effective domains and the evolution of this density is, correspondingly, linked to the nature of the domain growth. When the phase transition is complete and there is no longer sufficient thermal energy available for the field to fluctuate into the false ground-state, the topological defects are frozen into the field.

Consider an ensemble of systems evolving from one of a set of disordered states whose relative probabilities are known, to an ordered state as indicated above. If the phase change begins at time t_0 then, for $t > t_0$, it is possible in principle to calculate the probability $p_t[\Phi]$ that the complex field $\phi(\mathbf{x}, t)$ takes the value $\Phi(\mathbf{x})$ at time t . Throughout, it will be convenient to decompose Φ into real and imaginary parts as $\Phi = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2)$ (and ϕ accordingly). This is because we wish to track the field as it falls from the unstable ground-state hump at the centre of the potential to the ground-state manifold in Cartesian field space.

The calculation of $p_t[\Phi]$ will be performed later in our simple model. For the moment, consider it given. The question is, how can we infer the string densities and the density correlations from $p_t[\Phi]$? That we can calculate them at all is a consequence of the fact, noted earlier, that the string core is a line of field zeroes. This is equally true for both relativistic and non-relativistic $O(2)$ theories. The zeroes of Φ_a ($a=1,2$) which define the vortex positions form either closed loops or ‘infinite’ string i.e.string that does not intersect itself. Following Halperin¹² we define the *topological line density* $\vec{\rho}(\mathbf{r})$ by

$$\vec{\rho}(\mathbf{r}) = \sum_n \int ds \frac{d\mathbf{R}_n}{ds} \delta^3[\mathbf{r} - \mathbf{R}_n(s)]. \quad (1)$$

In (2.1) ds is the incremental length along the line of zeroes $\mathbf{R}_n(s)$ ($n=1,2,..$) and $\frac{d\mathbf{R}_n}{ds}$ is a unit vector pointing in the direction which corresponds to positive winding number. Only winding numbers $n = \pm 1$ are considered, with higher winding numbers understood as describing multiple zeroes. If dA_j is an incremental two-dimensional surface containing the point \mathbf{r} , whose normal is in the j th direction, then $\rho_j(\mathbf{r})$ is the net density of strings (i.e the density of strings *minus* the density of antistrings on dA_j).

Ensemble averaging $\langle F[\Phi] \rangle_t$ at time t is understood as averaging over the field probabilities $p_t[\Phi]$ as

$$\langle F[\Phi] \rangle_t = \int \mathcal{D}\Phi p_t[\Phi] F[\Phi]. \quad (2)$$

In general this ensemble averaging is not thermal averaging since, out of equilibrium, we have no Boltzmann distribution. We shall only consider situations in which

$$\langle \rho_j(\mathbf{r}) \rangle_t = 0, \quad (3)$$

i.e. an equal likelihood of a string or an antistring passing through an infinitesimal area. However, the line density correlation functions

$$C_{ij}(\mathbf{r}; t) = \langle \rho_i(\mathbf{r}) \rho_j(\mathbf{0}) \rangle_t \quad (4)$$

will be non-zero, and give information on the persistence length of strings.

It follows that, in terms of the zeroes of $\Phi(\mathbf{r})$, $\rho_i(\mathbf{r})$ can be written as

$$\rho_i(\mathbf{r}) = \delta^2[\Phi(\mathbf{r})] \epsilon_{ijk} \partial_j \Phi_1(\mathbf{r}) \partial_k \Phi_2(\mathbf{r}), \quad (5)$$

where $\delta^2[\Phi(\mathbf{r})] = \delta[\Phi_1(\mathbf{r})] \delta[\Phi_2(\mathbf{r})]$. The coefficient of the δ -function in Eq.5 is the Jacobian of the transformation from line zeroes to field zeroes. It permits us to define a further line density that we shall also find useful, the *total line density* $\bar{\rho}(\mathbf{r})$

$$\bar{\rho}_i(\mathbf{r}) = \delta^2[\Phi(\mathbf{r})] |\epsilon_{ijk} \partial_j \Phi_1(\mathbf{r}) \partial_k \Phi_2(\mathbf{r})|. \quad (6)$$

Unlike the case for $\rho_i(\mathbf{r})$

$$n(t) = \langle \bar{\rho}_i(\mathbf{r}) \rangle_t > 0 \quad (7)$$

and measures the *total* string density in the direction i , without regard to string orientation. The isotropy of the initial state guarantees that $n(t)$ is independent of the direction i . We note that the Jacobian factor multiplying the field δ -functions in Eq.5 and Eq.6 guarantees that random field zeroes with no vorticity will not be counted.

In general, the best that we can do is write $\langle \bar{\rho}_i(\mathbf{r}) \rangle_t$ as

$$\begin{aligned} \langle \bar{\rho}_i(\mathbf{r}) \rangle_t &= \int \mathcal{D}\Phi \, p_t[\Phi] \, \delta^2[\Phi(\mathbf{r})] |\epsilon_{ijk} \partial_j \Phi_1(\mathbf{r}) \partial_k \Phi_2(\mathbf{r})| \\ &= \int \mathcal{D}\alpha \, \langle |\epsilon_{ijk} \partial_j \Phi_1(\mathbf{r}) \partial_k \Phi_2(\mathbf{r})| e^{i \int d\mathbf{x} \alpha_a \Phi_a} \rangle_t. \end{aligned} \quad (8)$$

The simple model that we shall propose later assumes that $p_t[\Phi(\mathbf{r})]$ is Gaussian. In this case the system is solvable, as we shall now show. Let us assume that

$$\begin{aligned} \langle \Phi_a(\mathbf{r}) \rangle_t &= 0, \\ \langle \Phi_a(\mathbf{r}) \partial_j \Phi_b(\mathbf{r}) \rangle_t &= 0, \end{aligned} \quad (9)$$

and, further, that

$$\begin{aligned} \langle \Phi_a(\mathbf{r}) \Phi_b(\mathbf{r}') \rangle_t &= W_{ab}(|\mathbf{r} - \mathbf{r}'|; t) \\ &= \delta_{ab} W(|\mathbf{r} - \mathbf{r}'|; t), \end{aligned} \quad (10)$$

is diagonal. All other connected correlation functions are taken to be zero. Then all ensemble averages are given in terms of $W(r; t)$, where $r = |\mathbf{r}|$. In particular, $\langle \bar{\rho}_i(\mathbf{r}) \rangle_t$ separates as

$$\langle \bar{\rho}_i(\mathbf{r}) \rangle_t = \langle \delta^2[\Phi(\mathbf{r})] \rangle_t \langle |\epsilon_{ijk} \partial_j \Phi_1(\mathbf{r}) \partial_k \Phi_2(\mathbf{r})| \rangle_t \quad (11)$$

or, equivalently, from Eq.8

$$\langle \bar{\rho}_i(\mathbf{r}) \rangle_t = \int \mathcal{D}\alpha \, \langle |\epsilon_{ijk} \partial_j \Phi_1(\mathbf{r}) \partial_k \Phi_2(\mathbf{r})| \rangle_t \langle e^{i \int d\mathbf{x} \alpha_a \Phi_a} \rangle_t. \quad (12)$$

It follows¹², on first performing the α integration in the second factor, that

$$n(t) = \frac{1}{2\pi} \left| \frac{W''(0; t)}{W(0; t)} \right|, \quad (13)$$

where primes on W denote differentiation with respect to r . Thus

$$n(t) = O\left(\frac{1}{\xi^2}\right), \quad (14)$$

where ξ is the length at time t that sets the scale in $W(r; t)$.

The density-density correlation functions

$$C_{ij}(\mathbf{r}; t) = A(r; t)\delta_{ij} + B(r; t)\left(\frac{r_i r_j}{r^2} - \delta_{ij}\right), \quad (15)$$

have a much more complicated realisation¹² in terms of W and its derivatives. We shall be looking for signs of anticorrelation, which enables us to determine the persistence length of strings in the network (i.e. how bendy they are). The bendier, the more string that will occur in small loops. This is important, since large scale structure formation in the universe requires a certain amount of infinite string.

Nothing that we have said so far discriminates between vortices in a relativistic or a non-relativistic medium. This distinction appears in the definition of $\langle \dots \rangle_t$, to which we now turn.

3. The Relationship Between Relativistic and Non-Relativistic Media

In order to compare vortex formation in relativistic and non-relativistic media it is convenient to develop the formalism of the relativistic field to accomodate both. For simplicity we assume in each case that the system is initially in *equilibrium* in the disordered phase. The interpolation between relativistic and non-relativistic regimes is then effected by introducing the chemical potential μ into the relativistic theory, coupled to the conserved charge Q (particle number minus antiparticle number) arising from the $O(2)$ symmetry. Provided μ is small in comparison to a particle mass the introduction of such a potential will have little effect on a phase transition for the relativistic theory initiated by quenching the system from a high temperature T_0 to, effectively, zero temperature. However, on increasing μ prior to the transition it becomes more costly to produce antiparticles and, if the initial temperature T_0 is decreased to a value much less than μ , such antiparticles as are produced are frozen out. The system is then one of non-relativistic particles at a temperature much less than the particle rest masses. In this nonrelativistic regime the transition is induced by a quench in μ itself or equivalently, since μ determines the density, by a density (pressure) quench.

There are two separate problems. The first is to show explicitly how the non-relativistic limit occurs. The second is to relate this non-relativistic limit (of a

relativistic field) to a conventional a priori non-relativistic field theory of many-body physics. However, before we can do either we need to discuss non-equilibrium field dynamics.

3.1. Field Dynamics with a Chemical Potential

From the viewpoint given above the chemical potential is seen as determining the initial conditions for the subsequent dynamics, for which we adopt the closed time path method (Schwinger-Keldysh formalism)^{13,14,15,16}. We have already assumed that, at the initial time t_0 , we are in a disordered state with $\langle \phi \rangle = 0$. Our ignorance is parametrised by the probability distribution $p_{t_0}[\Phi]$ that, at time t_0 , $\phi(t_0, \mathbf{x}) = \Phi(\mathbf{x})$. For the moment we take it as given. Whether we are in a relativistic or non-relativistic regime is largely encoded in $p_{t_0}[\Phi]$. The subsequent, essentially generic, non-equilibrium field evolution is driven by a rapid change in the environment. Specifically, for $t > t_0$ the action for the field is taken to be

$$S[\phi] = \int d^4x \left(\frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2} m^2(t) \phi_a^2 - \frac{1}{4} \lambda(t) (\phi_a^2)^2 \right). \quad (16)$$

where $m(t)$, $\lambda(t)$ describe the evolution of the parameters of the theory under external influences, to which the field responds. There is no penalty in using the Lorentz covariant Eq.16, even for non-relativistic media. If, initially, there are no antiparticles at low temperature, there will remain no antiparticles. As with Φ , it is convenient to decompose ϕ in terms of two massive real scalar fields ϕ_a , $a = 1, 2$ as $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, in terms of which $S[\phi]$ shows a global $O(2)$ invariance, broken by the mass term if $m^2(t)$ is negative.

The change of phase that begins at time t_0 will, by the mechanism indicated earlier, lead to the appearance of vortices. We saw in the previous section that the vortex distributions at later times $t_f > t_0$ can be read off from the probability $p_{t_f}[\Phi_f]$ that the measurement of ϕ will give the value Φ_f . The evolution of $p_t[\Phi]$ from t_0 to t_f is most simply written as a closed time-path integral in which the field ϕ is integrated along the closed path $C_+ \oplus C_-$ of Fig.1, where $\phi = \phi_+$ on C_+ and $\phi = \phi_-$ on C_- . If $\mathcal{D}\phi_\pm = \prod_{a=1}^2 \mathcal{D}\phi_{\pm,a}$ and spatial labels are suppressed then

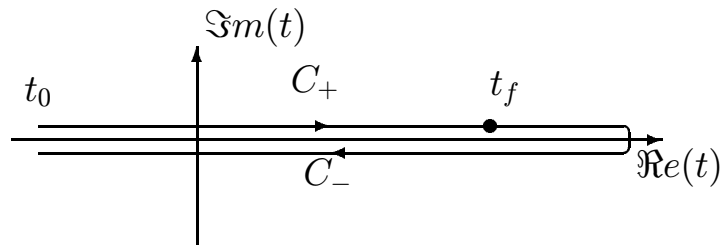


Figure 1: The closed timepath contour $C_+ \oplus C_-$.

$$p_{t_f}[\Phi_f] = \int \mathcal{D}\Phi p_{t_0}[\Phi] \int_{\phi_{\pm}(t_0)=\Phi} \mathcal{D}\phi_+ \mathcal{D}\phi_- \delta[\phi_+(t_f) - \Phi_f] \exp\left\{i\left(S[\phi_+] - S[\phi_-]\right)\right\}. \quad (17)$$

where $\delta[\phi_+(t) - \Phi_f]$ is a delta functional, imposing the constraint $\phi_+(\mathbf{x}, t) = \Phi_f(\mathbf{x})$ for each \mathbf{x} . This is no more than the statement that, for a given initial state, the probability amplitude is given by the integration along C_+ , and its complex conjugate (which, when multiplied with it, gives the probability) is given by the integration back along C_- . The \pm two-field notation is misleading in that it suggests that the ϕ_+ ($= \phi_{a,+}$) and ϕ_- fields are decoupled. That this is not so follows immediately from the fact that $\phi_+(t_f) = \phi_-(t_f)$.

To return to the initial conditions, the simple analytic results of the previous section could only be achieved if $p_t[\Phi]$ were Gaussian and, therefore, that $p_{t_0}[\Phi]$ itself be Gaussian. This may not be unrealistic. The simplest such distribution has Φ Boltzmann distributed at time t_0 at a temperature of $T_0 = \beta_0^{-1}$ and chemical potential μ according to a *free*-field Hamiltonian H_0 , which we take as

$$H_0 = \int d^3x \left[\frac{1}{2} \pi_a^2 + \frac{1}{2} (\nabla \phi_a)^2 + \frac{1}{2} m^2 \phi_a^2 \right]. \quad (18)$$

where $\pi_a = \dot{\phi}_a$ in real time. We stress that Gaussian does not necessarily mean *free*. Most simply, particles need to interact before they can equilibrate. More importantly, the free-field Gaussian approximation adopted here can be extended to include interactions self-consistently in a Hartree approximation¹⁷.

Since chemical potentials are not usually relevant to relativistic bosons a few words are in order. The $O(2)$ invariance of H_0 leads to a conserved Noether current with conserved charge

$$\begin{aligned} Q &= \int d^3x (\phi_2 \pi_1 - \pi_2 \phi_1) \\ &= \int d^3x (\phi_2 \dot{\phi}_1 - \dot{\phi}_2 \phi_1). \end{aligned} \quad (19)$$

The numerical value of Q is the number of particles minus the number of antiparticles. The thermal probability distribution $p_{t_0}[\Phi]$ is thus taken to be

$$p_{t_0}[\Phi] = \langle \Phi, t_0 | e^{-\beta_0(H_0 - \mu Q)} | \Phi, t_0 \rangle. \quad (20)$$

We note that interactions with weak coupling will not change $p_{t_0}[\Phi]$ significantly.

There are two ways to proceed. The first, which to us is the most natural, accepts H_0 as determining the temporal evolution of the fields, and relegates μ to the initial boundary conditions on the fields. From this viewpoint, $p_{t_0}[\Phi]$ can be written as the imaginary-time path integral

$$p_{t_0}[\Phi] = \int_{B_\mu[\Phi]} \mathcal{D}\phi \exp\left\{iS_0[\phi]\right\}, \quad (21)$$

for a corresponding action

$$S_0[\phi] = \int d^4x \left[\frac{1}{2} (\partial_\nu \phi_a) (\partial^\nu \phi_a) - \frac{1}{2} m^2 \phi_a^2 \right]. \quad (22)$$

The boundary condition $B_\mu[\Phi]$ incorporates the chemical potential. In terms of the Q eigenstates, the complex field $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ and its adjoint, $B_\mu[\Phi]$ is

$$B_\mu[\Phi] : \phi(t_0) = \Phi = e^{-\beta_0\mu q}\phi(t_0 - i\beta_0), \quad (23)$$

where $q = 1$ is the ϕ -field eigenvalue of Q and the time-integral in Eq.22 is taken in imaginary time from t_0 to $t_0 - i\beta_0$. In terms of the same complex fields we have

$$S_0[\phi] = \int d^3x [(\partial_\nu\phi^*)(\partial^\nu\phi) - m^2\phi^*\phi] \quad (24)$$

and, with $\pi = (\pi_1 - i\pi_2)/\sqrt{2}$,

$$H_0 = \int d^3x [\pi^*\pi + (\nabla\phi^*)(\nabla\phi) + m^2\phi^*\phi]. \quad (25)$$

However, for calculating vortices a decomposition in terms of ϕ_a, π_a is usually preferable.

Although we are just setting an initial condition the effect is, inevitably, to give an action $S_0[\phi]$ of the form of $S[\phi]$ of Eq.16. This permits the interpretation that the action $S[\phi]$ is valid for all times t , with the proviso that the system is in thermal equilibrium for $t < t_0$, during which period the mass $m(t)$ takes the constant value m and $\lambda(t) = 0$.

On relabelling the integration variable ϕ of Eq.21 by ϕ_3 , we now have the explicit form for $p_{t_f}[\Phi_f]$:-

$$\begin{aligned} p_{t_f}[\Phi_f] &= \int \mathcal{D}\Phi \int_{B_\mu[\Phi]} \mathcal{D}\phi_3 e^{iS_0[\phi_3]} \int_{\phi_\pm(t_0)=\Phi} \mathcal{D}\phi_+ \Phi \mathcal{D}\phi_- e^{i(S[\phi_+] - S[\phi_-])} \delta[\phi_+(t_f) - \Phi_f] \\ &= \int_{B_\mu} \mathcal{D}\phi_3 \mathcal{D}\phi_+ \mathcal{D}\phi_- \exp\left\{iS_0[\phi_3] + i(S[\phi_+] - S[\phi_-])\right\} \delta[\phi_+(t_f) - \Phi_f], \end{aligned} \quad (26)$$

where the boundary condition B_μ is now (in terms of the field combinations $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$)

$$B_\mu : \phi_+(t_0) = e^{-\beta_0\mu q}\phi_3(t_0 - i\beta_0). \quad (27)$$

More succinctly, $p_{t_f}[\Phi_f]$ can be written as the time ordering of a single field doublet:-

$$p_{t_f}[\Phi_f] = \int_{B_\mu} \mathcal{D}\phi e^{iS_C[\phi]} \delta[\phi_+(t_f) - \Phi_f], \quad (28)$$

along the contour $C = C_+ \oplus C_- \oplus C_3$ of Fig.2, extended to include a third imaginary leg, where ϕ takes the values ϕ_+, ϕ_- and ϕ_3 on C_+, C_- and C_3 respectively, for which S_C is $S[\phi_+], S[\phi_-]$ and $S[\phi_3]$, for which last case $m(t) = m, \lambda(t) = 0, t \in C_3$.

As a final step in these formal manipulations we see that expression Eq.28 enables us to write the Φ -field ensemble averages $\langle \dots \rangle_t$ in terms of the ϕ -field Wightman functions. Consider the generating functional:-

$$Z_\mu[j_+, j_-, j_3] = \int_{B_\mu} \mathcal{D}\phi \exp\left\{iS_C[\phi] + i \int j\phi\right\}, \quad (29)$$

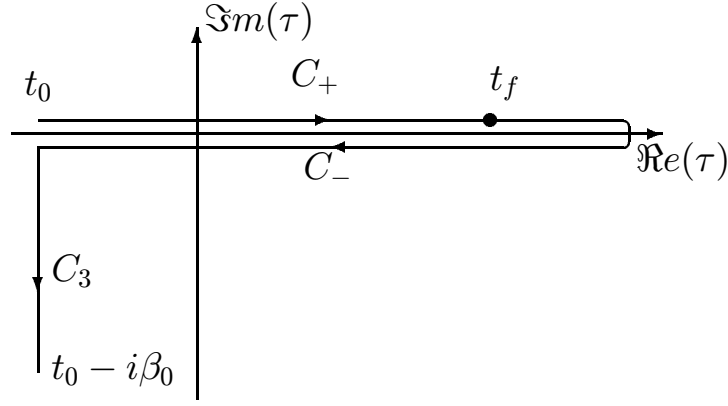


Figure 2: A third imaginary leg

where $\int j\phi$ is shorthand for:-

$$\int j\phi \equiv \int_0^\infty dt [j_+(t)\phi_+(t) - j_-\phi_-(t)] + \int_0^{-i\beta} j_3(t)\phi_3(t) dt, \quad (30)$$

omitting spatial arguments. Then introducing $\alpha_a(\mathbf{x})$ where $a = 1, 2$, we find:-

$$\begin{aligned} p_{t_f}[\Phi] &= \int \mathcal{D}\alpha \int_{B_\mu} \mathcal{D}\phi \exp\{iS_C[\phi]\} \exp\left\{i \int d^3x \alpha_a(\mathbf{x}) [\phi_+(t_f, \mathbf{x}) - \Phi(\mathbf{x})]_a\right\} \\ &= \int \mathcal{D}\alpha \exp\left\{-i \int \alpha_a \Phi_a\right\} Z_\mu[\bar{\alpha}, 0, 0], \end{aligned} \quad (31)$$

where $\bar{\alpha}$ is the source $\bar{\alpha}(\mathbf{x}, t) = \alpha(\mathbf{x})\delta(t - t_f)$. As with $\mathcal{D}\phi$, $\mathcal{D}\alpha$ denotes $\prod_1^N \mathcal{D}\alpha_a$.

Ensemble averages are now expressible in terms of Z_μ . Of particular relevance,

$$\begin{aligned} W_{ab}(|\mathbf{r} - \mathbf{r}'|; t) &= \langle \Phi_a(\mathbf{r}) \Phi_b(\mathbf{r}') \rangle_t \\ &= \langle \phi_a(\mathbf{r}, t) \phi_b(\mathbf{r}', t) \rangle, \end{aligned} \quad (32)$$

the equal-time Wightman function with the given thermal boundary conditions. Because of the time evolution there is no time translation invariance in the double time label.

Not surprisingly, $p_t[\Phi]$ can only be calculated explicitly in very simple circumstances, like our Gaussian approximation, but before we do that we still have to extract the relativistic limit. Since the chemical potential is embedded in the equilibrium boundary conditions and, like temperature, can only be defined in equilibrium, this is essentially an equilibrium problem and, for the moment, we forget the dynamics.

3.2. Deriving the Non-Relativistic Limit

The expression Eq.28 is valid for all μ and all T_0 but, as it stands, is not sympathetic to the isolation of a non-relativistic limit in a form with which we are familiar.

Nonetheless, the extraction of a non-relativistic regime from it is not difficult. Instead of working with $p_{t_0}[\Phi]$ directly, we can work with the partition function Z_μ , as we saw in the previous section. The partition function for this equilibrium theory is (for doublet sources j_a restricted to the C_3 -contour) $Z_\mu[0, 0, j_3]$ of Eq.29, written

$$Z_\mu[j] \equiv Z_\mu[0, 0, j] = \int_{B_\mu} \mathcal{D}\phi \exp\left\{iS_0[\phi] + i \int j_a \phi_a\right\}, \quad (33)$$

where $S_0[\phi]$ is given in Eq.22 and we integrate only along the contour C_3 . As before, $\mathcal{D}\phi \equiv \mathcal{D}\phi_1 \mathcal{D}\phi_2$. On rotating to Euclidean time $\tau = it$, $Z_\mu[j]$ becomes

$$Z_\mu[j] = \langle \exp\left\{\int j_a \phi_a\right\} \rangle = \int_{B_\mu} \mathcal{D}\phi \exp\left\{-S_{0,E}[\phi] - \int j_a \phi_a\right\}, \quad (34)$$

where $S_{0,E}[\phi]$ is the relativistic Euclidean action

$$S_{0,E}[\phi] = \int_0^{\beta_0} d\tau \int d^3x \left[\frac{1}{2} \dot{\phi}_a^2 + \frac{1}{2} (\nabla \phi_a)^2 + \frac{1}{2} m^2 \phi_a^2 \right]. \quad (35)$$

and the sum is taken over fields $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ satisfying the boundary condition B_μ : $\phi(\mathbf{x}, \tau) = e^{-\beta_0 \mu} \phi(\mathbf{x}, \tau - \beta_0)$ in imaginary time. The dot now means differentiation with respect to τ , $\dot{\phi}_a = \partial_\tau \phi_a$.

The bracket $\langle \dots \rangle$ here denotes the thermal average

$$\langle F[\phi] \rangle = \text{Tr}\{e^{-\beta_0(H_0 - \mu Q)} F[\phi]\}. \quad (36)$$

where H_0 is given in Eq.18. Since it is time-independent we have dropped the t -suffix.

$Z_\mu[j]$ can be calculated as it stands. However, because of the chemical potential in the boundary conditions it is not immediately obvious how to identify the phase of the system. This is clarified by adopting a second approach to chemical potentials, in which we transfer the chemical potential term μQ to the Hamiltonian, to create an effective Hamiltonian $H_0 - \mu Q$. Although $H_0 - \mu Q$ no longer generates time translations of the physical fields ϕ and ϕ^* , it does generate time translations of the effective field $\tilde{\phi}(t) = e^{i\mu t} \phi(t)$ (and its adjoint). To see the advantages of removing μ from the boundary conditions we express Z_μ of Eq.33 not as a sum over ϕ -field histories but as a sum over $\tilde{\phi}$ -field histories. Although not usually posed in this way the details are well-understood. For example, see Kapusta²⁰, Haber & Weldon²¹, and Bernstein et al.²².

With the Jacobian unity, $Z_\mu[j]$ becomes

$$Z_\mu[j] = \int_{B_0} \mathcal{D}\tilde{\phi} \exp\left\{-S_\mu[\tilde{\phi}] - \int j_a \tilde{\phi}_a\right\}, \quad (37)$$

where $S_\mu[\phi]$ is now the effective action

$$S_\mu[\phi] = \int_0^{\beta_0} d\tau \int d^3x \left[\frac{1}{2} \dot{\phi}_a^2 + \frac{1}{2} (\nabla \phi_a)^2 + \frac{1}{2} m_0^2 \phi_a^2 + i\mu(\phi_2 \dot{\phi}_1 - \dot{\phi}_2 \phi_1) \right]. \quad (38)$$

in which

$$m_0^2 = m^2 - \mu^2 \quad (39)$$

It follows from Eq.23 that, with μQ bundled with H_0 the $\tilde{\phi}_a$ integrations are now taken over *periodic* configurations with boundary conditions $B_0 : \tilde{\phi}_a(\mathbf{x}, \tau) = \tilde{\phi}_a(\mathbf{x}, \tau - \beta_0)$ in imaginary time, with no μ -dependence. Most importantly, the zeroes of $\tilde{\phi}$ are the zeroes of ϕ , leading to identical line densities.

This displacement of μ from the boundary conditions to the action has enabled us to replace the classical potential by an effective potential in which the state of the system is more transparent. Semiclassically, when $m_0^2 > 0$ (i.e. $\mu^2 < m^2$) the $O(2)$ symmetry is unbroken. This describes our disordered initial state. However, once $m_0^2 < 0$ (i.e. $\mu^2 > m^2$) the free theory is unstable. This is a signal that the $O(2)$ symmetry is broken and a transition to an ordered phase has occurred. Thus, effectively, it is m_0^2 , rather than m^2 that carries the time dependence, changing from positive to negative at $t = t_0$.

Semiclassically, the non-relativistic limit is trivial. Although we achieve it by following the phase boundary $m_0^2 = 0$ from high T and low μ to low T and high μ the decomposition of m_0^2 is immaterial. Only in the cross-terms $i\mu(\phi_2\dot{\phi}_1 - \dot{\phi}_2\phi_1)$ will the change be noticed. Even this is misleading since, as we shall see, this term plays no role in our Gaussian approximation.

All that remains is a problem of interpretation. Let us first consider the relativistic regime in which the initial environment is very hot, with $T \gg m \gg \mu$. In this case, with μ irrelevant, the symmetry is broken by a change in m^2 . Suppose that, on the completion of the transition, $m_0^2 \simeq m^2$ takes the value $m_0^2 = -M^2 < 0$. If, in this relativistic case, the final temperature is very low, there are then no thermal effects once interactions have been taken into account and M is a physical parameter, determining the Higgs mass, $m_H = \sqrt{2}M$. However, because of thermal radiative effects $m_0(t \leq 0) = m_0$ is not a physical parameter. In practice, there is no loss in taking it as the effective scalar field mass at temperature $T_0 = \beta_0^{-1}$, despite the fact that such a mass is defined in terms of large scale fluctuation averages (the effective potential). That is, in the mean-field approximation,

$$m_0^2 = -M^2 \left(1 - \frac{T^2}{T_0^2}\right). \quad (40)$$

The results will turn out to be largely independent of m_0 provided it is comparable to M , which is the case if we do not quench from too close to the transition. On the other hand, for the non-relativistic theory, for which $T/m \ll 1$, the parameter m can always be identified with the boson mass. In this case we change the sign of m_0^2 in the action by a change in μ from $\mu^2 < m^2$ to $\mu^2 > m^2$, a density quench. To preserve uniformity of notation, we take $m_0^2(t) = m_0^2$ when $t \leq t_0$ and $m_0^2(t) = -M^2 < 0$ once the transition is complete.

3.3. Extracting the Non-Relativistic Field Theory

At this stage we could proceed directly to our Gaussian fluctuation model. However, it is advantageous to show how the unfamiliar equation Eq.38 can be cast in a form standard to non-relativistic many-body theory. In fact, one of the difficulties of making straightforward comparisons between early universe physics and many-body physics is this difference in formalism, a reflection of the freezing out of antiparticles in the non-relativistic regime. Thus, for example, a non-relativistic order field, with its interpretation as a probability amplitude for finding particles, is necessarily complex. As is apparent, our main tools in relativistic particle physics are path integrals which, conventionally, are not separated into particle and antiparticle sectors. This split has to be made if we are to recover the familiar non-relativistic as a field theory in its own right.

To do this we rewrite Z_μ of Eq.37 as a sum over histories in the full phase-space of the fields $\tilde{\phi}_a$ and their conjugates $\tilde{\pi}_a$. As we shall see, it is just this extended sum over histories that is needed for isolating the non-relativistic limit of the theory.

As a phase-space integral, $Z_\mu[j]$ takes the form

$$Z_\mu[j] = \int_{B_0} \mathcal{D}\tilde{\pi}_1 \mathcal{D}\tilde{\pi}_2 \mathcal{D}\tilde{\phi}_1 \mathcal{D}\tilde{\phi}_2 \exp\left\{-\tilde{S}_{\mu,E}[\tilde{\pi}, \tilde{\phi}] - \int j_a \tilde{\phi}_a\right\}, \quad (41)$$

where

$$-\tilde{S}_{\mu,E}[\pi, \phi] = \int_0^{\beta_0} d\tau \left\{ \left(\int d^3x i\pi_a \dot{\phi}_a \right) - \left(H_0 - -\mu Q \right) \right\}. \quad (42)$$

We have made the integration over the *four* field degrees of freedom explicit. Although the $\tilde{\phi} - a$ -fields are constrained to be periodic, the $\tilde{\pi}_a$ integrations are unconstrained, and $\tilde{S}_{\mu,E}$ cannot be identified directly with the ‘effective’ action in the presence of a chemical potential. However, the $\tilde{\pi}$ integrations are readily performed to recreate Z_μ of Eq. 37.

It has been shown in the work of Kapusta, Bernstein et al.²² how, in the limit $m \gg T_0 \simeq -\mu_{\text{nr}}$, where $\mu_{\text{nr}} = \mu - m$, we recover the energy spectrum (and free energy) of a non-relativistic Bose gas. We shall now show that this same phase-space analysis permits us to recover the familiar non-relativistic partition function, for free bosons,

$$Z_{\mu_{\text{nr}}}[\eta^*, \eta] = \int_{b_0} \mathcal{D}\psi^* \mathcal{D}\psi \exp\left\{-S_{\mu_{\text{nr}}}[\psi^*, \psi] + \int (\eta\psi^* + \eta^*\psi)\right\}, \quad (43)$$

where the boundary condition b_0 denotes simple periodicity of the fields ψ, ψ^* in imaginary time with period β_0 . In Eq.43 $S_{\mu_{\text{nr}}}[\psi^*, \psi]$ is the non-relativistic action

$$\begin{aligned} S_{\mu_{\text{nr}}}[\psi^*, \psi] &= \int_0^{\beta_0} d\tau \int d^3x \left[\frac{1}{2}(\psi^* \dot{\psi} - \dot{\psi}^* \psi) + \frac{1}{2m}(\nabla\psi^*)(\nabla\psi) - \mu_{\text{nr}}\psi^*\psi \right] \\ &= \int_0^{\beta_0} d\tau \int d^3x \psi^* \left[\frac{\partial}{\partial\tau} - \frac{1}{2m}\nabla^2 - \mu_{\text{nr}} \right] \psi \end{aligned} \quad (44)$$

in the order field ψ and its conjugate.

The non-relativistic field $\psi(\mathbf{x}, t)$ is understood as annihilating a particle at the point \mathbf{x} , whereas $\psi^*(\mathbf{x}, t)$ creates a particle at \mathbf{x} . Although the particles are distinct from the antiparticles there are no antiparticles in $Z_{\mu_{\text{nr}}}$. $|\psi|^2$ measures the local particle density, a role not available to $|\phi|^2$ of the relativistic field theory. Of course, we can decompose ψ as $\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$ but, for once, we decline to do so since ψ_1 and ψ_2 have no obvious interpretation.

$Z_\mu[j]$ of Eq.37 and $Z_{\mu_{\text{nr}}}[\eta^*, \eta]$ of Eq.43 cannot be identical, coming from different origins, but they are closely related in the regime $m \gg T_0 \simeq -\mu_{\text{nr}}$. To see this, we rewrite $Z_\mu[j]$ of Eq.41 in terms of the complex fields ϕ, ϕ^*, π, π^* as

$$Z_\mu[j, j^*] = \int \mathcal{D}\pi^* \mathcal{D}\pi \mathcal{D}\phi^* \mathcal{D}\phi \exp \left\{ -\tilde{S}_{\mu, E}[\pi, \phi] - \int (j^* \phi + j \phi^*) \right\}, \quad (45)$$

where we have dropped all *tilde* labels from the fields and

$$-- \tilde{S}_{\mu, E}[\pi, \phi] = \int_0^{\beta_0} d\tau \left\{ \int d^3x \left(i\pi \dot{\phi} + i\pi^* \dot{\phi}^* \right) - \left(H_0 - -\mu Q \right) \right\}, \quad (46)$$

in which H_0 is given in Eq.25 and Q is now, equivalently

$$Q = i \int d^3x (\phi^* \pi^* - \pi \phi). \quad (47)$$

We could proceed as before, and perform the π, π^* integrations, which would give us $Z_\mu[j]$ of Eq.37 in complex field form, but this is nothing new. Instead, we perform canonical transformations from the phase-space fields ϕ, ϕ^*, π, π^* to a new set of fields $\Psi, \Psi^*, \bar{\Psi}, \bar{\Psi}^*$ that, in the non-relativistic limit, annihilate and create particles and antiparticles. Specifically,

$$\begin{aligned} \Psi &= \sqrt{\frac{m}{2}} \phi + \frac{i}{\sqrt{2m}} \pi^*, \\ \Psi^* &= \sqrt{\frac{m}{2}} \phi^* - \frac{i}{\sqrt{2m}} \pi, \end{aligned} \quad (48)$$

annihilate and create non-relativistic particles in this limit, whereas

$$\begin{aligned} \bar{\Psi} &= \sqrt{\frac{m}{2}} \phi^* + \frac{i}{\sqrt{2m}} \pi, \\ \bar{\Psi}^* &= \sqrt{\frac{m}{2}} \phi - \frac{i}{\sqrt{2m}} \pi^*, \end{aligned} \quad (49)$$

annihilate and create non-relativistic *antiparticles* respectively.

To confirm that this is the case we invert the above. The field measure

$$\mathcal{D}\pi^* \mathcal{D}\pi \mathcal{D}\phi^* \mathcal{D}\phi = \mathcal{D}\Psi^* \mathcal{D}\Psi \mathcal{D}\bar{\Psi}^* \mathcal{D}\bar{\Psi} \quad (50)$$

is unchanged. Inserting these decompositions in $Z_\mu[j, j^*]$ of Eq.45 (where, for simplicity, we switch off the sources j, j^*) gives Z_μ of the form

$$Z_\mu \equiv Z_\mu[0, 0] = \int_{B_0} \mathcal{D}\Psi^* \mathcal{D}\Psi \mathcal{D}\bar{\Psi}^* \mathcal{D}\bar{\Psi} \exp\{-\tilde{S}_{\mu,E}\} \quad (51)$$

where $\tilde{S}_{\mu,E}$ of Eq.46 is now a functional of $\Psi, \bar{\Psi}$, etc., but B_0 is a boundary condition on the ϕ fields, and *not* the Ψ fields.

Then, on direct substitution $\tilde{S}_{\mu,E}$ is

$$\begin{aligned} \tilde{S}_{\mu,E} &= \int_0^{\beta_0} d\tau \int d^3x \Psi^* \left[\frac{\partial}{\partial \tau} - \frac{1}{2m} \nabla^2 + (m - \mu) \right] \Psi \\ &+ \int_0^{\beta_0} d\tau \int d^3x \bar{\Psi}^* \left[\frac{\partial}{\partial \tau} - \frac{1}{2m} \nabla^2 + (m + \mu) \right] \bar{\Psi} \\ &+ \int_0^{\beta_0} d\tau \int d^3x \frac{1}{2m} \left\{ (\nabla \bar{\Psi})(\nabla \Psi) + (\nabla \bar{\Psi}^*)(\nabla \Psi^*) \right\}, \end{aligned} \quad (52)$$

after removing total derivatives. This can be written in terms of the non-relativistic action $S_{\mu_{nr}}[\psi^*, \psi]$ of Eq.44 as

$$\tilde{S}_{\mu,E} = S_{\mu_{nr}}[\Psi^*, \Psi] + S_{-\mu-m}[\bar{\Psi}^*, \bar{\Psi}] + S_{0,I}[\Psi^*, \Psi, \bar{\Psi}^*, \bar{\Psi}], \quad (53)$$

where $S_{0,I}$, coupling particles to antiparticles, is the third term of equation Eq.52 and $\mu_{nr} = \mu - m$. $S_{\mu_{nr}}[\Psi^*, \Psi]$ and $S_{-\mu-m}[\bar{\Psi}^*, \bar{\Psi}]$ each possess an $O(2)$ invariance, for the individual conservation of particle and antiparticle number. The $O(2)$ symmetry of the antiparticle action $S_{-\mu-m}[\bar{\Psi}^*, \bar{\Psi}]$ remains unbroken. It is the $O(2)$ symmetry of the particle action $S_{\mu_{nr}}$ of Eq.44 which leads to non-relativistic vortices on its breaking. Away from the non-relativistic limit the $O(2) \otimes O(2)$ symmetry of $\tilde{S}_{\mu,E}$ is reduced to the $O(2)$ of the relativistic theory (the conservation of particle minus antiparticle number) by $S_{0,I}$.

To see this pattern we observe that, on partial integration, $S_{0,I}$ can be thought of as providing sources for $\bar{\Psi}$ and $\bar{\Psi}^*$ in Z_μ of Eq.51, which can be integrated out. This gives Z_μ as an integral over the non-relativistic partition with chemical potential $-m - \mu$,

$$Z_\mu = \int_{B_0} \mathcal{D}\Psi^* \mathcal{D}\Psi \exp\{-S_{\mu_{nr}}[\Psi^*, \Psi]\} Z_{-m-\mu}[-\nabla^2 \Psi/2m, -\nabla^2 \Psi^*/2m] \quad (54)$$

in the terminology of Eq.43 where, again, the periodic boundary condition is on the ψ fields. For the free-field case in hand, $Z_{-m-\mu}$ in the integrand is Gaussian in its sources. For the non-relativistic regime, in which $|\mu - m| \ll m$, whence $-\mu - m \approx -2m$, the factor $Z_{-m-\mu} \approx Z_{-2m}$ in the integrand of Eq.54 can be expanded in powers of momenta. For particle kinetic energies $\mathbf{k}^2/2m \ll m$, $Z_{-m-\mu} \approx 1$. For typical $k = |\mathbf{k}|$, with $k^2/2m = O(T)$, the errors are $O(k^2/m^2) = O(T/m)$. From Eq.48 and Eq.49, we identify the semi-classical ϕ with Ψ as

$$\phi = \frac{1}{\sqrt{2m}} \Psi + \frac{1}{\sqrt{2m}} \langle \bar{\Psi}^* \rangle \simeq \frac{1}{\sqrt{2m}} \Psi \quad (55)$$

on neglecting the anti-particle field fluctuations. As a result, zeroes in ϕ are also zeroes in Φ . Further, on neglecting these fluctuations, $\phi \simeq \pi^*/2m$ showing that, in the non-relativistic limit, π is approximately periodic in imaginary time. Thus the boundary condition B_0 , denoting periodic ϕ fields, becomes the boundary condition b_0 for periodic Ψ fields, whence we achieve our goal of showing that, in the non-relativistic limit, the relativistic partition function

$$Z_\mu \approx Z_{\mu_{\text{nr}}} = \int_{b_0} \mathcal{D}\Psi^* \mathcal{D}\Psi \exp\{-S_{\mu_{\text{nr}}}[\Psi^*, \Psi]\}, \quad (56)$$

where $\mu_{\text{nr}} = \mu - m$ as before. The inclusion of sources will not change this result, although we shall have to include sources that couple to the π^*, π fields.

This demonstration of how calculations with the relativistic Z_μ of Eq.45 are indistinguishable from those of the non-relativistic $Z_{\mu_{\text{nr}}}$ when $\mu \approx m$ can be extended to the case of symmetry breaking, although we shall not be able to go beyond the simplest Gaussian approximation in this talk. Further details are given in our work¹¹.

Now that we have demonstrated how to recover the non-relativistic field theory, we return to the relativistic formalism of Eq.38, for large and small μ .

4. A Gaussian Model for Vortex Formation

Having established the role of the chemical potential in letting us interpolate between relativistic and non-relativistic regimes in the initial conditions (with a corresponding interpolation between relativistic and non-relativistic field theories), we are now in a position to determine the effect of these initial conditions on vortex production in a simple model of Gaussian field fluctuations.

We have already assumed that the initial conditions correspond to a disordered state in equilibrium for $t < t_0$. The action Eq.16 was proposed in a context in which the chemical potential was relegated to the boundary condition. As we observed earlier, if we wish to represent it so as to match the action Eq.38, in which the chemical potential is manifest, we can accommodate changes in m or changes in μ by the requirement that, at $t = t_0$, $m_0^2(t) = m^2(t) - \mu^2(t)$ changes sign. Most simply, we assume that, for $t > t_0$, $m_0^2(t)$ takes the *negative* value $m_0^2(t) = -M^2 < 0$ *immediately*. That is, the potential at the origin has been instantaneously inverted everywhere, breaking the global $O(2)$ symmetry. If $\lambda(t) = \lambda$ is very *weak* then, for times $Mt < \ln(1/\lambda)$, the ϕ -field, falling down the hill away from the metastable vacuum, will not yet have experienced the upturn of the potential, before the point of inflection. Thus, for these small times, $\lambda(t)$ can also be set to zero, and $p_t[\Phi]$ is Gaussian, as required.

In the relativistic regime $\mu \ll m$, and the t -dependence of m_0 is entirely carried by m , as in Eq.40. On the other hand, in the non-relativistic limit, when $\mu \simeq m$ and m is unchange in time, m_0^2 changes from

$$m_0^2(t) = m_0^2 = -2m\mu_0 > 0, \quad (57)$$

say, when $t \leq t_0$, to

$$m_0^2(t) = -M^2 = -2m\mu_f < 0, \quad (58)$$

when $t > t_0$. Henceforth, we take $t_0 = 0$.

The onset of the phase transition at time $t = t_0$ is characterised by the instabilities of long wavelength fluctuations permitting the growth of correlations. Although the initial value of $\langle \phi \rangle$ over any volume is zero, we anticipate that the resulting evolution will lead to domains of constant $\langle \phi \rangle$ phase, whose boundaries will trap vortices.

Of course, instantaneous change is physically impossible. Consider small amplitude fluctuations of ϕ_a , at the top of the parabolic potential hill. Long wavelength fluctuations, for which $|\mathbf{k}|^2 < M^2$, begin to grow exponentially. If their growth rate $\Omega(k) = \sqrt{M^2 - |\mathbf{k}|^2}$ is much slower than the rate of change of the environment which is causing the quench, then those long wavelength modes are unable to track the quench. It will turn out that the time-scale at which domains appear in this instantaneous quench is $t_d = O(M^{-1})$. As long as the time taken to implement the quench is comparable to t_d and much less than $t_f = O(M^{-1} \ln(1/\lambda))$ the approximation is relevant. We note that, in the non-relativistic regime, for which $\Omega(k)$ has the same definition,

$$\begin{aligned} \Omega^2(k) &= M^2 - |\mathbf{k}|^2 = 2m \left(\mu_f - \frac{k^2}{2m} \right) \\ &= 2m(\mu_f - \epsilon(k)). \end{aligned} \quad (59)$$

Thus the momentum restriction $|\mathbf{k}| < M$ is just $\epsilon(k) < \mu_f$.

We are now in a position to evaluate $p_t[\Phi]$, or rather $W_{ab}(r; t)$, for $t > 0$, and calculate the defect density accordingly. Before we quote the result we note that the $i\mu(\phi_2\dot{\phi}_1 - \dot{\phi}_2\phi_1)$ term in $S_\mu[\phi]$ of Eq.38 couples the $a = 1$ and $a = 2$ fields ϕ_a together and, in general, $G_{ab}(\mathbf{r} - \mathbf{r}'; t, t') = \langle \phi_a(\mathbf{r}, t)\phi_a(\mathbf{r}', t') \rangle$ is *not* diagonal in the $O(2)$ labels. However, for *equal* times diagonal behaviour is restored as $G_{ab}(\mathbf{r}; t, t) = \delta_{ab}G(\mathbf{r}; t, t)$ and the $i\mu(\phi_2\dot{\phi}_1 - \dot{\phi}_2\phi_1)$ term can effectively be discarded. This leaves us in a situation for which Eq.9 and Eq.10 are satisfied and the results of Halperin are directly applicable. That is, W_{ab} is diagonal,

$$W_{ab}(|\mathbf{r} - \mathbf{r}'|; t, t) = \delta_{ab}W(|\mathbf{r} - \mathbf{r}'|; t, t), \quad (60)$$

whence $W(|\mathbf{r} - \mathbf{r}'|; t) = \langle \phi_a(\mathbf{r}, t)\phi_a(\mathbf{r}', t) \rangle$ (no summation), the thermal Wightman function for either ϕ_1 or ϕ_2 .

But for the chemical potential, this situation of inverted harmonic oscillators was studied many years ago by Guth and Pi¹⁸ and Weinberg and Wu¹⁹. In the context of domain formation, we refer to the recent work of Boyanovsky et al.¹⁷, and our own^{8,10}. For the case in hand, if we make a separation into the unstable long wavelength modes, for which $|\mathbf{k}| < M$, and the short wavelength modes $|\mathbf{k}| > M$, then $W(r; t)$ is

$$W(r; t) = \int_{|\mathbf{k}| < M} d^3k e^{i\mathbf{k}\cdot\mathbf{x}} C(k; \mu) \left[1 + A(k)(\cosh(2\Omega(k)t) - 1) \right]$$

$$+ \int_{|\mathbf{k}|>M} d^3k e^{i\mathbf{k}\cdot\mathbf{x}} C(k; \mu) \left[1 + a(k)(\cos(2w(k)t) - 1) \right] \quad (61)$$

with $r = |\mathbf{x}|$ and

$$\begin{aligned} \Omega^2(k) &= M^2 - |\mathbf{k}|^2, \\ w^2(k) &= -M^2 + |\mathbf{k}|^2, \\ A(k) &= \frac{1}{2} \left(1 + \frac{\omega^2(k)}{\Omega^2(k)} \right), \\ a(k) &= \frac{1}{2} \left(1 - \frac{\omega^2(k)}{w^2(k)} \right). \end{aligned} \quad (62)$$

All the μ -dependence is contained in the factor

$$\mathcal{C}(k, \mu_0) = \frac{1}{2\omega(k)} \left[\frac{e^{-\beta_0\omega} - e^{\beta_0\omega}}{e^{\beta_0(m+\mu_0)} - e^{-\beta_0\omega} - e^{\beta_0\omega} + e^{-\beta_0(m+\mu_0)}} \right]. \quad (63)$$

where

$$\omega^2(k) = |\mathbf{k}|^2 + m_0^2. \quad (64)$$

In the zero chemical potential limit $\mu = m_0 + \mu_0 = 0$, $\mathcal{C}(k, \mu_0)$ takes the familiar form

$$\mathcal{C}(k, \mu_0) = \frac{1}{2\omega(k)} \coth(\beta_0\omega(k)/2) \quad (65)$$

On the other hand, in the non-relativistic limit $\epsilon(k) = \mathbf{k}^2/2m \simeq \omega - m$ and $\mu_0 \ll m$ $\mathcal{C}(k, \mu_0)$ is the equally familiar Bose distribution

$$\mathcal{C}(k, \mu_0) \simeq \frac{1}{2\omega(k)} \left(\frac{1}{1 - e^{-\beta_0(\epsilon(k) - \mu_0)}} \right). \quad (66)$$

Further, in the high temperature relativistic limit $T_0 \gg m_0$, $\mathcal{C}(k, \mu_0)$ of Eq.65 simplifies as

$$\frac{1}{2\omega(k)} \coth(\beta_0\omega(k)/2) \simeq \frac{T_0}{\mathbf{k}^2 + m_0^2}. \quad (67)$$

Equally, in the non-relativistic limit when $\epsilon(k) - \mu_0 \ll T_0$,

$$\begin{aligned} \mathcal{C}(k, \mu_0) &\simeq \frac{T_0}{\epsilon(k) + |\mu_0|} \\ &\propto \frac{T_0}{\mathbf{k}^2 + m_0^2} \end{aligned} \quad (68)$$

on using the definition of m_0 given earlier in Eq.58. From Eq.13 onwards it follows that the overall scale of W is immaterial to the vortex density. As a result, in these regimes the relativistic and non-relativistic $W(r, t)$ are *identical*, once we take the identifications of Eq.58 into account.

We observe that, if we were to use $W(r; t)$ of Eq.61 as it stands, then both $W(0; t)$ and $W''(0; t)$ necessarily suffer from ultraviolet divergences. However, the string thickness at the end of the quench will be $O(M^{-1})$. It is the zeroes coarse-grained to this scale that will provide the subsequent network. Thus, if we do not probe the field zeroes within a string we need consider only the first term in Eq.61. There is another point. Even before the quench begins there is a high density of line zeroes coarse-grained to this same scale $O(M^{-1})$ in the initial equilibrium phase. However, these modes are entirely transient. If we were to calculate the correlations of $\rho_i(\mathbf{x})$ at *different* times t and t' , we would find rapidly oscillating behaviour with period $\Delta t = O(m^{-1})$. On the other hand, a calculation of the density correlations at different times from the unstable modes in Eq.61 does not give oscillatory, but damped, behaviour. It is the residue of the strings produced by the unstable modes that survives to produce the network, and the transient strings can be ignored. Henceforth, we retain only the first term

$$W(r; t) = \int_{|\mathbf{k}| < M} d^3 k e^{i\mathbf{k}\cdot\mathbf{x}} \mathcal{C}(k, \mu_0) \left[1 + A(k) (\cosh(2\Omega(k)t) - 1) \right] \quad (69)$$

of Eq.61. In the two critical regimes where Eq.67 and Eq.68 are valid, $W(r; t)$ is again the same for both cases.

Even though the approximation is only valid for small times, there is a regime $Mt \geq 1$, for small couplings, in which t is large enough for $\cosh(2Mt) \approx \frac{1}{2} \exp(2Mt)$ and yet Mt is still smaller than the time $O(\ln 1/\lambda)$ at which the fluctuations begin to sample the ground-state manifold. In this regime

$$W(r; t) \simeq \int_{|\mathbf{k}| < M} d^3 k \mathcal{C}(k, \mu) e^{i\mathbf{k}\cdot\mathbf{x}} A(k) e^{2\Omega(k)t}. \quad (70)$$

In these circumstances the integral at time t is dominated by a peak in the integrand $k^2 e^{2\Omega(k)t}$ at k around k_c , where

$$tk_c^2 = M \left(1 + O\left(\frac{1}{Mt}\right) \right), \quad (71)$$

and we have assumed M and m_0 to be comparable. The effect of changing β_0 is only visible in the $O(1/Mt)$ term. In fact, we are being unnecessarily restrictive in wanting to preserve the identical behaviour of Eq.67 and Eq.68. All that is required for identical *leading* behaviour in relativistic and non-relativistic regimes is that $\mathcal{C}(k, \mu_0)$ varies slowly in the vicinity of the peak of the integrand at $tk_c^2 \simeq M$. [$A(k)$ is already slowly varying]. This is the case when, allowing for coefficients $O(1)$,

$$1 < \frac{\mu_f}{|\mu_0|} = \frac{M^2}{m_0^2} \ll tM < \ln\left(\frac{1}{\lambda}\right) \quad (72)$$

where, in the same spirit, we have taken $|\mu_0| < \mu_f$. The upper bound on tM is a reminder that interactions are always present, and the Gaussian approximation

must fail as soon as the field fluctuations have extended to the true ground-states at the minima of the potential. The lower bound is necessary for the integrand to be peaked strongly so that the saddle-point approximation is valid. As long as Eq.72 is basically correct, any difference between the relativistic and non-relativistic regimes will be non-leading.

Assuming these limits, we recover what would have been our first naive guess for a correlation function based on the growth of the unstable modes,

$$W(r; t) \simeq \int_{|\mathbf{k}| < M} d^3k e^{i\mathbf{k}\cdot\mathbf{r}} e^{2\Omega(k)t}, \quad (73)$$

We understand the dominance of wavevectors at $k_c^2 = M/t$ in the integrand as indicating the formation of domains of mean size

$$\xi(t) = O(\sqrt{t/M}) \quad (74)$$

once $Mt > 1$. Specifically, we take $\xi(t) = 2\sqrt{t/M}$. Once $Mt > 1$ then $\xi(t) > M^{-1}$, where M^{-1} characterises the cold vortex radius. In the weak coupling approximation individual domains become large enough to accomodate many vortices before the approximation breaks down. There is no difficulty with causality since domains increase in size as $\dot{\xi} = \frac{1}{\sqrt{Mt}} < 1$. On neglecting terms exponentially small in Mt , $W(r; t)$ of Eq.73 can be further rewritten as

$$W(r; t) \simeq e^{2Mt} \int_0^\infty dk \text{sinc}(kr) k^2 e^{-tk^2/M} \quad (75)$$

$$= W(0; t) \exp\left(-\frac{r^2}{\xi^2(t)}\right) \quad (76)$$

where

$$W(0; t) \approx C \frac{e^{2Mt}}{(Mt)^{3/2}}, \quad (77)$$

for some C. The exponential growth of $W(0; t)$ in t reflects the way the field amplitudes fall off the hill centred at $\Phi = 0$. With the peaking in wavelength $l = k^{-1}$ understood as indicating the appearance of domains of characteristic linear dimension $\xi(t)$, the Gaussian in r is a reflection of the rms variation $\Delta\xi$ in domain size ξ . This variation is large. If we isolate the Gaussian saddle-point in Eq.73 as

$$W(r; t) \simeq e^{2Mt} \int_0^M dk \text{sinc}(kr) k_c^2 e^{-(k-k_c)^2/2(\Delta k)^2} \quad (78)$$

then

$$\frac{\Delta\xi}{\xi} = \frac{\Delta k}{k_c} = \frac{1}{2}. \quad (79)$$

To calculate the number density of vortices at early times we insert the expression Eq.75 for W into the equations derived earlier, to find

$$n(t) = \frac{1}{\pi} \frac{1}{\xi(t)^2}. \quad (80)$$

We note that the dependence on time t of both the density and density correlations is only through the correlation length $\xi(t)$. We have a *scaling* solution in which, as the domains of coherent field form and expand, the interstring distance grows accordingly. Since the only way the defect density can decrease without the background space-time expanding is by defect-antidefect annihilation, we deduce that the coalescence of domains proceeds by the annihilation of small loops of string. However, because the density of vortices only depends on $\xi(t)$ in this early stage, the fraction of string in ‘infinite’ string remains constant. Thus, at the same time as small loops disappear, other loops must rearrange themselves so that the length of ‘infinite’ string decreases accordingly. Finally, there is roughly one string zero per coherence area, a long held belief for whatever mechanism.

There is one final concern. A necessary condition for this rolling down of the field to be valid is that the initial field fluctuations should be small enough that there is no significant probability that the field is already in the true vacuum. As long as we do not begin the quench from too close to the transition, there is, in fact, no difficulty. See our work¹⁰ for more details.

4.1. Vortex Density Correlations

In addition to the gross vortex density Eq.80 we can calculate the density-density correlation functions $C_{ij}(r; t)$ of Eq.15, identical for both the relativistic plasma and the non-relativistic medium.

Yet again, as in the case of the density $n(t)$, the t -dependence of C_{ij} only occurs implicitly through $\xi(t)$. The simple analytic form of Eq.75 enables us to calculate A and B , up to exponentially small terms in Mt , as

$$\begin{aligned} A(r; t) &= \frac{2}{\pi^2 \xi^4(t)} \frac{e^{-2r^2/\xi^2(t)}}{(1 - e^{-2r^2/\xi^2(t)})^2} \left[(1 - e^{-2r^2/\xi^2(t)}) - 2 \frac{r^2}{\xi^2(t)} \right] < 0, \\ B(r; t) &= \frac{2}{\pi^2 \xi^4(t)} \frac{e^{-2r^2/\xi^2(t)}}{(1 - e^{-2r^2/\xi^2(t)})} > 0, \end{aligned} \quad (81)$$

Suppose that $\mathbf{r} = (0, 0, r)$. Then

$$\begin{aligned} C_{11} = C_{22} &= A(r; t) - B(r; t) \\ &= -\frac{2}{\pi^2 \xi^4(t)} \left(\frac{2r^2}{\xi^2(t)} \right) \frac{e^{-2r^2/\xi^2(t)}}{(1 - e^{-2r^2/\xi^2(t)})^2} < 0. \end{aligned} \quad (82)$$

That is, we have *anticorrelation* of densities for parallel directions (and positive B for orthogonal ones).

It is useful to expand Eq.81 and Eq.82 for $r < \xi$. On normalising by a factor of n^2 ,

$$\frac{A(r; t)}{n^2(t)} = -1 + O\left(\frac{r^2}{\xi^2}\right) < 0,$$

$$\frac{B(r; t)}{n^2(t)} = \left(\frac{\xi^2(t)}{r^2} \right) + O(1) > 0. \quad (83)$$

For $r > \xi$ there is exponential falloff but, from Eq.83 we see that, in units of $n(t)^{-2}$, the anticorrelation is large. Since strings with a long persistence length would imply *positive* parallel correlations, we can interpret these anticorrelations (and others) as a reflection of an increased string bendiness. Although it is difficult to be precise, this suggests a significant amount of string in small loops. This is an important issue since, as we noted, early universe cosmology requires infinite string if string is to be the source of large-scale structure. In practice, some is enough.

There is another, indirect, way in which we can see the tendency for more string to be in small loops than we might have thought. In numerical simulations of string networks, the rule of thumb for a *regular* domain structure in field phase is that a significant fraction of string, if not most, is infinite string.

However, as we saw earlier in Eq.79, we do not have a regular domain structure in our model but have domains with a large variance $\Delta\xi/\xi = \frac{1}{2}$. Unfortunately, the domain structure that we have here is not yet appropriate for a direct comparison since, as well as the domain size, the field magnitude has a variance about $\langle |\phi| \rangle = O(M e^{Mt})$. The work of Guth and Pi¹⁸ shows that this can be parametrised by an effective dispersion in t in $\langle |\phi| \rangle$ of $\Delta t = O(M^{-1})$. Nonetheless, consider an idealised case in which domain growth stops instantaneously because of back-reaction at some time t_f . The distribution of strings will then be as above for $\xi = \xi(t_f)$, while the field adjusts to the vacuum manifold, without changing phase, in each domain. We would expect some string-antistring annihilation to continue while this adjustment occurs, so that the $n(t)$ calculated previously is an overestimate of the string density at the end of the transition. However, the domains will still be of varying size with variance still given approximately by Eq.79. The inclusion of domain variance in numerical simulation of string networks (albeit in a different way from that proposed here) shows²³ that, the greater the variance, the more string is in small loops. Beyond observing that there seems to be some infinite string, we will say no more.

All our results are for quenches that go from significantly above the transition to significantly below it. These give the *smallest* value of $\Delta\xi/\xi$ possible and, plausibly, the best chance of producing infinite string. Had we begun closer to the transition, where the initial fluctuations are larger, the dispersion in domain size, and consequently the fraction of string in small loops, would be even greater.

5. Conclusions

In this paper we have shown how global O(2) vortices appear, at a quench from the ordered to disordered state, as a consequence of the growth of unstable Gaussian long wavelength fluctuations. Most importantly, in the light of discussions about the extent to which vortex production in low-temperature many-body systems simulates vortex production in the early universe, our model supports the analogy. Specifically, with our simple assumptions, vortex production is *identical* in both a

relativistic high-temperature quench, in which the initial state is characterised by $T \gg m$, and in a non-relativistic density quench in which the initial state is described by $T \ll m$ (to freeze out antiparticles) with a chemical potential $\mu_{\text{nr}} \simeq T$. All that is required is an appropriate translation of the parameters from the one case to the other. To see this, we have shown how to isolate the non-relativistic particle/antiparticle sectors of the relativistic path-integrals that we use in our calculations. The $O(2)$ symmetry, whose breaking leads to the production of vortices, leads to the conservation of particle number minus antiparticle number in the former case, and to the conservation of particle number in the latter.

In our simple model of Gaussian fluctuations the resulting string configurations scale as a function of the correlation length $\xi(t) = O(t^{\frac{1}{2}})$, at about one vortex/correlation area. This is compatible with the Kibble mechanism for vortex production on domain boundaries upon phase separation. For a weak coupling theory the domain cross-sections are significantly larger than a vortex cross-section at the largest times for which the approximations are valid. However, there is a large variance in their size, with $\Delta\xi/\xi = \frac{1}{2}$. Because of this there is more string in small loops than we might have anticipated.

We stress that our model can, at best, describe weak coupling systems for the short times while the domains are growing before the defects freeze out. This is unsatisfactory for most early universe applications and for low-temperature many-body systems. However, we know in principle¹⁷ how to include back-reaction (still within the context of a Gaussian approximation) to slow down domain growth as the field fluctuations spread to the ground-state manifold. The identity of the weak-coupling results of the two regimes should survive to this case also, although it will probably lead to different conclusions from those above. Further, there is no difficulty in principle of embedding these results in a FRW metric, along the lines of²⁴. This is being actively pursued.

1. T.W.B. Kibble, *J. Phys.* **A9** (1976) 1387.
2. E.P.S Shellard and A. Vilenkin, *Cosmic Strings and other Topological Defects* (Cambridge University Press, 1994)
3. P.C. Hendry, N.S. Lawson, R.A.M. Lee, P.V.E. McClintock and C.D.H. Williams, *Nature* **368** (1994) 314.
4. V.M.H. Ruutu, M. Krusius, U. Parts, G.E. Volovik, *Neutron Mediated Vortex Nucleation in Rotating Superfluid $^3\text{He} - B$* , (Low Temperature Laboratory, Helsinki University of Technology, 02150 Espoo, Finland), in preparation.
5. P.C. Hendry, N.S. Lawson, R.A.M. Lee, P.V.E. McClintock and C.D.H. Williams, to be published in the Proceedings of the NATO Advanced Study Institute on *Topological Defects*, Newton Institute, Cambridge 1994, (Plenum Press, to appear).
6. W.H. Zurek, *Nature* **317** (1985) 505, *Acta Physica Polonica* **B24** (1993) 1301.
7. W.H. Zurek, to be published in the Proceedings of the NATO Advanced Study Institute on *Topological Defects*, Newton Institute, Cambridge 1994, (Plenum Press, to appear).

8. A.J. Gill and R.J. Rivers, *Phys. Rev.* **D51** (1995) 6949.
9. R.J. Rivers and T.S. Evans, *The Production of Strings and Vortices at Phase Transitions*, to be published in the Proceedings of the NATO Advanced Study Institute on *Topological Defects*, Newton Institute, Cambridge 1994, (Plenum Press, to appear).
10. A.J. Gill, T.S. Evans and R.J. Rivers, *Vortex Production in Non-Relativistic and Relativistic Media*, Imperial College Preprint, Imperial/TP/94-95/48, July 1995.
11. A.J. Gill, T.S. Evans and R.J. Rivers, *The Chemical Potential and the Non-Relativistic Limit*, Imperial College Preprint (in preparation).
12. B.I. Halperin, *Statistical Mechanics of Topological Defects*, published in *Physics of Defects*, proceedings of Les Houches, Session XXXV 1980 NATO ASI, editors Balian, Kléman and Poirier (North-Holland Press, 1981) p.816.
13. J. Schwinger, *J. Math. Phys.* **2** (1961) 407.
14. K.T. Mahanthappa and P.M. Bakshi, *J.M. Phys.* **4** (1963) 1; *ibid*, (1963) 12.
15. L.V. Keldysh, *Sov. Phys. JETP* **20** (1965) 1018.
16. K-C. Chou, Z-B. Su, B-L. Hao and L. Yu, *Phys. Rep.* **118** (1985) 1.
17. D. Boyanovsky, Da-Shin Lee and Anupam Singh, *Phys. Rev.* **D48** (1993) 800.
18. A. Guth and S-Y. Pi, *Phys. Rev.* **D32** (1985) 1899.
19. E.J. Weinberg and A. Wu, *Phys. Rev.* **D36** (1987) 2474.
20. J.I. Kapusta, *Finite Temperature Field Theory*, (Cambridge University Press, 1989).
J.I. Kapusta, *Phys. Rev.* **D24** (1981) 426.
21. H.E. Haber and H.A. Weldon, *Phys. Rev.* **D25** (1982) 502.
22. J. Bernstein and S. Dodelson, *Phys. Rev. Lett.* **66** (1991) 683.
K.M. Benson, J. Bernstein and S. Dodelson, *Phys. Rev.* **D44** (1991) 2480.
23. A. Yates and T.W.B. Kibble, Imperial College preprint Imperial/TP/94-95/49.
24. D. Boyanovsky, H.J. de Vega, R. Holman, *Phys. Rev.* **D49** (1994) 2769.