

## On singular solutions in multidimensional gravity

Vladimir D. Ivashchuk and Vitaly N. Melnikov

*Center for Gravitation and Fundamental Metrology, VNIIMS  
3/1 M. Ulyanovoy str., Moscow, 117313, Russia  
e-mail: ivas@cvsu.uucp.free.net*

### ABSTRACT

It is proved that the Riemann tensor squared is divergent as  $\tau \rightarrow 0$  for a wide class of cosmological metrics with non-exceptional Kasner-like behaviour of scale factors as  $\tau \rightarrow 0$ , where  $\tau$  is synchronous time. Using this result it is shown that non-trivial generalization of the spherically-symmetric Tangherlini solution to the case of  $n$  Ricci-flat internal spaces [13] has a divergent Riemann tensor squared as  $R \rightarrow R_0$ , where  $R_0$  is parameter of length of the solution. Multitemporal naked singularities are also considered.

PACS numbers: 04.20, 04.40.

Moscow 1995

# 1 Introduction

In the recent decade there has been a great interest to multidimensional models of classical and quantum gravity (see, for example, [1]-[3] and references therein). This interest was stimulated mainly by the investigations on the theory of supergravity and superstrings [4], i.e. the ideas of the unification of interactions gave a "new life" to the ideas of Kaluza and Klein.

As it was shown in papers [5]-[8] devoted to multidimensional cosmological models with perfect fluid a large variety of the exact solutions have a Kasner-like asymptotical behaviour for small values of the synchronous time parameter  $\tau \rightarrow 0$ . Here we do not consider the exceptional solutions such as exponential and power-law inflationary solutions [8]. The solutions with oscillatory (or stochastic behaviour) as  $\tau \rightarrow 0$  also are not covered by the presented scheme.

In this paper we use the Riemann tensor squared as an indicator of the singular behaviour of the cosmological solutions as  $\tau \rightarrow 0$ . In Sec. 2 we present an explicit formula for the quadratic invariant (Riemann tensor squared) of the metric defined on the product of spaces with the scale factors depending on the points of the first space. In Sec. 3 the quadratic invariant is presented for a wide class of cosmological metrics describing the evolution of  $n$  spaces of arbitrary dimensions. It is proved (see Proposition 2) that, when all spaces are 1-dimensional, the Riemann tensor squared is positive and divergent as  $t = \tau \rightarrow 0$  for all non-trivial (non-Milne-like) configurations. In Subsec. 3.2 this result is generalized to the case of Ricci-flat spaces. In Subsec. 3.3 the main theorem concerning the divergency of the Riemann tensor squared for a wide class of cosmological metrics with non-exceptional Kasner-like behaviour of scale factors as  $\tau \rightarrow 0$  is proved. In Sec. 4 we apply the obtained result to the generalization of the spherically-symmetric Tangherlini solution [10] to the case of  $n$  Ricci-flat internal spaces [13]. In Subsec. 4.1 the multitemporal generalization of the Tangherlini solution is considered [14]. This solution is shown to describe the (multitemporal) naked singularity when the parameters are non-exceptional.

## 2 General formalism

Let  $(M, g)$  be a manifold  $M$  with the metric  $g$ . The squared Riemann tensor

$$I[g] \equiv R_{MNPQ}[g]R^{MNPQ}[g] \quad (2.1)$$

is a smooth real-valued function on  $M$ . For smooth function  $\phi : M \rightarrow R$  we also define two smooth functions on  $M$

$$U[g, \phi] \equiv g^{MN}(\partial_M \phi)\partial_N \phi, \quad (2.2)$$

$$V[g, \phi] \equiv g^{M_1 N_1} g^{M_2 N_2} [\nabla_{M_1}(\partial_{M_2} \phi) + (\partial_{M_1} \phi)\partial_{M_2} \phi] \times \\ [\nabla_{N_1}(\partial_{N_2} \phi) + (\partial_{N_1} \phi)\partial_{N_2} \phi], \quad (2.3)$$

where  $\nabla = \nabla[g]$  is covariant derivative with respect to  $g$ . The scalar invariants (2.1)-(2.3) play an important role in what follows. Now, we consider the manifold

$$M = M_0 \times M_1 \times \dots \times M_n \quad (2.4)$$

with the metric

$$g = g^{(0)} + \sum_{i=1}^n \exp[2\phi^i(x)]g^{(i)}, \quad (2.5)$$

where  $g^{(0)} = g_{\mu\nu}^{(0)}(x)dx^\mu \otimes dx^\nu$  is metric on  $M_0$ ,  $g^{(i)}$  is metric on  $M_i$  and  $\phi^i : M_0 \rightarrow \mathbf{R}$  is smooth function,  $i = 1, \dots, n$ .

**Proposition 1.** The Riemann tensor squared for the metric (2.5) has the following form

$$\begin{aligned} I[g] &= I[g^{(0)}] + \sum_{i=1}^n \{e^{-4\phi^i} I[g^{(i)}] - 4e^{-2\phi^i} U[g^{(0)}, \phi^i] R[g^{(i)}] \\ &\quad - 2N_i U^2[g^{(0)}, \phi^i] + 4N_i V[g^{(0)}, \phi^i]\} \\ &\quad \sum_{i,j=1}^n 2N_i N_j [g^{(0),\mu\nu} (\partial_\mu \phi^i) \partial_\nu \phi^j]^2, \end{aligned} \quad (2.6)$$

where  $U$ - and  $V$ -invariants are defined in (2.2), (2.3) and  $R[g^{(i)}]$  is scalar curvature of  $g^{(i)}$  and  $N_i = \dim M_i$  is dimension of  $M_i$ ,  $i = 1, \dots, n$ .

**Sketch of proof.** For  $n = 1$  the relation (2.6) may be verified by a straightforward calculation. For  $n > 1$  relation (2.6) may be proved by induction (on  $n$ ) using the following decomposition formulas

$$U[g^{(0)} + \exp(2\phi^1(x))g^{(1)}, \phi(x)] = U[g^{(0)}, \phi(x)], \quad (2.7)$$

$$V[g^{(0)} + \exp(2\phi^1(x))g^{(1)}, \phi(x)] = V[g^{(0)}, \phi(x)] + N_1 [g^{(0),\mu\nu} (\partial_\mu \phi^1) \partial_\nu \phi]^2 \quad (2.8)$$

( $x \in M_0$ ).

For the scalar curvature of the metric (2.5) we get

$$\begin{aligned} R[g] &= R[g^{(0)}] + \sum_{i=1}^n e^{-2\phi^i} R[g^{(i)}] - \\ &\quad \sum_{i,j=1}^n (N_i \delta_{ij} + N_i N_j) g^{(0),\mu\nu} (\partial_\mu \phi^i) \partial_\nu \phi^j \\ &\quad - 2 \sum_{i=1}^n N_i \Delta[g^{(0)}] \phi^i, \end{aligned} \quad (2.9)$$

where  $\Delta[g^{(0)}]$  is Laplace-Beltrami operator corresponding to  $g^{(0)}$  (see also [15]).

**Remark 1.** In (2.6) and in what follows we use the following condensed notations:  $I[g] = I[g](x)$ ,  $I[g^{(\nu)}] = I[g^{(\nu)}](x_\nu)$ , for  $x \in M$ ,  $x_\nu \in M_\nu$ ,  $\nu = 0, \dots, n$  and analogously for scalar curvatures.

### 3 Multidimensional cosmology

Here we are interested in the special case of (2.4), (2.5) with  $M_0 = (t_1, t_2)$ ,  $t_1 < t_2$ . Thus, we consider the metric

$$g_c = -B(t)dt \otimes dt + \sum_{i=1}^n A_i(t)g^{(i)}, \quad (3.1)$$

defined on the manifold

$$M = (t_1, t_2) \times M_1 \times \dots \times M_n. \quad (3.2)$$

Here, like in (2.4), (2.5)  $g^{(i)}$  is a metric on  $M_i$  and  $B(t), A_i(t) \neq 0, i = 1, \dots, n$ .

From Proposition 1 we obtain the Riemann tensor squared for the metric (3.1) (see also [14])

$$\begin{aligned} I[g_c] = & \sum_{i=1}^n \{A_i^{-2} I[g^{(i)}] + A_i^{-3} B^{-1} \dot{A}_i^2 R[g^{(i)}] - \frac{1}{8} N_i B^{-2} A_i^{-4} \dot{A}_i^4 \\ & + \frac{1}{4} N_i B^{-2} (2A_i^{-1} \ddot{A}_i - B^{-1} \dot{B} A_i^{-1} \dot{A}_i - A_i^{-2} \dot{A}_i^2)^2\} \\ & + \frac{1}{8} B^{-2} [\sum_{i=1}^n N_i (A_i^{-1} \dot{A}_i)^2]^2. \end{aligned} \quad (3.3)$$

(We recall that  $\dim M_i = N_i, i = 1, \dots, n$ .)

For scalar curvature of (3.1) we get from (2.9)

$$\begin{aligned} R[g_c] = & \sum_{i=1}^n \{e^{-2x^i} R[g^{(i)}] + \\ & e^{-2\gamma} N_i [2\dot{x}^i + \dot{x}^i (\sum_{j=1}^n N_j \dot{x}^j - 2\dot{\gamma}) + (\dot{x}^i)^2]\}, \end{aligned} \quad (3.4)$$

where  $B = e^{2\gamma}$  and  $A_i = e^{2x^i}, i = 1, \dots, n$ .

### 3.1 $(n + 1)$ -dimensional Kasner solution

Let us consider the metric on  $\mathbf{R}_+ \times \mathbf{R}^n$

$$g = -dt \otimes dt + \sum_{i=1}^n t^{2\alpha_i} dx^i \otimes dx^i, \quad (3.5)$$

where  $t > 0, -\infty < x^i < \infty$  and  $\alpha_i$  are constants,  $i = 1, \dots, n$ . From (3.3) we get

$$I[g] = 2F(\alpha)t^{-4}, \quad (3.6)$$

where

$$F(\alpha) = \sum_{i=1}^n [2\alpha_i^2 (\alpha_i - 1)^2 - \alpha_i^4] + [\sum_{i=1}^n \alpha_i^2]^2. \quad (3.7)$$

Now we impose the following restrictions on the parameters  $\alpha_i$

$$\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \alpha_i^2 = 1. \quad (3.8)$$

The metric (3.5) with the restrictions (3.8) imposed satisfies the vacuum Einstein equations (or, equivalently,  $R_{MN}[g] = 0$ ). It is a trivial generalization of the well-known Kasner solution. In this case

$$F(\alpha) = \Phi(\alpha) = \Phi_n(\alpha) \equiv \sum_{i=1}^n [\alpha_i^4 - 4\alpha_i^3] + 3. \quad (3.9)$$

We define a Milne set as

$$\mathcal{M} = \mathcal{M}_n = \{(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\} \subset \mathcal{E}, \quad (3.10)$$

where

$$\mathcal{E} = \mathcal{E}_n \equiv \{\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{R}^n \mid \sum_{i=1}^n \alpha_i = \sum_{i=1}^n \alpha_i^2 = 1\}. \quad (3.11)$$

Notice, that  $\mathcal{E}$  is  $(n-2)$ -dimensional ellipsoid for  $n > 2$  ( $\mathcal{E} \simeq \mathbf{S}^{n-2}$ ).

For  $n = 1$ ,  $\mathcal{M} = \mathcal{E} = \{(1)\}$  and we are lead to well-known Milne solution

$$g_M = -dt \otimes dt + t^2 dx^1 \otimes dx^1. \quad (3.12)$$

We recall that by the coordinate transformation  $y^0 = t \cosh x^1$ ,  $y^1 = t \sinh x^1$  the metric (3.12) is reduced to the Minkowsky metric  $\eta = -dy^0 \otimes dy^0 + dy^1 \otimes dy^1$  in the upper light cone  $y^0 > |y^1|$ .

For  $\alpha = (\dots, 0, 1_i, 0, \dots) \in \mathcal{M}$  we get a trivial extension of the Milne metric:

$$g_m = -dt \otimes dt + t^2 dx^i \otimes dx^i + \sum_{j \neq i}^n dx^j \otimes dx^j, \quad (3.13)$$

$i = 1, \dots, n; n > 1$ .

**Proposition 2.** Let  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{E}$ . Then  $\Phi(\alpha) \geq 0$  and  $\Phi(\alpha) = 0$  if and only if  $\alpha \in \mathcal{M}$ .

**Proof.** For  $n = 1, 2$  the proposition is trivial. So, we consider the case  $n > 2$ . Let

$$\Phi| = \Phi|_{\mathcal{E}} : \mathcal{E} \longrightarrow \mathbf{R} \quad (3.14)$$

be the restriction of the function  $\Phi$  (3.9) on  $\mathcal{E}$  (3.11). Since  $\mathcal{E}$  is a smooth submanifold in  $\mathbf{R}^n$  (see (3.8)) the function  $\Phi|$  is also smooth ( $\Phi| = \Phi \circ i$ , where  $i : \mathcal{E} \longrightarrow \mathbf{R}^n$  is canonical embedding). The manifold  $\mathcal{E}$  is compact (it is isomorphic to  $\mathbf{S}^{n-2}$ ). Let  $\text{Min} = \text{Min}(\Phi|)$  is the set of points of (absolute) minimum of  $\Phi|$  and  $\text{Ext} = \text{Ext}(\Phi|)$  is the set of points of extremum of  $\Phi|$ . The set  $\text{Min}$  is non-empty:  $\text{Min} \neq \emptyset$ , since  $\Phi|$  is a continuous real-valued function defined on the compact topological space  $\mathcal{E}$ . Clearly, that  $\text{Min} \subset \text{Ext}$ .

First we find  $\text{Ext}$  using the standard scheme of conditional extremum. We consider the function

$$\hat{\Phi}(\alpha, \lambda, \mu) = \Phi(\alpha) + \mu(\sum_{i=1}^n \alpha_i^2 - 1) + \lambda(\sum_{i=1}^n \alpha_i - 1) \quad (3.15)$$

where  $\mu, \lambda \in \mathbf{R}$ . The point  $\alpha$  belongs to  $\text{Ext}$  if and only if there exist  $\lambda, \mu \in \mathbf{R}$  such that  $(\alpha, \mu, \lambda)$  is a point of extremum of the function (3.15), i.e. the relations (3.8) and

$$\frac{\partial \hat{\Phi}}{\partial \alpha_i} = 4\alpha_i^3 - 12\alpha_i^2 + 2\mu\alpha_i + \lambda = 0, \quad (3.16)$$

$i = 1, \dots, n$ , are satisfied. From (3.8) and (3.16) we obtain

$$4 \sum_{i=1}^n \alpha_i^3 - 12 + 2\mu + \lambda n = 0, \quad (3.17)$$

$$4 \sum_{i=1}^n \alpha_i^4 - 12 \sum_{i=1}^n \alpha_i^3 + 2\mu + \lambda = 0, \quad (3.18)$$

and hence

$$4\Phi(\alpha) = \lambda(n-1). \quad (3.19)$$

Let us consider the cubic equation

$$4y^3 - 12y^2 + 2\mu y + \lambda = 4(y-y_1)(y-y_2)(y-y_3) = 0. \quad (3.20)$$

We prove that for given  $\mu$  and  $\lambda$  the cubic equation (3.20) should have three different real roots. Indeed, there should be at least two different real roots since otherwise  $\alpha_1 = \dots = \alpha_n$  but this is impossible due to the Kasner constraints (3.8). The third root should be also real and we are lead to the following three possibilities for the roots: i)  $y_1 = y_2 < y_3$ ; ii)  $y_1 < y_2 = y_3$ ; iii)  $y_1 < y_2 < y_3$ . It follows from (3.20) that

$$y_1 + y_2 + y_3 = 3 \quad (3.21)$$

and hence  $y_3 > 1$ . But due to (3.8)  $\alpha_i \leq 1$  and as a consequence the possibilities i) and ii) can not be fulfilled (otherwise  $\alpha_i = y_1$  for all  $i$ ). Thus  $y_1 < y_2 < y_3$  and due to (3.16) and  $y_3 > 1$  we obtain

$$\{x|x = \alpha_i, i = 1, \dots, n\} = \{y_1, y_2\}. \quad (3.22)$$

Solving (3.8) for  $\alpha$  satisfying (3.22) we get

$$y_1 = y_1(m_1, m_2) = \frac{m_1 - \sqrt{\Delta}}{m_1 n}, \quad (3.23)$$

$$y_2 = y_2(m_1, m_2) = \frac{m_2 + \sqrt{\Delta}}{m_2 n}, \quad (3.24)$$

$$\Delta = m_1 m_2 (n-1). \quad (3.25)$$

Here  $m_a$  is the number of  $\alpha_i$  equal to  $y_a$ ,  $a = 1, 2$ . Clearly, that  $m_1 + m_2 = n$  and  $m_1, m_2 \geq 1$ . It follows from (3.19) and (3.20) that

$$\lambda = 4\Phi(\alpha)/(n-1) = -4y_1 y_2 y_3, \quad (3.26)$$

$$\mu = 2(y_1 y_2 + y_2 y_3 + y_3 y_1). \quad (3.27)$$

In fact we find the expression for the set of extremum

$$\text{Ext} = E_1 \sqcup \dots \sqcup E_n, \quad (3.28)$$

where

$$E_k = \{(\alpha_1 = y_2, \dots, \alpha_k = y_2, \alpha_{k+1} = y_1, \dots, \alpha_n = y_1) \text{ and all permutations}\}, \quad (3.29)$$

$k = 1, \dots, n-1$  and  $y_1 = y_1(n-k, k)$ ,  $y_2 = y_2(n-k, k)$  (see (3.23), (3.24)). Clearly, that the number of elements in Ext is  $2^n - 2$ .

For  $\alpha \in E_k$ ,  $k > 1$ , we have

$$\Phi(\alpha) > 0. \quad (3.30)$$

This can be readily verified using the inequalities  $y_1(n-k, k) < 0$ ,  $0 < y_2(n-k, k) < y_3(n-k, k)$ ,  $k > 1$ , and the relation (3.26). For  $\alpha \in E_1$

$$\Phi(\alpha) = 0. \quad (3.31)$$

Using the inclusion  $\text{Min} \subset \text{Ext}$  and the relations (3.28), (3.30) and (3.31) we obtain

$$\text{Min} = \text{Min}(\Phi) = E_1 = \mathcal{M}. \quad (3.32)$$

The proposition 2 follows from the relations (3.31) and (3.32).

Remark 2. It will be proved in a separate publication that  $\Phi|$  (see (3.14)) is a Morse function.

From (3.6), (3.9) and Proposition 2 we get for the Kasner metric (3.5) with  $\alpha \in \mathcal{E} \setminus \mathcal{M}$

$$I[g](t, \vec{x}(t)) \rightarrow +\infty, \quad \text{as } t \rightarrow +0. \quad (3.33)$$

for arbitrary curve  $\vec{x}(t)$ .

### 3.2 Kasner-like solutions with Ricci-flat spaces

Here we consider the following metric [9]

$$g = -dt \otimes dt + \sum_{i=1}^n t^{2\alpha_i} c_i g^{(i)}, \quad (3.34)$$

defined on the manifold (3.2) with  $(t_1, t_2) = (0, +\infty) = \mathbf{R}_+$ , where  $(g^{(i)}, M_i)$  are Ricci-flat internal spaces, i.e.  $R_{m_i n_i}[g^{(i)}] = 0$ , and  $c_i \neq 0$  are constants,  $i = 1, \dots, n$ ,  $n \geq 2$ . Parameters  $\alpha_i$  satisfy the relations

$$\sum_{i=1}^n N_i \alpha_i = \sum_{i=1}^n N_i \alpha_i^2 = 1. \quad (3.35)$$

The metric (3.34) with the restrictions (3.35) imposed satisfies the vacuum Einstein equations.

For the metric (3.34), (3.35) we get from (3.3)

$$I[g] = \sum_{i=1}^n t^{-4\alpha_i} c_i^{-2} I[g^{(i)}] + 2\Phi_*(\alpha) t^{-4}, \quad (3.36)$$

where

$$\Phi_*(\alpha) \equiv \sum_{i=1}^n N_i [\alpha_i^4 - 4\alpha_i^3] + 3. \quad (3.37)$$

Analogously to (3.10) we introduce the Milne set

$$\mathcal{M}_* = \{\alpha | \alpha = (\dots, 0, 1_i, 0, \dots), N_i = 1\} \subset \mathcal{E}_*, \quad (3.38)$$

where

$$\mathcal{E}_* \equiv \{\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{R}^n | \sum_{i=1}^n N_i \alpha_i = \sum_{i=1}^n N_i \alpha_i^2 = 1\}. \quad (3.39)$$

For  $n > 2$   $\mathcal{E}_*$  is  $(n - 2)$ -dimensional ellipsoid.

Example 1. The set (3.38) is empty:  $\mathcal{M}_* = \emptyset$ , if and only if  $N_i > 1$  for all  $i$ .

Example 2. For  $N_1 = \dots = N_n = 1$  we have  $\mathcal{M}_* = \mathcal{M}$  (see (3.10)).

**Proposition 3.** Let  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{E}_*$ . Then  $\Phi_*(\alpha) \geq 0$  and  $\Phi_*(\alpha) = 0$  if and only if  $\alpha \in \mathcal{M}_*$ .

**Proof.** Here we consider the function  $\Phi = \Phi_N$  (3.9) corresponding to  $N = \sum_{i=1}^n N_i$ . For  $\alpha \in \mathcal{E}_*$  we have

$$\Phi_*(\alpha) = \Phi_N(\beta(\alpha)), \quad (3.40)$$

where the set  $\beta(\alpha) = \beta = (\beta_1, \dots, \beta_N)$  is defined by the relations

$$\begin{aligned} \beta_1 &= \dots = \beta_{N_1} = \alpha_1, \\ \dots \\ \beta_{N-N_n+1} &= \dots = \beta_N = \alpha_n. \end{aligned} \quad (3.41)$$

It is evident that  $\beta \in \mathcal{E}_N$  (see (3.11)). The Proposition 3 follows from the Proposition 2, relation (3.40) and the equivalence

$$\beta(\alpha) \in \mathcal{M} \iff \alpha \in \mathcal{M}_*. \quad (3.42)$$

Here  $\mathcal{M} = \mathcal{M}_N$  is the Milne set corresponding to  $N$ .

**Proposition 4.** Let  $g$  be the metric (3.34) with the set  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{E}_* \setminus \mathcal{M}_*$  and

$$I[g^{(i)}] \geq 0, \quad (3.43)$$

for all  $i = 1, \dots, n$ . Then

$$I[g](t, f(t)) \rightarrow +\infty, \quad \text{as } t \rightarrow +0, \quad (3.44)$$

for any function

$$f : \mathbf{R}_+ \longrightarrow M_1 \times \dots \times M_n. \quad (3.45)$$

If the condition (3.43) is not imposed the relation (3.44) takes place if  $f(t) \rightarrow f_0 \in M_1 \times \dots \times M_n$  as  $t \rightarrow +0$ .

**Proof.** The first part of the proposition follows from (3.36), (3.43) and the inequality  $\Phi_*(\alpha) > 0$  for  $\alpha \notin \mathcal{M}_*$  (see proposition 3). The second part of the proposition follows from continuity of the functions  $I[g^{(i)}]$  on  $M_i$ , the inequalities  $\alpha_i < 1$ ,  $i = 1, \dots, n$ , and the relation (3.36). Indeed, the functions  $I[g^{(i)}](f^i(t))$ ,  $i = 1, \dots, n$ , have limits as  $t \rightarrow +0$ , and hence are bounded. Here  $f(t) = (f^1(t), \dots, f^n(t))$ . Thus, the second term in the right hand side of (3.36) is dominating in the limit  $t \rightarrow +0$  and we are lead to (3.44).

### 3.3 The solutions with asymptotically Kasner behaviour

Here we consider the metric

$$g = -wd\tau \otimes d\tau + \sum_{i=1}^n A_i(\tau)g^{(i)}, \quad (3.46)$$



defined on the manifold

$$(0, T) \times M_1 \times \dots \times M_n, \quad (3.47)$$

where  $T > 0$  and  $w = \pm 1$ . We suppose that the metric  $g^{(i)}$  on the manifold  $M_i$  satisfy the following conditions: the functions  $I[g^{(i)}]$ ,  $wR[g^{(i)}]$ , are bounded from below, i.e

$$I[g^{(i)}](x_i) \geq C_i \quad (3.48)$$

for all  $x_i \in M_i$  and

$$wR[g^{(i)}](x_i) \geq D_i \quad (3.49)$$

for all  $x_i \in M_i$ ,  $i = 1, \dots, n$ .

**Remark 3.** Clearly that the conditions (3.48), (3.49) are satisfied for compact manifold  $M_i$ . The first condition is also satisfied when the metric  $g^{(i)}$  has the Euclidean signature: in this case  $C_i = 0$ .

We also suppose that the scale factors  $A_i : (0, T) \rightarrow \mathbf{R}$  are smooth functions ( $A_i(\tau) \neq 0$ ), satisfying the following asymptotical relations

$$A_i(\tau) = c_i \tau^{2\alpha_i} [1 + o(1)], \quad (3.50)$$

$$\dot{A}_i(\tau) = c_i \tau^{2\alpha_i - 1} [2\alpha_i + o(1)], \quad (3.51)$$

$$\ddot{A}_i(\tau) = c_i \tau^{2\alpha_i - 2} [2\alpha_i(2\alpha_i - 1) + o(1)], \quad (3.52)$$

as  $\tau \rightarrow +0$ , where  $c_i \neq 0$ ,  $\alpha_i$  are constants,  $i = 1, \dots, n$ . We recall that the notation  $\varphi(\tau) = o(1)$  as  $\tau \rightarrow +0$  means that  $\varphi(\tau) \rightarrow 0$  as  $\tau \rightarrow +0$ .

**Remark 4.** The relations (3.51) and (3.52) should not obviously follow from (3.50). A simple counterexample is

$$A_i(\tau) = 1 + \tau \sin \frac{1}{\tau^2}. \quad (3.53)$$

But if

$$A_i(\tau) = c_i \tau^{2\alpha_i} [1 + \varphi_i(\tau)], \quad (3.54)$$

where  $c_i \neq 0$  and

$$\varphi_i(\tau) = o(1), \quad \tau \dot{\varphi}_i(\tau) = o(1), \quad \tau^2 \ddot{\varphi}_i(\tau) = o(1), \quad (3.55)$$

as  $\tau \rightarrow +0$ ,  $i = 1, \dots, n$ , then the relations (3.50)-(3.52) are satisfied.

**Theorem.** Let  $g$  be the metric (3.46) defined on the manifold (3.47), where the metrics  $g^{(i)}$ ,  $i = 1, \dots, n$ , satisfy the relations (3.48) and (3.49). Let the scale factors  $A_i(\tau)$ ,  $i = 1, \dots, n$ , satisfy the relations (3.50)-(3.52), where the set of parameters  $\alpha = (\alpha_i)$  satisfies the Kasner-like relations (3.35) ( $N_i = \dim M_i$ ) and is non-exceptional, i.e.  $\alpha \notin \mathcal{M}_*$  ( $\mathcal{M}_*$  is defined in (3.38)). Then

$$I[g](\tau, x) \rightarrow +\infty, \quad \text{as } \tau \rightarrow +0, \quad (3.56)$$

uniformly on  $x \in M_1 \times \dots \times M_n$ .

**Proof.** From (3.3) we get

$$I[g] = I_1[g] + I_2[g] + I_3[g], \quad (3.57)$$

where

$$I_1[g] = \sum_{i=1}^n A_i^{-2} I[g^{(i)}], \quad (3.58)$$

$$I_2[g] = \sum_{i=1}^n A_i^{-3} \dot{A}_i^2 wR[g^{(i)}], \quad (3.59)$$

$$I_3[g] = \sum_{i=1}^n \left\{ -\frac{1}{8} N_i A_i^{-4} \dot{A}_i^4 + \frac{1}{4} N_i (2A_i^{-1} \ddot{A}_i - A_i^{-2} \dot{A}_i^2)^2 \right\} \\ + \frac{1}{8} \left[ \sum_{i=1}^n N_i (A_i^{-1} \dot{A}_i)^2 \right]^2. \quad (3.60)$$

From (3.50)-(3.52) and (3.60) we obtain

$$I_3[g] = [2\Phi_*(\alpha) + o(1)]\tau^{-4}, \quad (3.61)$$

as  $\tau \rightarrow +0$ , where  $\Phi_*(\alpha)$  is defined in (3.37). We note that due to  $\alpha \notin \mathcal{M}_*$  and Proposition 3

$$\Phi_*(\alpha) > 0. \quad (3.62)$$

From (3.50), (3.51) we get

$$\frac{\dot{A}_i^2}{A_i^3} = c_i^{-1} \tau^{-2-2\alpha_i} [4\alpha_i^2 + o(1)], \quad (3.63)$$

as  $\tau \rightarrow +0$ . We note also that due to  $\alpha \notin \mathcal{M}_*$

$$\alpha_i < 1, \quad (3.64)$$

$i = 1, \dots, n$ . Let

$$\delta = \min_i (1 - \alpha_i) > 0. \quad (3.65)$$

Then it follows from (3.63) that there exists  $\tau_1 > 0$  such that

$$0 \leq \frac{\dot{A}_i^2}{A_i^3} < \tau^{-4+\delta}, \quad (3.66)$$

for all  $\tau < \tau_1$ ,  $i = 1, \dots, n$ . From (3.49) and (3.66) we get

$$\frac{\dot{A}_i^2}{A_i^3} wR[g^{(i)}](x_i) \geq -|D_i| \tau^{-4+\delta} \quad (3.67)$$

for all  $x_i \in M_i$ ,  $\tau < \tau_1$ ,  $i = 1, \dots, n$ , and hence

$$I_2[g](\tau, x) \geq -A\tau^{-4+\delta}, \quad (3.68)$$

for all  $\tau < \tau_1$  and  $x \in M_1 \times \dots \times M_n$  ( $A = \sum_{i=1}^n |D_i|$ ). Analogously we get from (3.50)

$$A_i^{-2} = c_i^{-2} \tau^{-4\alpha_i} [1 + o(1)], \quad (3.69)$$

and consequently there exists  $\tau_2 > 0$  such that

$$A_i^{-2} < \tau^{-4+\delta} \quad (3.70)$$

for all  $\tau < \tau_2$ ,  $i = 1, \dots, n$ . Using (3.48) and (3.70) we obtain (analogously to (3.68))

$$I_1[g](\tau, x) \geq -B\tau^{-4+\delta}, \quad (3.71)$$

for all  $\tau < \tau_2$  and  $x \in M_1 \times \dots \times M_n$  ( $B = \sum_{i=1}^n |C_i|$ ). It follows from (3.61), (3.68), (3.71) that

$$I[g](\tau, x) \geq \tau^{-4}[2\Phi_*(\alpha) - (A+B)\tau^\delta + o(1)] \rightarrow +\infty, \quad (3.72)$$

as  $\tau \rightarrow +0$ . This imply the relation (3.56). Theorem is proved.

Remark 5. When the relations (3.48) and (3.49) are not imposed the relation (3.56) is valid (at least) for any (fixed)  $x \in M_1 \times \dots \times M_n$ .

## 4 Spherically symmetric solutions with Ricci-flat internal spaces

Now we apply the obtained above results to the following scalar vacuum solution [8]

$$g = -f^a dt \otimes dt + f^{b-1} dR \otimes dR + f^b R^2 d\Omega_d^2 + \sum_{i=1}^n f^{a_i} B_i g^{(i)}, \quad (4.1)$$

$$\exp(2\varphi) = B_\varphi f^{a_\varphi}, \quad (4.2)$$

defined on the manifold

$$M = (R_0, +\infty) \times \mathbf{R} \times \mathbf{S}^d \times M_1 \times \dots \times M_n, \quad (4.3)$$

where  $(M_i, g^{(i)})$  are Ricci-flat internal spaces,  $\dim M_i = N_i$ ,  $i = 1, \dots, n$ ,  $d\Omega_d^2$  is the canonical metric on  $d$ -dimensional sphere  $\mathbf{S}^d$  ( $d \geq 2$ ) and  $f = f(R) = 1 - (R_0/R)^{d-1}$ . Here  $R_0, B_\varphi, B_i > 0$  are constants and the parameters  $b, a, a_1, \dots, a_n$  satisfy the relations

$$b = (1 - a - \sum_{i=1}^n a_i N_i) / (d - 1), \quad (4.4)$$

$$(a + \sum_{i=1}^n a_i N_i)^2 + (d - 1)(a^2 + a_\varphi^2 + \sum_{i=1}^n a_i^2 N_i) = d. \quad (4.5)$$

The solution (4.1)-(4.3) is a scalar-vacuum multispace generalization of the Tangherlini solution [10]. In the parametrization of the harmonic-type variable this solution was presented earlier in [16, 14]. For  $a_\varphi = 0$  see also [13, 14]. Some special cases were considered earlier in [12] (for  $d = 2$ ,  $a_\varphi = 0$ ) and [11] ( $n = 1$  and  $d = 2$ ).

The metric and scalar field from (4.1), (4.2) satisfy the field equations

$$R_{MN}[g] = \partial_M \varphi \partial_N \varphi, \quad (4.6)$$

$$\Delta[g]\varphi = 0, \quad (4.7)$$

corresponding to the action

$$S = \int d^D x \sqrt{|g|} \{R[g] - \partial_M \varphi \partial_N \varphi g^{MN}\}. \quad (4.8)$$

Now, we introduce a new variable

$$\tau = \tau(R) = \int_{R_0}^R dx [f(x)]^{(b-1)/2}. \quad (4.9)$$

The integral in (4.9) is convergent since due (4.4) and (4.5)

$$b > -1. \quad (4.10)$$

The map (4.9) defines a diffeomorphism from  $(R_0, +\infty)$  to  $\mathbf{R}_+$ . We consider the diffeomorphism

$$\sigma : M' \longrightarrow M, \quad (4.11)$$

generated by (4.9):  $\sigma(\tau, t, \dots) = (R(\tau), t, \dots)$ , where

$$M' = \mathbf{R}_+ \times \mathbf{R} \times \mathbf{S}^d \times M_1 \times \dots \times M_n. \quad (4.12)$$

The substitution (4.9) into (4.1) gives a metric on the manifold (4.12) (the dragging of (4.1) by the map (4.11))

$$\begin{aligned} \sigma^* g &= d\tau \otimes d\tau + A_0(\tau) g^{(0)} \\ &+ \sum_{i=1}^n A_i(\tau) g^{(i)} - A_{-1}(\tau) dt \otimes dt, \end{aligned} \quad (4.13)$$

where  $g^{(0)} = d\Omega_d^2$  and

$$A_i(\tau) = [f(R(\tau))]^{a_i}, \quad (4.14)$$

$$A_0(\tau) = R^2(\tau) [f(R(\tau))]^b, \quad (4.15)$$

$i = -1, 1, \dots, n$ ;  $a_{-1} = a$ . From (4.9) and the asymptotical behaviour

$$f(R) \sim \frac{(d-1)}{R_0} (R - R_0), \quad \text{as } R \rightarrow R_0 \quad (4.16)$$

we get

$$R - R_0 \sim (c_* \tau)^{2/(b+1)}, \quad f(R(\tau)) \sim c \tau^{2/(b+1)}, \quad (4.17)$$

as  $\tau \rightarrow +0$ , where  $c_*, c$  are constants. From (4.17) we get for the scale factors (4.14), (4.15) and the scalar field the following asymptotical relations

$$A_i(\tau) \sim c_i \tau^{2\alpha_i}, \quad (4.18)$$

$$A_0(\tau) \sim c_0 R_0^2 \tau^{2\alpha_0}, \quad (4.19)$$

$$\exp(2\varphi(\tau)) \sim c_\varphi \tau^{2\alpha_\varphi}, \quad (4.20)$$

as  $\tau \rightarrow +0$ , where  $c_i, c_0, c_\varphi$  are constants, and

$$\alpha_i = a_i/(b+1), \quad \alpha_0 = b/(b+1), \quad (4.21)$$

$$\alpha_\varphi = a_\varphi/(b+1), \quad (4.22)$$

$i = -1, 1, \dots, n$ . The parameters (4.21), (4.22) are correctly defined due to (4.10) and satisfy the Kasner-like relations

$$\sum_{\nu=-1}^n N_\nu \alpha_\nu = 1, \quad (4.23)$$

$$\sum_{\nu=-1}^n N_\nu \alpha_\nu^2 + \alpha_\varphi^2 = 1. \quad (4.24)$$

Here  $N_{-1} = 1$  and  $N_0 = d$ .

Now we consider the case  $\alpha_\varphi = 0$  (or equivalently  $a_\varphi = 0$ ). Let

$$\mathcal{M}_1 = \{\alpha = (\alpha_{-1}, \dots, \alpha_n) = (\dots, 0, 1_\nu, 0, \dots), N_\nu = 1\} \subset \mathbf{R}^{n+2}, \quad (4.25)$$

$$\mathcal{T} = \{a. = (a_{-1}, a_1, \dots, a_n) = (\dots, 0, 1_\nu, 0, \dots), N_\nu = 1\} \subset \mathbf{R}^{n+1}. \quad (4.26)$$

Clearly, that  $\mathcal{M}_1 \subset \mathcal{E}_2$  and  $\mathcal{T} \subset \mathcal{E}_1$ , where  $\mathcal{E}_1 \subset \mathbf{R}^{n+1}$  and  $\mathcal{E}_2 \subset \mathbf{R}^{n+2}$  are  $n$ -dimensional ellipsoids defined by relations (4.5) and (4.23), (4.24) respectively. It is not difficult to verify that the function  $\alpha = \alpha(a.)$  from (4.21) defines the diffeomorphism  $\mathcal{E}_1 \rightarrow \mathcal{E}_2$  and

$$\alpha(a.) \in \mathcal{M}_1 \iff a. \in \mathcal{T}. \quad (4.27)$$

**Proposition 5.** Let  $\sigma^*g$  be the metric (4.13)-(4.15) with the parameters (4.21), (4.22) satisfying the relations (4.23), (4.24) and obeying the restrictions:  $\alpha_\varphi = 0$ ,  $\alpha = (\alpha_{-1}, \dots, \alpha_n) \notin \mathcal{M}_1$ . Let Ricci-flat internal spaces  $(M_i, g^{(i)})$ ,  $i = 1, \dots, n$ , satisfy the self-boundness conditions (3.48). Then

$$I[\sigma^*g](\tau, y) \rightarrow +\infty, \text{ as } \tau \rightarrow +0, \quad (4.28)$$

uniformly on  $y \in \mathbf{R} \times \mathbf{S}^d \times M_1 \times \dots \times M_n$ .

**Proof.** We denote  $(M_{-1}, g^{(-1)}) = (\mathbf{R}, -dt \otimes dt)$  and  $(M_0, g^{(0)}) = (\mathbf{S}^d, d\Omega_d^2)$ . Due to assumption of the proposition, flatness of  $(-1)$ -space and the relations

$$I[g^{(0)}] = 2d(d-1) = 2R[g^{(0)}], \quad (4.29)$$

the conditions of the Theorem ((3.48), (3.49)) are satisfied for all spaces  $(M_\nu, g^{(\nu)})$ ,  $\nu = -1, \dots, n$ . All scale factors  $A_\nu(\tau)$ ,  $\nu = -1, \dots, n$ , and their first and second derivatives satisfy the Kasner-like asymptotical conditions of the theorem (see (3.50)-(3.52)). This may be proved using asymptotical relations (4.18), (4.19) and (4.9). Thus, the proposition 5 follows from the Theorem.

Using equivalence (4.27) and the relation  $I[\sigma^*g](\tau, y) = I[g](R(\tau), y)$  we may reformulate the Proposition 5 for the metric (4.1).

**Proposition 6.** Let  $g$  be the metric (4.1) with the parameters satisfying (4.4), (4.5) and  $a_\varphi = 0$ ,  $(a, a_1, \dots, a_n) \notin \mathcal{T}$  (see (4.26)). Let Ricci-flat internal spaces  $(M_i, g^{(i)})$ ,  $i = 1, \dots, n$ , satisfy the self-boundness conditions (3.48). Then

$$I[g](R, y) \rightarrow +\infty, \text{ as } R \rightarrow R_0, \quad (4.30)$$

uniformly on  $y \in \mathbf{R} \times \mathbf{S}^d \times M_1 \times \dots \times M_n$ .

Remark 6. Due to (3.3), (4.29) and the flatness of  $t$ -space  $I[g]$  does not depend on  $y_j \in M_j$ ,  $j = -1, 0$ .

Remark 7. From (4.6) we obtain

$$R[g] = g^{MN} \partial_M \varphi \partial_N \varphi = a_\varphi^2 f^{-1-b} (f')^2. \quad (4.31)$$

where  $f = f(r) = 1 - (R_0/r)^{d-1}$ . Using (4.31), (4.10) and the relation

$$f' = (d-1)R_0^{d-1} r^{-d} \quad (4.32)$$

we obtain for  $a_\varphi \neq 0$

$$R[g](r, y) \rightarrow +\infty, \text{ as } r \rightarrow R_0. \quad (4.33)$$

## 4.1 Multitemporal generalization of Tangherlini solution

Now we consider the special case of the solution (4.1)-(4.3) with  $n-1$  one-dimensional internal spaces (extra times). This solution defined on the manifold

$$M = (R_0, +\infty) \times \mathbf{R}^n \times \mathbf{S}^d, \quad (4.34)$$

reads

$$g = - \sum_{i=1}^n f^{a_i} B_i dt^i \otimes dt^i + f^{b-1} dR \otimes dR + f^b R^2 d\Omega_d^2, \quad (4.35)$$

$$\exp(2\varphi) = B_\varphi f^{a_\varphi}, \quad (4.36)$$

$f = f(R) = 1 - (R_0/R)^{d-1}$ , where  $R_0, B_\varphi, B_i > 0$  are constants and the parameters  $b, a_1, \dots, a_n$  satisfy the relations

$$b = (1 - \sum_{i=1}^n a_i) / (d-1), \quad (4.37)$$

$$(\sum_{i=1}^n a_i)^2 + (d-1)(a_\varphi^2 + \sum_{i=1}^n a_i^2) = d. \quad (4.38)$$

Let us consider the case  $a_\varphi = 0$  [14]. The "Tangherlini set" in thi case (see (4.26))

$$\mathcal{T} = \{(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\} \subset \mathbf{R}^n. \quad (4.39)$$

consists of  $n$  points. As was shown in [14] the multitemporal horizon takes place only for  $(a_1, \dots, a_n) \in \mathcal{T}$ . For  $(a_1, \dots, a_n) \notin \mathcal{T}$  we get from Proposition 6

$$I[g](R) \rightarrow +\infty, \text{ as } R \rightarrow R_0, \quad (4.40)$$

i.e. the solution (4.35) describes a multitemporal naked singularity. (The relation (4.40) was stated previously in [14].) This singular solution is unstable under monopole perturbations: this follows from the recent (more general) result of Bronnikov *et al* [17, 16, ?].

When  $a_\varphi \neq 0$ , we get from (4.33)

$$R[g](r) \rightarrow +\infty, \text{ as } r \rightarrow R_0. \quad (4.41)$$

It may be shown that in this case we also have a multitemporal naked singularity. It should be noted also that for the case  $n = 2$  the considered multitemporal solutions were recently generalized in [17] for more complicated model.

### **Acknowledgement**

One of the authors (V.D.I) is grateful to A.F.Ionov, Yu.S. Vladimirov, and organizers of the First Ionov School in Yaroslavl for invitation and kind hospitality.

## References

- [1] Yu.S.Vladimirov. Physical Space-Time Dimension and Unification of Interactions. Moscow University Press 1987 [in Russian].
- [2] Yu. S. Vladimirov, The Space-Time: Explicit and Hidden Symmetries, Nauka, Moscow, 1989 [in Russian].
- [3] V. N. Melnikov, in: Itogi Nauki i Tekhniki. Classical Field Theory and Theory of Gravity vol 1 Gravitation and Cosmology. (Moscow: VINITI, 1991) p. 49 [in Russian]; preprint CBPF-NF-051/93, Rio de Janeiro, Brazil, 1993.
- [4] M.B.Green, J.H.Schwarz and E.Witten, "Superstring Theory" Cambridge University Press., Cambridge, 1987.
- [5] V.D.Ivashchuk and V.N.Melnikov, *Int. J. Mod. Phys. D* **3** (1994), 795.
- [6] U.Bleyer, V.D.Ivashchuk, V.N.Melnikov and A.I.Zhuk, *Nucl. Phys. B* **429** (1994), 177.
- [7] V.D.Ivashchuk and V.N.Melnikov, *Class. Quantum Grav.* **12** (1995) 809.
- [8] V.D.Ivashchuk and V.N.Melnikov, *Gravitation and Cosmology* **1** No 2 (1995) 133.
- [9] V.D.Ivashchuk, *Phys. Lett. A* **170** (1992) 16.
- [10] F.R.Tangherlini, *Nuovo Cimento* **27** (1963), 636.
- [11] K. A. Bronnikov, V. D. Ivashchuk, Abstr. Rep. of VIII Soviet Grav. Conf, Erevan, EGU, 1988, p. 70.
- [12] K. A. Bronnikov, V. D. Ivashchuk and V.N.Melnikov, Abstr. Rep. of VIII Soviet Grav. Conf, Erevan, EGU, 1988, p. 158.
- [13] S. B. Fadeev, V. D. Ivashchuk and V. N. Melnikov, *Phys. Lett. A* 161 (1991) 98.
- [14] V.D.Ivashchuk and V.N.Melnikov, *Class. Quantum Grav.* **11** (1994) 1793.
- [15] V.A.Berezin, G.Domenech, M.L.Levinas, C.O.Lousto and N.D.Umerez, *Gen. Relativ. Grav.* **21** (1989), 1177. K.A.Bronnikov, U. Bleyer, V.N.Melnikov and S.B.Fadeev, *Nachrite fur Fisik* (1994) ?.
- [16] K.A.Bronnikov and V.N.Melnikov, *Annals of Physics* (N.Y.) **239** (1995) 40.
- [17] K.A.Bronnikov, *Gravitation and Cosmology* **1** No 1 (1995) 67.