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Energy Eigenvalues For Supersymmetric Potentials via Quantum Hamilton-Jacobi Formalism

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ABSTRACT

Using quantum Hamilton-Jacobi formalism of Leacock and Padgett, we show how to obtain the exact eigenvalues for supersymmetric (SUSY) potentials.

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1. Introduction

In classical mechanics, the Hamilton-Jacobi formalism is used to arrive at the frequencies of periodic systems, through the action variable, without having to obtain the complete solution of the equation of motion [1]. Analogously, quantum Hamilton-Jacobi (QHJ) formalism has been developed [2] where the quantum action variable yields the exact energy spectra of the bound state problems, avoiding an explicit solution of the corresponding Schrodinger equation. The QHJ equation is equivalent to the Schrodinger equation and is written in terms of the quantum momentum function (QMF), $p(x,E)$, which is the logarithmic derivative of the wave function. The quantization of the energy E arises from a suitable contour integral of the QMF, equated to integral multiple of $\hbar$. In this paper we apply this formalism as developed by Leacock and Padgett to obtain exact energy eigenvalues for potentials which exhibit supersymmetry (SUSY) and shape invariance (SI) [3]. It is well known that for potentials having the above two properties, one can find out the eigenvalues algebraically.

Our motivation in carrying out this explicit computation has been (a) to gain a better understanding of the working of the QHJ formalism and (b) to see what kind of singularities are possible for the QMF which play a significant role, as will be seen later, in determining the eigenvalues. It is hoped that the singularity structure of QMF will enable us to analyse exactness or nonexactness of the well known SUSY WKB formula [3,4].

The quantum Hamilton-Jacobi equation for a given potential $V(x)$ is given by

$$\begin{aligned} \frac{\hbar}{i}\frac{\partial^2 W(x,E)}{\partial x^2} + \left(\frac{\partial W(x,E)}{\partial x}\right)^2 &= E - V(x) \\ &\equiv p_c^2(x,E), \end{aligned} \tag{1}$$

where we have set $2m = 1$ and where, $W(x,E)$ is the quantum Hamilton's charac-

teristic function. The quantum momentum function $p(x,E)$ is defined as

$$p(x,E) \equiv \frac{\partial W(x,E)}{\partial x}. \tag{2}$$

The QMF is required to satisfy the boundary condition

$$\lim_{\hbar \to 0} p(x,E) \to p_c(x,E).$$

Here $p_c(x,E)$ is the classical momentum function, and is defined in such a way that it has the value $+\sqrt{E-V(x)}$, just below the branch cut which joins the two turning points.

Explicitly, the QMF is related to the wavefunction by

$$p = -i\hbar \ \frac{1}{\psi}\frac{\partial \psi}{\partial x} \ .$$

For the n-th energy level, ψ has n zeros between the classical turning points. These zeros of the wave function correspond to simple poles of the momentum function with residue given by $\hbar/i$ for each of these zeros. The location of these poles is energy dependent and we shall refer to them as the moving poles. Other poles whose location does not depend on energy will be refered to as the fixed poles of QMF. Let C be a contour enclosing the moving poles between the classical turning points. The integral

$$J(E) \equiv \frac{1}{2\pi}\oint_C p(x,E)dx, \tag{3}$$

called the quantum action variable is obviously $J(E)$ is equal to $n\hbar$. Thus

$$J(E) = n\hbar, \tag{4}$$

when inverted for E, gives the exact energy eigenvalues.

In this paper we show how the QHJ method gives exact eigenvalues for SUSY potentials. SUSY quantum mechanics (QM) has been studied extensively in the

last decade. For a review of SUSY in QM, the reader is referred to recent review article by Cooper, Khare and Sukhatme [3]. In SUSY QM one considers a pair of potentials defined by ($2m = 1$).

$$V_{\pm} = \omega^2(x) \pm \hbar \frac{d}{dx}\omega(x), \tag{5}$$

where $\omega(x)$ is called the super-potential. The corresponding Hamiltonians

$$H_{\pm} = p^2 + V_{\pm}(x),$$

can be factorized as

$$H_{-} = A^{\dagger}A \tag{6}$$

$$H_{+} = AA^{\dagger}, \tag{7}$$

where

$$A = \hbar \frac{d}{dx} + \omega(x) \tag{8}$$

$$A^{\dagger} = -\hbar \frac{d}{dx} + \omega(x). \tag{9}$$

Whenever the function $\exp(-\int^x \omega(y)dy)$ is square integrable, it represents the ground state wave function of H_{-}. Let ψ_0 denote this ground state wave function. Then we have

$$\omega(x) = -\hbar \frac{1}{\psi_0}\frac{\partial \psi_0}{\partial x}\,. \tag{10}$$

The eigenvalues and eigenfunctions for the two Hamiltonians $H_{\pm}$ are related. Using the intertwining relations $AH_{-} = H_{+}A$ and $A^{\dagger}H_{+} = H_{-}A^{\dagger}$, it can be easily shown that $E^{(-)}_{n+1} = E^{(+)}_{n}$ (apart from the ground state wavefunction satisfying $A\psi^{(-)}_0 = 0$). Hence $E^{(-)}_{n+1}$ and $E^{(+)}_{n}$ are the energy eigenvalues for H_{-} and H_{+} respectively. It has been known that in addition to SUSY, if a potential has a discrete reparameterization invariance called shape invarience, the corresponding eigenvalues and

eigenfunctions can be explicitly obtained algebraically. In the next section we give the details of the steps in QHJ formalism required to solve one of these potentials. In Sec. 3. we shall take up the solution of a class of SUSY potentials.

2. Eckart Potential

In case of SI potentials both V_+ and V_- are of the same functional form, albeit with different values of the parameter. In the following, we will consider only the V_- and apply the QHJ formalism to obtain the corresponding eigenvalues. In this section we shall obtain the energy eigenvalues for bound states of the Eckart potential given by

$$V(x) = A^2 + \frac{B^2}{A^2} + A(A - \alpha\hbar)\mathrm{cosech}^2\alpha x - 2B\coth\alpha x\,. \qquad (11)$$

$$(x \geq 0)$$

The corresponding superpotential is given by

$$\omega(x) = -A\coth\alpha x + \frac{B}{A}, \qquad (12)$$

where A and B are constants and the quantum Hamilton- Jacobi equation is given by

$$p^2(x,E) - i\hbar\frac{\partial p(x,E)}{\partial x} = E - A^2 - \frac{B^2}{A^2} - A(A - \alpha\hbar)\mathrm{cosech}^2\alpha x + 2B\coth\alpha x\ . \qquad (13)$$

A simple way to obtain the eigenvalues of Eckart hamiltonian is to make use of the transformation

$$y = \coth\alpha x. \qquad (14)$$

Under this mapping $p(x,E)$ becomes a function of y when x is expressed in terms of y. We shall continue to use p to denote the function of y so obtained. Thus

$p(y,E)$ means the QMF $p(x,E)$, expressed in terms of the variable y. *This will be understood for all change of variables to be considerd below.*

The quantization condition (4) then becomes

$$I_{C_1} \equiv \frac{1}{2\pi\alpha}\oint_{C_1}\frac{p(y,E)dy}{1-y^2} = n\hbar, \tag{15}$$

where C_1 is the image of the contour C of (3), enclosing the turning point, under the mapping $x \to y = \coth\alpha x$. The QHJ equation written in terms of y is given as

$$p^2 - i\hbar\alpha\frac{dp}{dy}(1+y)(1-y) = E - A^2 - \frac{B^2}{A^2} - A(A-\alpha\hbar)(y+1)(y-1) + 2By \tag{16}$$

Note that the above mapping $y = \coth\alpha r$ introduces additional singularities in the integrand at $y \pm 1$. The contour integral I_{C_1} is calculated by deforming the contour, (see fig 1.) so as to enclose all the singular points of the integrand. The poles of QMF can be located using available results [5] on complex zeroes of solutions of linear differential equations. They are easily found by inspection for all the problems of interest in this paper. For evaluation of the integral (15) consider the contour integral I_{Γ_R} for a circle Γ_R of radius R which is taken to be large enough so that outside Γ_R, $p(y)$ has no singularities. The singular points of the integrand are the poles at $y = \pm 1$ and the moving poles enclosed inside C_1. Therefore, we have

$$\frac{1}{2\pi\alpha}\oint_{\Gamma_R}\frac{pdy}{1-y^2} = \frac{1}{2\pi\alpha}\left(\oint_{C_1}\frac{pdy}{1-y^2} + \oint_{\gamma_1}\frac{pdy}{1-y^2} + \oint_{\gamma_2}\frac{pdy}{1-y^2}\right). \tag{17}$$

Denoting the integrals in (17) as I_{Γ_R}, I_{C_1}, I_{γ_1} and I_{γ_2} (fig.1), we rewrite (17) as

$$I_{\Gamma_R} = I_{C_1} + I_{\gamma_1} + I_{\gamma_2}. \tag{18}$$

The contour integral I_{Γ_R} is calculated by one more change of variable to $z = 1/y$. In terms of the variable z the integral I_{Γ_R} becomes

$$I_{\Gamma_R} = \oint_{\Gamma_R}\frac{pdy}{1-y^2} \tag{19}$$

$$= \oint_{\gamma_0} \frac{pdz}{1-z^2} \tag{20}$$

$$\equiv I_{\gamma_0} \tag{21}$$

where γ_0 is a small circle in the z-plane enclosing only one singular point $z = 0$. It is worth reminding that both the contours are in the anticlockwise direction and the singularity at $y \to \infty$ is mapped to the singularity at $z = 0$.

Therefore (15), (18) and (21) give

$$n\alpha\hbar = \frac{1}{2\pi}\oint_{C_1} \frac{pdy}{1-y^2} = \frac{1}{2\pi}\left(\oint_{\gamma_0} \frac{pdz}{1-z^2} - \oint_{\gamma_1} \frac{pdy}{1-y^2} - \oint_{\gamma_2} \frac{pdy}{1-y^2}\right). \tag{22}$$

The calculation of various integrals requires behavior of the momentum function near the singular points. In particular, we need the value of the function at $y = \pm 1$ and its residue at $z = 0$. These are calculated by substituting appropriate Taylor or Laurent series expansion of p in the QHJ equation and solving for the first few coefficients in the series expansion. For example, for calculation of the contour integral around $y = \pm 1$, the series expansion of $p(y, E)$ around $y = \pm 1$ is used. The integrand suggests that we need to calculate only the coefficient of the constant term. We illustrate this for computing the integrals I_{γ_1}; for this purpose we at first expand $p(y, E)$ as

$$p(y, E) = \alpha_0 + \alpha_1(y-1) + \alpha_2(y-1)^2 + \cdots\cdots$$

and substite the expansion in (16). Comparing the terms independent of y on both sides gives

$$\alpha_0^2 = E - A^2 - \frac{B^2}{A^2} + 2B \tag{23}$$

Let β_0 be the constant term in the expansion of $p(y, E)$ in powers of $(y + 1)$. Then β_0 is similarly determined and is given by

$$\beta_0^2 = E - A^2 - \frac{B^2}{A^2} - 2B \tag{24}$$

Note that, there are two roots for α_0 and β_0, corresponding to the two signs of the square of the right hand side in (23)and (24). Similarly for obtaining the integral I_{γ_0}, we need to compute the residue of $p(z,E)$ at $z=0$. We expand $p(z,E)$ as

$$p(z,E) = b_1/z + a_0 + a_1 z + \cdots \tag{25}$$

and substitute (25) the QHJ equation written in terms of the variable z

$$p^2 - i\alpha\hbar\left(\frac{1-z^2}{z^2}\right)\frac{dp}{dz} = E - A^2 - \frac{B^2}{A^2} - A(A-\alpha\hbar)\left(\frac{1-z^2}{z^2}\right) + \frac{2B}{z}. \tag{26}$$

Comparing the coefficients of $1/z^2$ on the two sides gives,

$$b_1 = \frac{-i\alpha\hbar \pm i(\alpha\hbar - 2A)}{2} \tag{27}$$

We select the correct root for α_0, β_0 and b_1 from (27), (23) and (24) by comparing them with the answer for $E=0$. It is straight forward to obtain the value of b_1, α_0 and β_0 for $E=0$, by recalling that for zero energy the QMF is related to the superpotential by $p(x, E=0) = i\omega(x)$. Writing the superpotential $\omega(x)$ in terms of z, we get

$$\omega(z) = -\frac{A}{z} + \frac{B}{A}. \tag{28}$$

The residue of $\omega(z)$ at $z=0$ is $-A$. Comparing this answer with value of b_1, we see that the correct choice of b_1 is given by $b_1 = -iA$ for all E. Similarly one looks for the coefficient of expansion of w in powers of $y \pm 1$ and this is compared with the values of α_0 and β_0 respectively for $E=0$. It is found that these roots have relative opposite signs and are given by

$$\alpha_0 = -\sqrt{E - A^2 - \frac{B^2}{A^2} + 2B} \tag{29}$$

$$\beta_0 = \sqrt{E - A^2 - \frac{B^2}{A^2} - 2B} \tag{30}$$

It may be noted that the correct sign of the residues calculated can also be fixed by taking $\hbar \to 0$ and looking at the behaviour of p_c near the point of interest. This

procedure, as originally suggested by Leacock and Padgett, is a bit complicated. For SUSY potentials under consideration in this article we have found it useful to follow the alternate procedure as explained above.

The contour integrals I_{γ_0}, I_{γ_1} and I_{γ_2} are, therefore, computed to be

$$I_{\gamma_1} = \frac{i\alpha_0}{2\alpha}, \qquad I_{\gamma_2} = \frac{i\beta_0}{2\alpha}, \qquad \text{and} \qquad I_{\gamma_0} = \frac{A}{\alpha}.$$

Thus the energy eigenvalues are obtained from

$$I_{C_1} = n\hbar = \frac{A}{\alpha} - \frac{i\beta_0}{2\alpha} - \frac{i\alpha_0}{2\alpha} \tag{31}$$

which on further simplification gives

$$E_n = A^2 + \frac{B^2}{A^2} - \frac{B^2}{(n\alpha\hbar + A)^2} - (n\alpha\hbar + A)^2 \ , \tag{32}$$

One can arrive at the above energy eigenvalues using other mappings. Use of a different mapping will be considered in the next section and we shall then show how to obtain the bound state eigenvalues for other SUSY potentials.

3. Other SUSY potentials

In this section we will show how the QHJ method could be used for other SUSY potentials. We will use $y = \exp(i\alpha x)$ mapping for the SUSY potentials involving trigonometric functions. The remaining potentials involving hyperbolic functions $y = \exp(\alpha x)$ will be used. The treatment of each of these cases runs parallel to the treatment given to the Eckart potential in the previous section except for a new point which requires special attention. The use of the mapping $y = \exp(\alpha x)$ gives rise to extra energy dependent poles in $p(x, E)$ in the non-classical region. Computation of the exact energy eigen-values requires the knowledge of an integral along a contour enclosing these energy dependent poles. In all the cases investigated in this paper this integral can be related to the integral around the contour which encloses poles in the classical region on the real axis.

In this section at first we shall work out the eigenvalues for the Eckart potential again, using the mapping $y = \exp(\alpha x)$, but concentrating only on the new points as compared to the treatment in the previous section.

The corresponding super potential for the Eckart potential written in terms of the variable $y = \exp(\alpha x)$ is

$$\omega(y) = -A\left(\frac{y^2+1}{y^2-1}\right) + \frac{B}{A}, \tag{33}$$

and the QHJ equation is

$$p^2 - i\hbar\alpha y\frac{dp}{dy} = E - A^2 - \frac{B^2}{A^2} - \frac{4A(A-\alpha\hbar)y^2}{(y^2-1)^2} + \frac{2B(y^2+1)}{(y^2-1)}. \tag{34}$$

Now the equation

$$E - A^2 - \frac{B^2}{A^2} - \frac{4A(A-\alpha\hbar)y^2}{(y^2-1)^2} + \frac{2B(y^2+1)}{(y^2-1)} = 0, \tag{35}$$

has four solutions for the turning points. These are shown as A, B, A' B' (see Fig. 2). The moving poles of p are (i) on the real axis and between A and B and (ii)

on the real axis between A' and B'. It should however be noted, that two of these turning points A' and B' are in the non-classical region. We note that the symmetry $y \to -y$ in (35) interchanges A with A' and B with B'. This symmetry implies

$$I_{C_1} = I_{C_2}, \tag{36}$$

where C_1 and C_2 are contours enclosing A, B, and A', B' respectively as shown in Fig. 2. Next note that now $y = 0$ is a pole of the integrand in the action integral (see (38) below). Also $p(y)$ has poles at $y = 1$ and $y = -1$ because the right hand side of QHJ is singular at $y = 1$ and $y = -1$. Introducing I_{Γ_R} for the large circle Γ_R enclosing all the singular points we arrive at

$$I_{\Gamma_R} = I_{C_1} + I_{C_2} + I\gamma_1 + I_{\gamma_2} + I_{\gamma_3}. \tag{37}$$

Here $I_{\gamma_1}, I_{\gamma_2}, I_{\gamma_3}$ are integrals along contours $\gamma_1, \gamma_2, \gamma_3$ which enclose the singular points $y = 1, y = -1$ and $y = 0$ respectively. Therefore, using (36), the quantization condition

$$I_{C_1} \equiv \frac{1}{2\pi\alpha} \oint_{C_1} \frac{dy}{y} p(y, E) = n\hbar. \tag{38}$$

becomes

$$I_{\Gamma_R} - \sum_p I_{\gamma_p} = 2n\hbar. \tag{39}$$

The rest of the calculation is same as in Sec. 2. and one easily arrives at

$$E_n = A^2 + \frac{B^2}{A^2} - \frac{B^2}{(n\alpha\hbar + A)^2} - (n\alpha\hbar + A)^2. \tag{40}$$

The calculation of eigenvalues for other SUSY potentials proceeds in a similar fashion. The results are summarized in Table I and Table II. The range of the variable x is $-\infty < x < \infty$, unless indicated otherwise. Expressions listed in the coloumn four are αI_{γ_p} are the values for a contour γ_p enclosing only the singular point indicated in the third column of the table. The value of I_{γ_p} for the pole at ∞ stands for

the value I_{Γ_R}. For the fixed poles at $y = 0$ and ∞ the square roots in the residue are found to have relatively opposite signs, on comparision with the coefficients of corresponding terms in the expansion of $\omega(x)$ for $E = 0$. The eigen-values listed in the last column are obtained using (39).

4. Conclusion:

In conclusion, we have explicitly worked out the eigenvalues for the SUSY potentials using the QHJ method. Apart from checking the correctness of the formalism, this exercise provides insight into solvability of these potentials. The main effort involved in use of this scheme lies in selecting the correct root for the residues needed. This problem was solved here by comparing the answers obtained from the QHJ for $E = 0$ with that obtained from the superpotential. In general the choice of the correct root for the residue can be made by using the boundary condition on QMF, viz., $p \to p_c$ in the limit $\hbar \to 0$ and where the branch of p_c is selected in such a way that it corresponds to the positive sign just below the cut on the real axis joining the physical turning points.

The Leacock-Padgett method is a powerful method for obtaining the eigenvalues analytically as well as numerically and can be applied to other potentials. It is worth pointing out that for the potentials considered in this article, the SUSY-WKB approximation gives exact answer. Since QHJ as shown here also gives exact answer, it is natural to enquire as to the relation between these two approaches.This will throw light on the question of exactness of the SUSY WKB formula. This investigation will be reported elsewhere.

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References

1. H. Goldstein, Classical Mechanics (Addison-Wesley, New York,1950).

2. R.A. Leacock and M.J. Padgett, Phys. Rev. **D28**, 2491 (1983)
   R.A.Leacock and M.J. Padgett, Phys. Rev. Lett. **50**, 3 (1983).

3. F. Cooper, A. Khare and U. Sukhatme Phys. Reports **251**, 267 (1995).

4. A. Comtet, A. Bandrauk and D.K. Campbell, Phys. Lett. **B150**, 159 (1985).

5. E. L. Ince , *Ordinary Differential Equations*, Dover Publcations N.Y (1956).

For all the potentials listed in this table the mapping $y = \exp(i\alpha x)$ is used.

Name of potential	Potential	Location of fixed poles	αI_{γ_p}	Eigen value
		0	$\sqrt{E+A^2}$	
Scarf I (Trignometric)	$-A^2+(A^2+B^2+A\alpha\hbar)\times$ $\sec^2\alpha x - B(2A+\alpha\hbar)\times$ $\sec\alpha x \tan\alpha x$ $(-\pi/2 \le \alpha x \le \pi/2)$	i	$-A+B$	$(A+n\alpha\hbar)^2$ $-A^2$
		$-i$	$-(A+B)$	
		∞	$-\sqrt{E+A^2}$	
		0	$-\sqrt{E+A^2-B^2/A^2+2iB}$	
Rosen-Morse-I (trignometric)	$A(A-\alpha\hbar)\mathrm{cosec}^2\alpha x$ $-A^2+B^2/A^2$ $+2B\cot\alpha x$ $(0 \le \alpha x \le \pi)$	1	A	$+B^2/A^2$ $A^2-(A+n\alpha\hbar)^2$
		-1	A	$-B^2/(A+n\alpha\hbar)^2$
		∞	$\sqrt{E+A^2-B^2/A^2-2iB}$	

For all the potentials listed in this table the mapping $y = \exp(\alpha x)$ is used.

Name of potential	Potential	Location of fixed poles	αI_{γ_p}	Eigen value
Scarf II (hyperbolic)	$A^2 + (B^2 - A^2 - A\alpha\hbar)\times \operatorname{sech}^2 \alpha x + B(2A + \alpha\hbar)\times \operatorname{sech}\alpha x \tanh \alpha x$	0	$-i\sqrt{E - A^2}$	$A^2 - (A - n\alpha\hbar)^2$
		i	$iB - A$	
		$-i$	$-iB - A$	
		∞	$i\sqrt{E - A^2}$	
Rosen - Morse II (Hyperbolic)	$A^2 + B^2/A^2 - A(A + \alpha\hbar)\operatorname{sech}^2 \alpha x + 2B \tanh \alpha x$	0	$-i\sqrt{E - A^2 - B^2/A^2 + 2B}$	$A^2 + B^2/A^2 - (A - n\alpha\hbar)^2$
		i	$-A$	$-B^2/(A - n\alpha\hbar)^2$
		$-i$	$-A$	
		∞	$i\sqrt{E - A^2 - B^2/A^2 - 2B}$	
Eckart (Hyperbolic)	$A^2 + B^2/A^2 + A(A - \alpha\hbar)\operatorname{cosech}^2 \alpha x - 2B \coth \alpha x$ $(x \geq 0)$	0	$i\sqrt{E - A^2 - B^2/A^2 - 2B}$	$A^2 + B^2/A^2 - (A + n\alpha\hbar)^2$
		1	A	$-B^2/(A + n\alpha\hbar)^2$
		-1	A	
		∞	$-i\sqrt{E - A^2 - B^2/A^2 + 2B}$	
Generalised Poschl-Teller	$A^2 + (B^2 + A^2 + A\alpha\hbar)\times \operatorname{cosech}^2 \alpha x - B(2A + \alpha\hbar)\times \coth \alpha x \operatorname{cosech} \alpha x$ $(x \geq 0)$	0	$-i\sqrt{E - A^2}$	$A^2 - (A - n\alpha\hbar)^2$
		1	$-A + B$	
		-1	$-(A + B)$	
		∞	$i\sqrt{E - A^2}$	