

Fuzzy Mass Relations in the Standard Model

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Abstract

Recently Connes has proposed a new geometric version of the standard model including a non-commutative charge conjugation. We present a systematic analysis of the relations among masses and coupling constants in this approach. In particular, for a given top mass, the Higgs mass is constrained to lie in an interval. Therefore this constraint is locally stable under renormalization flow.

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1 Introduction

In his beautiful book [1], A. Connes applies non-commutative geometry to the standard model of particles. The last theorem of this book states that the ordinary Lagrangian of the standard model with three generations of leptons and quarks and one doublet of Higgs scalars has a natural algebraic interpretation. Its principal ingredients are two algebras and a generalized Dirac operator. From these, Connes constructs two differential algebras, two gauge potentials, their curvatures and the Euclidean Yang-Mills actions as scalar products of the curvatures with themselves. When applied to the commutative case — the commutative algebra of smooth functions on a four dimensional spacetime and the genuine Dirac operator — this Yang-Mills action and the covariantized Dirac action reproduce spinor electrodynamics. However, when applied to the tensor product of the commutative spacetime algebra with two non-commutative internal algebras these two Lagrangians reproduce exactly the Lagrangian of the standard model including the entire Higgs sector, i.e. the Klein-Gordon Lagrangian for the Higgs scalars, their Higgs potential and their Yukawa terms. In particular, the 18 free parameters of the standard model (which can be taken to be the three gauge couplings, g_1 , g_2 , g_3 , the masses of the W , of the Higgs, of three leptons and of six quarks, and four mixing parameters in the Cabbibo-Kobayashi-Maskawa matrix) remain free and are the only free physical parameters in the non-commutative approach.

In its original version, this non-commutative approach due to A. Connes and J. Lott [2],[3] had two major shortcomings: the need of two algebras with related bimodules and two extra $U(1)$ factors in the gauge group that had to be eliminated by two additional algebraic (unimodularity) conditions. Recently, Connes [4] has improved this framework by introducing a real structure of a spectral triplet $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, where \mathcal{A} is a *real* algebra represented on the Hilbert space \mathcal{H} and \mathcal{D} a Dirac operator on \mathcal{H} . The real structure is given by an anti-unitary operator J on \mathcal{H} which, in commutative geometry, is the charge conjugation. Now, the internal space of the standard model is described by *one* algebra $\mathcal{A} = \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C})$, \mathbb{H} is the algebra of quaternions, the Hilbert space is spanned by all leptons and quarks and the Dirac operator is given by the fermionic mass matrix. The hypothesis of the quoted theorem now becomes extremely simple. The gauge group G is the group of unitaries of \mathcal{A} , $G = SU(2) \times U(1) \times U(3)$. The gauge potential is a 1-form in *the* differential algebra of the triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and its Yang-Mills action together with the *Majorana* action suitably covariantized yields the action of the standard model with a doublet of Higgs scalars. There are again the 18 free parameters.

In the standard model, the space of parameters is a direct product of 18 intervals. In the non-commutative approach, the space of parameters which, according to Connes' theorem, has a non-empty interior, reveals an interesting shape. In particular, $\sin^2 \theta_w$ is bounded from above, the mass of the W lies essentially between the lightest and the heaviest fermion mass and the Higgs mass is bounded from below and above. Since all values of the interval are possible, we call this framing a *fuzzy mass relation*.

Throughout this paper, we assume that all fermion masses are different and that the Cabbibo-Kobayashi-Maskawa matrix is non-degenerate, i.e. has no proper invariant subspace.

Theorem.

i.

$$\sin^2 \theta_w < \frac{2}{3} \left(1 + \frac{1}{9} \left(\frac{g_2}{g_3} \right)^2 \right)^{-1}. \quad (1)$$

ii. If the heaviest lepton τ satisfies $m_\tau^2 < (m_t^2 + m_b^2 + m_c^2 + m_s^2 + m_u^2 + m_d^2)/3$, then, with e the lightest lepton, we have

$$m_e^2 < m_W^2 < (m_t^2 + m_b^2 + m_c^2 + m_s^2 + m_u^2 + m_d^2)/3. \quad (2)$$

iii.

$$m_{H_{min}}^2 < m_H^2 < m_{H_{max}}^2 \quad (3)$$

where $m_{H_{min}}^2$ and $m_{H_{max}}^2$ depend on all fermion masses but m_μ . The bounds are given by equations (13) and (15) and plotted in the figure.

$m_{H_{max}}^2 - m_{H_{min}}^2$ factorizes $(m_\tau^2 - m_e^2)$ and $(m_t^2 + m_b^2 + m_c^2 + m_s^2 + m_u^2 + m_d^2 - 3m_W^2)$. In particular, neglecting all fermion masses but m_τ and m_t , we have

$$m_{H_{max}} - m_{H_{min}} = \left[k \left(\frac{m_\tau}{m_t} \right)^2 + O \left(\left(\frac{m_\tau}{m_t} \right)^4 \right) \right] m_t$$

where k is given in equations (16-17) and is of order one for experimental values of m_W and m_t .

2 The geometric version of the standard model

The basis of non-commutative geometry is a (real) spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

\mathcal{A} is a real, associative algebra with unit 1 and involution $*$. The spacetime M is described by the infinite dimensional commutative algebra of smooth functions $f : M \rightarrow \mathbb{C}$ with involution $f^* = \bar{f}$, the complex conjugate. The internal space is described by a finite dimensional algebra whose group of unitaries $G := \{g \in \mathcal{A} \mid gg^* = g^*g = 1\}$ will contain the gauge group. In the case of the standard model, this choice is

$$\mathcal{A} = \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C}) \quad \text{with} \quad G = SU(2) \times U(1) \times U(3).$$

We denote by \mathbb{H} the algebra of quaternions, viewed as 2×2 matrices,

$$\begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \in \mathbb{H}, \quad x, y \in \mathbb{C}.$$

\mathcal{H} is a Hilbert space carrying a faithful representation ρ of the algebra \mathcal{A} . We also assume that \mathcal{H} is equipped with a chirality χ and a charge conjugation J . The chirality is a unitary operator of square one that commutes with the representation. Therefore χ decomposes the representation space into a left-handed piece $(1 - \chi)/2 \mathcal{H}$ and a right-handed piece $(1 + \chi)/2 \mathcal{H}$. The charge conjugation is an anti-unitary operator of square plus or minus one, depending on spacetime dimension and signature. Also depending on spacetime dimension and signature, J commutes or anticommutes with χ . We further assume that

- $\rho(a)$ commutes with $J\rho(\tilde{a})J^{-1}$, for all a, \tilde{a} in \mathcal{A} . (4)

The charge conjugation as well decomposes the representation space into two pieces, particles and anti-particles,

$$\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_L^c \oplus \mathcal{H}_R^c.$$

For a four dimensional spacetime, the Hilbert space consists of all square integrable (Dirac) spinors, a function f acting on a spinor ψ by multiplication, $(\rho(f)\psi)(x) := f(x)\psi(x)$. The chirality $\chi = \gamma_5$ decomposes a Dirac spinor into left- and right-handed (Weyl) spinors and the charge conjugation acts on ψ as $\psi^c := i\gamma^2\bar{\psi}$ where γ^2 is the second Dirac matrix and the bar denotes complex conjugation. The internal space counts as zero dimensional [1]. Its Hilbert space is finite dimensional and contains all fermions. For the standard model, we have

$$\mathcal{H}_L = (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}),$$

$$\mathcal{H}_R = ((\mathbb{C} \oplus \mathbb{C}) \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C} \otimes \mathbb{C}^N \otimes \mathbb{C}).$$

In each summand, the first factor denotes weak isospin doublets or singlets, the second N generations, $N = 3$, and the third denotes colour triplets or singlets.

Let us choose the following basis of $\mathcal{H} = \mathbb{C}^{90}$:

$$\begin{aligned} & \begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L, \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L; \\ & u_R, c_R, t_R, e_R, \mu_R, \tau_R; \\ & \begin{pmatrix} u \\ d \end{pmatrix}_L^c, \begin{pmatrix} c \\ s \end{pmatrix}_L^c, \begin{pmatrix} t \\ b \end{pmatrix}_L^c, \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L^c, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L^c, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L^c; \\ & u_R^c, c_R^c, t_R^c, e_R^c, \mu_R^c, \tau_R^c. \end{aligned}$$

Let $(a, b, c) \in \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C})$ be an element in the algebra \mathcal{A} . ρ acts on the above Hilbert space by

$$\rho(a, b, c) := \begin{pmatrix} \rho_w(a, b) & 0 \\ 0 & \bar{\rho}_s(b, c) \end{pmatrix}$$

with

$$\rho_w(a, b) := \begin{pmatrix} a \otimes 1_N \otimes 1_3 & 0 & 0 & 0 \\ 0 & a \otimes 1_N & 0 & 0 \\ 0 & 0 & B \otimes 1_N \otimes 1_3 & 0 \\ 0 & 0 & 0 & \bar{b}1_N \end{pmatrix}, \quad B := \begin{pmatrix} b & 0 \\ 0 & \bar{b} \end{pmatrix},$$

$$\rho_s(b, c) := \begin{pmatrix} 1_2 \otimes 1_N \otimes c & 0 & 0 & 0 \\ 0 & \bar{b}1_2 \otimes 1_N & 0 & 0 \\ 0 & 0 & 1_2 \otimes 1_N \otimes c & 0 \\ 0 & 0 & 0 & \bar{b}1_N \end{pmatrix}.$$

The chosen representation ρ will take into account weak interactions $\rho_w(a, b)$, $a \in \mathbb{H}$, $b \in \mathbb{C}$, and strong interactions $\rho_s(b, c)$, $c \in M_3(\mathbb{C})$, c for colour. This choice discriminates between leptons (colour singlets) and quarks (colour triplets). The role of $b \in \mathbb{C}$ appearing in both weak interactions $\rho_w(a, b)$ and strong interactions $\rho_s(b, c)$ is crucial to make $\rho(a, b, c)$ a representation of \mathcal{A} and is crucial for weak hypercharge computations. There is an apparent asymmetry between particles and anti-particles, the former are subject to weak, the latter to strong interactions. However, since particles and anti-particles are permuted by J via the fundamental property (4), the theory is invariant under charge conjugation.

The chirality operator and charge conjugation are

$$\chi = \begin{pmatrix} -1_{24} & 0 & 0 & 0 \\ 0 & +1_{21} & 0 & 0 \\ 0 & 0 & -1_{24} & 0 \\ 0 & 0 & 0 & +1_{21} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1_{45} \\ 1_{45} & 0 \end{pmatrix} C,$$

C being the complex conjugation.

The last item in the spectral triple is the (generalized) Dirac operator \mathcal{D} , a selfadjoint operator with the following properties:

- $\mathcal{D}\chi = -\chi\mathcal{D}$,
- $\mathcal{D}J = +J\mathcal{D}$,
- $[\mathcal{D}, \rho(a)]$ is bounded for all a in \mathcal{A} ,
- $[\mathcal{D}, \rho(a)]$ commutes with $J\rho(\bar{a})J^{-1}$, for all a, \bar{a} in \mathcal{A} .

For spacetime, \mathcal{D} is the genuine Dirac operator. For the internal space, \mathcal{D} is made up with the fermionic mass matrix \mathcal{M} ,

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{M} & 0 & 0 \\ \mathcal{M}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{M} \\ 0 & 0 & \mathcal{M}^* & 0 \end{pmatrix}.$$

Let us recall the mass matrix of the standard model:

$$\mathcal{M} = \begin{pmatrix} \begin{pmatrix} M_u \otimes 1_3 & 0 \\ 0 & M_d \otimes 1_3 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 \\ M_e \end{pmatrix} \end{pmatrix},$$

with

$$M_u := \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \quad M_d := C_{KM} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}, \quad M_e := \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}.$$

All indicated fermion masses are supposed positive and different. The Cabbibo-Kobayashi-Maskawa matrix C_{KM} is supposed non-degenerate in the sense that there is no simultaneous mass and weak interaction eigenstate.

Note that the strong interactions are vector-like: for all $b \in \mathbb{C}$ and $c \in M_3(\mathbb{C})$, $\rho_3(b, c)$ commutes with the corresponding restrictions of χ and \mathcal{D} .

A last ingredient of the general theory is another operator z on the Hilbert space. z is used to construct a gauge invariant scalar product $(\omega, \kappa) := \text{tr}(\omega^* \kappa z)$ for two forms ω, κ of equal degree in the differential algebra of the internal spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. Since the gauge couplings in usual Yang-Mills theories parameterize gauge invariant scalar products on the Lie algebra, z deserves the name ‘non-commutative coupling constant’. Here is the list of its properties:

- z is positive,
- $[z, \rho(a)] = [z, J\rho(a)J^{-1}] = 0, \quad a \in \mathcal{A},$
- $[z, \chi] = 0,$
- $[z, \mathcal{D}] = 0.$

For spacetime, z is simply a positive number times the identity. For the internal space of the standard model, the most general z involves $2(1+N) = 8$ strictly positive numbers $x, y_1, y_2, y_3, \tilde{x}, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3,$

$$z = \begin{pmatrix} z_w & 0 \\ 0 & \bar{z}_s \end{pmatrix},$$

$$z_w := \begin{pmatrix} x/3 \mathbf{1}_2 \otimes \mathbf{1}_N \otimes \mathbf{1}_3 & 0 & 0 & 0 \\ 0 & \mathbf{1}_2 \otimes y & 0 & 0 \\ 0 & 0 & x/3 \mathbf{1}_2 \otimes \mathbf{1}_N \otimes \mathbf{1}_3 & 0 \\ 0 & 0 & 0 & y \end{pmatrix},$$

$$z_s := \begin{pmatrix} \tilde{x}/3 \mathbf{1}_2 \otimes \mathbf{1}_N \otimes \mathbf{1}_3 & 0 & 0 & 0 \\ 0 & \mathbf{1}_2 \otimes \tilde{y} & 0 & 0 \\ 0 & 0 & \tilde{x}/3 \mathbf{1}_2 \otimes \mathbf{1}_N \otimes \mathbf{1}_3 & 0 \\ 0 & 0 & 0 & \tilde{y} \end{pmatrix},$$

$$y := \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{pmatrix}, \quad \tilde{y} := \begin{pmatrix} \tilde{y}_1 & 0 & 0 \\ 0 & \tilde{y}_2 & 0 \\ 0 & 0 & \tilde{y}_3 \end{pmatrix}.$$

The interpretation of these numbers is straightforward. The three y_j poise the weak interactions with the three lepton generations. The y_j enter independently because the Higgs scalar couples differently to the three leptons and in non-commutative geometry the Higgs is part of the gauge interactions. The three \tilde{y}_j poise the ‘strong’ interactions with the three lepton generations. They do not drop out because of the b in ρ_3 . However, as we shall see in equations (7-10), they will only enter as sum: strong interactions are unbroken and do not generate a Higgs. x and \tilde{x} poise weak and strong interactions with quarks. There is only one number per interaction because of the Cabbibo-Kobayashi-Maskawa mixing that we suppose non-degenerate.

In the standard model, the scalars turn out to sit in one isospin doublet - colour singlet φ and their potential is computed [5],

$$V(\varphi) = \frac{K}{16L^2} |\varphi|^4 - \frac{K}{2L} |\varphi|^2.$$

The coefficients depend on the coupling constants in z_w only — because strong interactions do not contribute to the spontaneous symmetry breaking— and on squares of the fermion masses, the Cabbibo-Kobayashi-Maskawa matrix drops out:

$$K := \frac{3}{2} \text{tr} [(M_u^* M_u)^2] x + \frac{3}{2} \text{tr} [(M_d^* M_d)^2] x + \text{tr} [M_u^* M_u M_d^* M_d] x + \frac{3}{2} \text{tr} [M_e^* M_e M_e^* M_e] y - \frac{1}{2} L^2 \left[\frac{1}{Nx + \text{tr } y} + \frac{1}{Nx + \text{tr } y/2} \right], \quad (5)$$

$$L := \text{tr} [M_u^* M_u] x + \text{tr} [M_d^* M_d] x + \text{tr} [M_e^* M_e] y. \quad (6)$$

At the same time, the Yang-Mills Lagrangians for isospin and colour come out respectively as

$$\begin{aligned} \frac{1}{2} \text{tr} [\rho(F_{2\mu\nu}, 0, 0)^* \rho(F_{2}^{\mu\nu}, 0, 0) z] &= \frac{2}{g_2^2} \frac{1}{4} \text{tr} [F_{2\mu\nu}^* F_{2}^{\mu\nu}], \quad F_{2\mu\nu} \in \{a \in \mathbb{H}, a^* = -a\}, \\ \frac{1}{2} \text{tr} [\rho(0, 0, F_{3\mu\nu})^* \rho(0, 0, F_{3}^{\mu\nu}) z] &= \frac{2}{g_3^2} \frac{1}{4} \text{tr} [F_{3\mu\nu}^* F_{3}^{\mu\nu}], \quad F_{3\mu\nu} \in \{c \in M_3(\mathbb{C}), c^* = -c\}, \end{aligned}$$

with gauge couplings therefore given by

$$\begin{aligned} g_2^{-2} &= Nx + \text{tr } y, \\ g_3^{-2} &= \frac{4}{3} N\tilde{x}. \end{aligned} \quad (7)$$

Consequently, we have the following mass relations:

$$m_W^2 = \frac{L}{Nx + \text{tr } y}, \quad (8)$$

$$m_H^2 = \frac{2K}{L}. \quad (9)$$

So far we have identified the $SU(2)$ of weak isospin and the $SU(3)$ of colour together with their gauge couplings. It remains to look at the $U(1)$ of hypercharge. The Lie algebra of the gauge group is $\mathfrak{g} = \{a \in \mathcal{A} \mid a^* = -a\} = su(2) \oplus u(1) \oplus su(3) \oplus u(1)$. Fortunately, the hypercharge generator Y is a linear combination of the two $U(1)$ generators $(0, i, 0)$, $(0, 0, i1_3)$:

$$Y = \frac{1}{i} \rho \left(0, \frac{i}{2}, \frac{i}{6} 1_3 \right).$$

To compute its gauge coupling, we have to recall that $U(1)$ gauge couplings are conventionally normalized differently than $SU(n)$ gauge couplings,

$$\frac{1}{2} \operatorname{tr} [Y^* Y z] =: \frac{1}{g_1^2} \frac{1}{4}.$$

Therefore,

$$g_1^{-2} = Nx + \frac{2}{9} N\tilde{x} + \frac{1}{2} \operatorname{tr} y + \frac{3}{2} \operatorname{tr} \tilde{y}. \quad (10)$$

A final remark of this section concerns the second, unwanted $U(1)$ which is generated by a linear combination orthogonal to Y . By imposing an algebraic condition, the Lie algebra \mathfrak{g} is reduced to the desired subalgebra $su(2) \oplus u(1)_Y \oplus su(3)$. The condition

$$\operatorname{tr} [J\rho(1_2, 0, 0)J^{-1} \rho(a, b, c)] = 0, \quad (a, b, c) \in \mathfrak{g}, \quad (11)$$

namely $4N(\operatorname{tr} c + \bar{b}) = 0$ selects precisely weak isospin, hypercharge and colour. This condition looks like a unimodularity condition because $J\rho(1_2, 0, 0)J^{-1}$ is a selfadjoint element in the commutant of $\rho(\mathcal{A})$. Despite this arbitrary choice of the element in the commutant, the condition (11) is equivalent to

$$\operatorname{tr} \left[(\rho(a, b, c) + J\rho(a, b, c)J^{-1}) P \right] = \operatorname{tr} [\rho_w(a, b) + \rho_s(b, c)] = 0,$$

where P is the projection on $\mathcal{H}_L \oplus \mathcal{H}_R$, the space of particles, and so appears more natural. Note that this condition is also related to the condition of vanishing anomalies [6].

3 Fuzzy relations among masses and coupling constants

3.1 Masses

Let us come back to the mass relations (8-9). Their coefficients (5-7) contain only squares of masses and we put

$$t := m_t^2, \quad W := m_W^2, \quad H := m_H^2, \quad \dots$$

Furthermore, the mass relations are homogeneous in the variables x, y_1, y_2, y_3 and we may set $x = 1/3$ without loss of generality. Then the two mass relations read

$$\begin{aligned}\frac{H}{W} + 1 &= \frac{C}{X} + 3\frac{Y}{X} - 2\frac{X}{1+X}, \\ X &= \sum_{j=0}^3 \alpha_j y_j,\end{aligned}\tag{12}$$

with the following abbreviations

$$\begin{aligned}C &:= \frac{t^2 + b^2 + c^2 + s^2 + u^2 + d^2}{W^2} + \frac{2}{3} \frac{tb + cs + ud}{W^2} - \frac{1}{3} \frac{q^2}{W^2}, \\ q &:= t + b + c + s + u + d, \\ \alpha_0 &:= q/3W, \quad \alpha_1 := e/W, \quad \alpha_2 := \mu/W, \quad \alpha_3 := \tau/W, \\ y_0 &:= 3x = 1, \\ X &:= \sum_{j=0}^3 y_j, \quad Y := \sum_{j=0}^3 \alpha_j^2 y_j.\end{aligned}$$

An immediate conclusion is that the W mass lies between the masses of the lightest and the heaviest fermion, more precisely, if the latter is a quark with non-degenerate Cabbibo-Kobayashi-Maskawa mixing in 3 generations, we have

$$e < W < (t + b + c + s + u + d)/3.$$

The following lemma justifies the choice of these new variables X and Y , since they are independent and bounded.

Lemma 1. *Let $\alpha_0, \alpha_1, \dots, \alpha_N$ be $N + 1$ real numbers, $N \geq 3$, satisfying the inequalities $0 < \alpha_1 < \dots < \alpha_N < 1 < \alpha_0$, and y_0, y_1, \dots, y_N be $N + 1$ strictly positive variables. Consider the domain in \mathbb{R}^{N+1} subject to the constraints $y_0 = 1$ and (12), namely*

$$D := \left\{ y = (1, y_1, \dots, y_N), y_j > 0, \sum_{j=0}^N (1 - \alpha_j) y_j = 0 \right\},$$

and define the variables $X := \sum_{j=0}^N y_j$, $Y := \sum_{j=0}^N \alpha_j^2 y_j$. Then,

- i. X and Y are independent on D .
- ii. On D , X and Y vary in the open intervals

$$\begin{aligned}X_{min} &:= \frac{\alpha_0 - \alpha_1}{1 - \alpha_1} < X < \frac{\alpha_0 - \alpha_N}{1 - \alpha_N} =: X_{max}, \\ Y_{min} &:= \alpha_0^2 + (\alpha_0 - 1) \frac{\alpha_1^2}{1 - \alpha_1} < Y < \alpha_0^2 + (\alpha_0 - 1) \frac{\alpha_N^2}{1 - \alpha_N} =: Y_{max}.\end{aligned}$$

Proof. i. follows from a non-vanishing functional determinant. In fact, it is sufficient to consider the case $N = 3$. We solve the constraint $\sum_{j=0}^3 (1 - \alpha_j) y_j = 0$:

$$y_3 = -\frac{1 - \alpha_0}{1 - \alpha_3} - \frac{1 - \alpha_1}{1 - \alpha_3} y_1 - \frac{1 - \alpha_2}{1 - \alpha_3} y_2,$$

we eliminate y_3 :

$$\begin{aligned} X &= \frac{\alpha_0 - \alpha_3}{1 - \alpha_3} + \frac{\alpha_1 - \alpha_3}{1 - \alpha_3} y_1 + \frac{\alpha_2 - \alpha_3}{1 - \alpha_3} y_2, \\ Y &= \left(\alpha_0^2 - \alpha_3^2 \frac{1 - \alpha_0}{1 - \alpha_3} \right) + \left(\alpha_1^2 - \alpha_3^2 \frac{1 - \alpha_1}{1 - \alpha_3} \right) y_1 + \left(\alpha_2^2 - \alpha_3^2 \frac{1 - \alpha_2}{1 - \alpha_3} \right) y_2, \end{aligned}$$

and compute the functional determinant

$$\det \begin{pmatrix} \frac{\partial X}{\partial y_1} & \frac{\partial X}{\partial y_2} \\ \frac{\partial Y}{\partial y_1} & \frac{\partial Y}{\partial y_2} \end{pmatrix} = \frac{(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_1 - \alpha_3)}{1 - \alpha_3} \neq 0.$$

To prove ii., we note that D is convex and bounded. Indeed, for $j = 1, \dots, N$ we have

$$(1 - \alpha_j) y_j < \sum_{j=1}^N (1 - \alpha_j) y_j = -(1 - \alpha_0) y_0 = \alpha_0 - 1,$$

and $0 < y_j < \frac{\alpha_0 - 1}{1 - \alpha_j}$. For every $n = 1, \dots, N$, let us define the vector

$$P_n := \left(1, 0, \dots, 0, \frac{\alpha_0 - 1}{1 - \alpha_n}, 0, \dots, 0 \right) \in \mathbb{R}^{N+1},$$

where the n dependent entry is in the n th position. Clearly, the P_n are in the closure of D and D is the interior of the convex envelope of the n vectors P_n : every $y \in D$ can be written as

$$y = \sum_{n=1}^N \lambda_n P_n \quad \text{with} \quad \lambda_n := \frac{1 - \alpha_n}{\alpha_0 - 1} y_n > 0 \quad \text{and} \quad \sum_{n=1}^N \lambda_n = 1$$

because of the constraint. Therefore

$$X = \sum_{j=0}^N y_j = \sum_{n=1}^N \lambda_n \left(1 + \frac{\alpha_0 - 1}{1 - \alpha_n} \right) = \sum_{n=1}^N \lambda_n \frac{\alpha_0 - \alpha_n}{1 - \alpha_n},$$

and as $(\alpha_0 - \alpha)/(1 - \alpha)$ is an increasing function of α ,

$$\frac{\alpha_0 - \alpha_1}{1 - \alpha_1} < X < \frac{\alpha_0 - \alpha_N}{1 - \alpha_N}.$$

Similarly, we obtain the bound on Y ,

$$Y = \sum_{j=0}^N \alpha_j^2 y_j = \sum_{n=1}^N \lambda_n \left(\alpha_0^2 + (\alpha_0 - 1) \frac{\alpha_n^2}{1 - \alpha_n} \right)$$

by noting that $\alpha^2/(1 - \alpha)$ is increasing in α :

$$\alpha_0^2 + (\alpha_0 - 1) \frac{\alpha_1^2}{1 - \alpha_1} < Y < \alpha_0^2 + (\alpha_0 - 1) \frac{\alpha_N^2}{1 - \alpha_N},$$

ending the proof of the lemma.

Note that as $\alpha_0, \alpha_1, \alpha_N$ vary, X_{min} and X_{max} take all values of $(1, \infty)$ and Y_{min} and Y_{max} take all values of (α_0^2, ∞) .

Let us now suppose that C is positive. Since

$$\frac{3}{2}W^2C = t^2 + b^2 + c^2 + s^2 + u^2 + d^2 - (t+b)(c+s+u+d) - (c+s)(u+d),$$

C is indeed positive in presence of the following hierarchy of quark masses: $u+d < \min\{c, s\}$, $c+2s < \min\{t, b\}$.

Then the function of two variables

$$f(X, Y) := \frac{C}{X} + 3\frac{Y}{X} - 2\frac{X}{1+X}$$

is decreasing in X and increasing in Y and we get the following bounds on the Higgs mass:

$$\tilde{H}_{min} := [f(X_{max}, Y_{min}) - 1]W < H < [f(X_{min}, Y_{max}) - 1]W =: H_{max}, \quad (13)$$

but we still must check the positivity of these bounds.

H_{max} is positive. Indeed, $-2X/(1+X) - 1 > -3$ for positive X and we shall verify

$$\left[\alpha_0^2 + (\alpha_0 - 1) \frac{\alpha_N^2}{1 - \alpha_N} \right] \frac{1 - \alpha_1}{\alpha_0 - \alpha_1} > 1.$$

As $\alpha^2/(1-\alpha)$ is increasing in $\alpha \in [0, 1]$, it is sufficient to prove the same inequality with α_N replaced by α_1 . Since

$$g(\alpha_1) := \frac{\alpha_0^2(1 - \alpha_1) + \alpha_1^2(\alpha_0 - 1)}{\alpha_0 - \alpha_1}$$

has negative derivative, $g'(\alpha_1) = 1 - \alpha_0 < 0$, we obtain $g(\alpha_1) > g(1) = 1$.

Concerning the lower bound, we remark that $\lim_{\tau \nearrow W} \tilde{H}_{min}/W = -3$, so we have to know when \tilde{H}_{min} is positive.

Lemma 2. *\tilde{H}_{min} is positive if and only if*

$$\alpha_N < \frac{X_+ - \alpha_0}{X_+ - 1} \quad (14)$$

$$\text{with } X_+ := (A - 1 + \sqrt{A^2 + 10A + 1})/6, \quad A := C + 3[\alpha_0^2 + (\alpha_0 - 1)\alpha_1^2/(1 - \alpha_1)].$$

Proof. For $X_{max} \in (1, \infty)$, $H_{min}/W = A/X_{max} - 2X_{max}/(1 + X_{max}) - 1$ is positive if and only if $-3X_{max}^2 + (A - 1)X_{max} + A > 0$. One root of this polynomial is negative, the other is X_+ and $X_+ > 1$ because $A > 3$. $X_{max} < X_+$ yields the desired upper bound on α_N .

Numerically, for $m_t = 176 \text{ GeV}$, the bound of (14) is $\alpha_3 < 0.92$, corresponding to $m_\tau < 76.83 \text{ GeV}$. In particular, since $X_+ > \alpha_0^2$ the condition $\alpha_3 < \frac{\alpha_0^2 - \alpha_0}{\alpha_0^2 - 1} = \frac{\alpha_0}{1 + \alpha_0}$ implies positive \tilde{H}_{min} . A fortiori, $m_\tau < \sqrt{Wt/(3W + t)} = 63 \text{ GeV}$, $m_t = 176 \text{ GeV}$, implies positive \tilde{H}_{min} .

From now on, we put

$$H_{min} = \tilde{H}_{min} \quad (15)$$

if the latter is positive.

Note that the bounds on the Higgs mass do not depend on the mass of the intermediate leptons (m_μ).

If $\Delta H := H_{max} - H_{min}$ denotes the length of the accessible interval for m_H^2 , one checks that

$$\frac{\Delta H}{W} = (\alpha_N - \alpha_1)(\alpha_0 - 1) \left[\frac{C}{(\alpha_0 - \alpha_N)(\alpha_0 - \alpha_1)} + 3 \frac{\alpha_0 + \alpha_N - \alpha_0 \alpha_N}{(\alpha_0 - \alpha_1)(1 - \alpha_N)} + 3 \frac{\alpha_1 + \alpha_N - \alpha_1 \alpha_N}{(\alpha_0 - \alpha_N)(1 - \alpha_1)} + \frac{2}{(\alpha_0 + 1 - 2\alpha_N)(\alpha_0 + 1 - 2\alpha_1)} \right].$$

Therefore the fuzziness disappears if and only if the sum of the squares of all six quark masses equals $3m_W^2$ or $m_\tau = m_e$. Indeed, neglecting all fermion masses but m_τ and m_t ,

$$\Delta m_H := m_{H_{max}} - m_{H_{min}} = \left[k \left(\frac{m_\tau}{m_t} \right)^2 + O \left(\frac{m_\tau}{m_t} \right)^4 \right] m_t, \quad (16)$$

$$k := \frac{\sqrt{3}}{2} \frac{r^3 + 3r^2 - 7r - 33}{r^3 + 5r^2 + 5r - 3} \sqrt{r \frac{r^2 + 2r - 1}{r + 3}}, \quad (16)$$

$$r := \left(\frac{m_t}{m_W} \right)^2. \quad (17)$$

For $m_t = 176 \text{ GeV}$, we have $k = 1.76$.

If there are only $N = 2$ generations of leptons and quarks (or likewise three generation of leptons and no quarks) then the Lemma 1 no longer holds since Y is a function of X . Nevertheless, the Higgs mass varies in an open interval, an analogue of equation (13) holds and Δm_H is governed by the mass difference $m_\mu - m_e$ [7]. If there is only $N = 1$ generation of leptons and quarks (or two generations of leptons and no quarks) then the bounds on the Higgs mass collapse, the mass relation becomes exact, i.e. an equality [5].

3.2 Coupling constants

In absence of strong interactions, there is a relation among the gauge couplings g_1 and g_2 [8] because then, only x and $\text{tr } y$ appear. Depending on the fermion content, this relation is exact or fuzzy. If there are only quarks in any number of generations, we have $\sin^2 \theta_w = 1/5$, and for only leptons in any number of generations, $\sin^2 \theta_w = 1/3$. For leptons *and* quarks the relation becomes fuzzy,

$$1/5 < \sin^2 \theta_w = \frac{x + \text{tr } y}{5x + 3 \text{tr } y} < 1/3.$$

However, without strong interactions, the geometric version of the standard model leads to wrong electric charges, up and down quark with opposite charges or charged neutrinos.

The addition of strong interactions cures this problem and introduces two more parameters, \tilde{x} and $\text{tr } \tilde{y}$. Consequently

$$\sin^2 \theta_w = \frac{g_2^{-2}}{g_1^{-2} + g_2^{-2}} = \frac{Nx + \text{tr } y}{2Nx + \frac{2}{9}N\tilde{x} + \frac{3}{2}\text{tr } y + \frac{3}{2}\text{tr } \tilde{y}}$$

is only bounded from above,

$$\sin^2 \theta_w < \frac{2}{3} \left(1 + \frac{1}{9} \left(\frac{g_2}{g_3} \right)^2 \right)^{-1}.$$

Note that the addition of right-handed neutrinos to the standard model [9] improves this constraint,

$$\sin^2 \theta_w < \frac{1}{2} \left(1 + \frac{1}{12} \left(\frac{g_2}{g_3} \right)^2 \right)^{-1}.$$

Using

$$\alpha_{em} := \frac{g_{em}^2}{4\pi} = \frac{g_2^2 \sin^2 \theta_w}{4\pi}$$

we rewrite the inequality (1):

$$\alpha_3 := \frac{g_3^2}{4\pi} > \frac{\alpha_{em}}{6(1 - 3/2 \sin^2 \theta_w)}.$$

It now says that the strong fine structure constant cannot be very small, $\alpha_3 > 0.002$. Experimentally it is around 0.11.

4 Conclusions

Non-commutative geometry explains the constraint $m_Z = m_W / \cos \theta_w$. Although being an equality it is stable under renormalization flow. Non-commutative geometry has three additional constraints: (2) explains why the top is so heavy, (3) predicts the Higgs mass and (1) constrains the weak mixing angle and the strong coupling constant. All three constraints are fuzzy, i.e. given by inequalities and therefore locally stable under renormalization flow [10]. Local stability should be sufficient since the theory contains no superheavy particle. Numerically the Higgs mass is predicted at $m_H = 280 \pm 33 \text{ GeV}$ for the current top mass of $m_t = 176 \pm 18 \text{ GeV}$, a prediction to be tested within ten years.

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References

- [1] A. Connes, *Non-Commutative Geometry*, Academic Press (1994)
- [2] A. Connes & J. Lott, *The metric aspect of non-commutative geometry*, in the proceedings of the 1991 Cargèse Summer Conference, eds.: J. Fröhlich et al., Plenum Press (1992)
- [3] D. Kastler, *A detailed account of Alain Connes' version of the standard model in non-commutative geometry, I and II*, Rev. Math. Phys. 5 (1993) 477
D. Kastler, *A detailed account of Alain Connes' version of the standard model in non-commutative geometry, III*, CPT-92/P.2824 (1992)
D. Kastler & M. Mebkhout, *Lectures on Non-Commutative Differential Geometry*, World Scientific, to be published
J. C. Várilly & J. M. Gracia-Bondía, *Connes' noncommutative differential geometry and the standard model*, J. Geom. Phys. 12 (1993) 223
J. Madore, *An Introduction to Noncommutative Differential Geometry and its Physical Applications*, Cambridge University Press (1995)
- [4] A. Connes, *Non-commutative geometry and reality*, IHES/M/95/52 (1995)
- [5] D. Kastler & T. Schücker, *The standard model à la Connes-Lott*, CPT-94/P.3091, hep-th/9412185 (1994)
D. Kastler & T. Schücker, *A detailed account of Alain Connes' version of the standard model in non-commutative geometry, IV*, Rev. Math. Phys., to appear
- [6] E. Alvarez, J. M. Gracia-Bondía & C. P. Martín, *Anomaly cancellation and the gauge group of the Standard Model in Non-Commutative Geometry*, hep-th/9506115 (1995)
- [7] B. Iochum, D. Kastler & T. Schücker, *Fuzzy Mass Relations for the Higgs*, hep-th/9506044, J. Math. Phys., to appear
- [8] B. Iochum & T. Schücker, *Yang-Mills-Higgs versus Connes-Lott*, hep-th/9501142, Comm. Math. Phys., to appear
- [9] J. M. Gracia-Bondía, *Connes' interpretation of the Standard model and massive neutrinos*, Phys. Lett. B, in press
G. Cammarata & R. Coquereaux, *Comments about Higgs fields, non-commutative geometry and the standard model*, CPT-95/P.3184, hep-th/9505192 (1995)
- [10] E. Alvarez, J. M. Gracia-Bondía & C. P. Martín, *A renormalisation group analysis of the NCG constraints $m_{top} = 2 m_W$, $m_{Higgs} = 3.14 m_W$* , Phys. Lett. 323B (1994) 259

Figure caption: Lower and upper bounds of the Higgs mass as a function of the top and τ masses, all other masses being set to their experimental values. For the experimental value, $m_\tau = 1.8 \text{ GeV}$, the two bounds differ by around 10^{-2} GeV in the indicated range of m_t .