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Solvable Potentials from Supersymmetric Quantum Mechanics

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Abstract

A recurrence relation of Riccati-type differential equations known in supersymmetric quantum mechanics is investigated to find exactly solvable potentials. Taking some simple *ansätze*, we find new classes of solvable potentials as well as reproducing the known shape-invariant ones.

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It is generally difficult to solve exactly the eigenvalue problem of a (time-independent) hamiltonian in quantum mechanics. Among the various methods developed for this purpose, a remarkably simple but powerful one is a ladder operator technique, whose typical example is a simple harmonic oscillator. If a hamiltonian has a discrete eigenvalue spectrum bounded from below, the energy eigenstates should be labeled by integers and formal rasing and lowering operators can be written in this basis. However, it does not provide a way to find the ladder operators explicitly for a given hamiltonian. A practical method of getting ladder operators has been studied by several authors([3]- [8]) using ideas of supersymmetric quantum mechanics([1], [2]) and a concept of shape-invariant potentials [3]. This approach has (re)produced many exactly solvable potentials. In this paper, we investigate the recurrence relation of Riccati-type differential equations arising in the supersymmetric quantum mechanics directly taking some *ansätze*. In addition to reproducing known shape-invariant potentials concisely, new classes of solvable potentials are found.

Let us begin by summarizing some results of supersymmetric quantum mechanics which are used for the present work(for a review, see [2] for example). Consider a family of one dimensional hamiltonians

$$H_k = \frac{p^2}{2} + V_k(x) , \qquad k = 0, 1, 2, \cdots , \tag{1}$$

where $p = -i\frac{\partial}{\partial x}$, which are connected by

$$H_{k+1} = B_k B_k^\dagger - E_k(1) = B_{k+1}^\dagger B_{k+1} , \qquad H_0 = B_0^\dagger B_0 . \tag{2}$$

Here B_k and $B_k^\dagger$ are defined using superpotentials $\alpha_k(x)$

$$B_k = \frac{1}{\sqrt{2}}\{\frac{\partial}{\partial x} + \alpha_k(x)\} , \qquad B_k^\dagger = \frac{1}{\sqrt{2}}\{-\frac{\partial}{\partial x} + \alpha_k(x)\} , \tag{3}$$

where $\alpha_k(x)$ is a real function such that the ground state wavefunction which is proportional to $exp(-\int \alpha_k(x)dx)$ should be normalizable. The nth eigenvalue of H_k is denoted by $E_k(n)$ $n = 0, 1, 2, \cdots$, and a relation between eigenvalues is given by

$$E_{k+1}(n) = E_k(n+1) - E_k(1) = E_0(n+k+1) - E_0(k+1) . \tag{4}$$

Note that all the ground state energies are defined to be zero. The $E_k(1)$ is assumed to be positive and is also denoted by s_k, standing for a level spacing between the kth and $(k+1)$th eigenstate of the starting hamiltonian H_0. It is worth noting that $E_0(n+1) = \sum_{k=0}^{n} s_k$, $n = 0, 1, 2, \cdots$ and $E_0(0) = 0$. Denoting the nth eigenfunction of H_k as $\psi_{k(n)}(x)$, important relations between the eigenfunctions follow from (2) as

$$
\begin{aligned}
\psi_{k+1(n)}(x) &= \{E_k(n+1)\}^{-1/2}\, B_k \psi_{k(n+1)}(x) \ , \\
B_k^\dagger\, \psi_{k+1(n)}(x) &= \{E_k(n+1)\}^{1/2} \psi_{k(n+1)}(x) \ .
\end{aligned}
\tag{5}
$$

Note that $B_k\, \psi_{k(0)}(x) = 0$. A repeated use of this relation gives

$$
B_0^\dagger B_1^\dagger \cdots B_k^\dagger\, \psi_{k+1(n)}(x) \;=\; \{E_0(n+k+1) E_1(n+k) \cdots E_k(n+1)\}^{1/2}\, \psi_{0(n+k+1)}(x) \ . \tag{6}
$$

Putting $n = 0$, this equation can be used to get the excited-state wavefunctions of H_0 from the knowledge of the ground states of systems related as superpartners.

Now, putting B_k and $B_k^\dagger$ in (3) into (2), a recurrence relation of Riccati-type differential equations is obtained as

$$
2V_{k+1}(x) \;=\; \alpha_{k+1}^2(x) - \alpha_{k+1}'(x) \;=\; \alpha_k^2(x) + \alpha_k'(x) - 2s_k \ , \tag{7}
$$

where the prime means a derivative with respect to x . To search for the possible family of $\alpha_k(x)$ and s_k satisfying the difference differential equation (7), it is convenient to put $\alpha_k(x)$ and $\alpha_{k+1}(x)$ as

$$
\alpha_k(x) \;=\; \frac{1}{2f_k'(x)}\{2s_k f_k(x) + f_k''(x)\} \ , \qquad \alpha_{k+1}(x) \;=\; \frac{1}{2f_k'(x)}\{2s_k f_k(x) - f_k''(x)\} \ , \tag{8}
$$

which solve (7) identically. It holds for non-zero s_k only, and in general, there can be added a free parameter σ_k, say, inside the curly brackets in (8). Note that $f_k(x)$ is proportional to the first excited-state wavefunction divided by that of the ground state of the kth hamiltonian. The consistency in the two equations of (8) requires

$$
\{f_k'(x) f_{k+1}'(x)\}' \;=\; 2\{s_k f_k(x) f_{k+1}'(x) \;-\; s_{k+1} f_k'(x) f_{k+1}(x)\} \ . \tag{9}
$$

We further rewrite equations (8), (9) by putting $f_k(x)$ as following :

$$f_k(x) \ = \ \tilde{g}_k(x)\, exp\,(\int \tilde{h}_k(x)dx\,)\ . \tag{10}$$

Note that there is an ambiguity in the choice of $\tilde{h}_k(x)$, or equivalently in $\tilde{g}_k(x)$: Adopting $g_k(x) = \tilde{g}_k(x)\, exp\,(\int \beta_k(x)dx\,)$, and $h_k(x) = \tilde{h}_k(x) - \beta_k(x)$, instead of tilded functions, $f_k(x)$ remains the same for arbitrary $\beta_k(x)$. We remove this ambiguity by choosing $\beta_k(x)$ such that $h_0(x) = 0$, and

$$f_k(x) \ = \ g_k(x)\, exp\,(\int h_k(x)dx\,)\ , \qquad f'_k(x) \ = \ f'_0(x)\, exp\,(\int h_k(x)dx\,)\ . \tag{11}$$

The consistency in this procedure requires for all $k = 0, 1, 2, \cdots$,

$$f'_0(x) \ = \ g'_k(x) \ + \ g_k(x)\, h_k(x)\ , \tag{12}$$

then the difference differential equation (7) is satisfied. Now, equation (9) becomes

$$f''_0(x) \ + \ \frac{1}{2}\, f'_0(x)\,(\, h_k(x) + h_{k+1}(x)\,) \ = \ s_k g_k(x) \ - \ s_{k+1} g_{k+1}(x)\ . \tag{13}$$

Solving this equation for $g_k(x)$ and inserting it into (12) yields

$$k f'''_0(x) \ + \ p_k(x) f''_0(x) \ + \ (\, q_k(x) + s_k - s_0\,)\, f'_0(x) \ - \ s_0 h_k(x) f_0(x) \ = \ 0\ , \tag{14}$$

where

$$p_k(x) \ = \ k h_k(x) \ + \ \theta_k(x)\ , \qquad q_k(x) \ = \ \theta'_k(x) \ + \ h_k(x)\theta_k(x)\ , \tag{15}$$

and

$$\theta_k(x) \ = \ \frac{1}{2} \sum_{j=0}^{k-1} (\ h_j(x) + h_{j+1}(x)\ )\ , \qquad \theta_0(x) \ = \ 0\ . \tag{16}$$

Since (14) is identically satisfied for $k = 0$, it is equivalent to the difference of $(k+1)$th and kth expressions in (14). Thus, with the difference denoted by $\Delta y_k \equiv y_{k+1} - y_k$, we finally obtain

$$f'''_0(x) \ + \ \Delta p_k(x) f''_0(x) \ + \ (\, \Delta q_k(x) + \Delta s_k\,)\, f'_0(x) \ - \ s_0 \Delta h_k(x) f_0(x) \ = \ 0\ . \tag{17}$$

It is equivalent to the difference differential equation (7), and we will try to find $f_0(x)$ and s_k consistently by making simple *ansätze* on $h_k(x)$ below. The superpotential in (8) is now given as

$$\alpha_k(x) \;=\; \frac{1}{f_0'(x)}\{\, s_0 f_0(x) \;-\; (\,k - \frac{1}{2}\,)\, f_0''(x)\,\} \;+\; \frac{1}{2}\, h_k(x) \;-\; \theta_k(x)\,. \tag{18}$$

Note that an overall constant factor of $f_0(x)$ does not affect the superpotential.

Let us solve (17). The simplest *ansatz* on $h_k(x)$ may be putting $h_k(x) = 0$ for all $k = 0, 1, 2, \cdots$. In this case, (17) becomes

$$f_0'''(x) \;+\; \Delta s_k\, f_0'(x) \;=\; 0\,. \tag{19}$$

This is solved consistently by putting

$$\Delta s_k \;=\; s\,, \tag{20}$$

where s is a constant parameter. It is easy to find $f_0'(x)$ in (19), and the superpotential is given in (18) with $h_k(x) = 0\,, \theta_k(x) = 0$ in this *ansatz*, that is,

$$\alpha_k(x) \;=\; \frac{1}{f_0'(x)}\{\, s_0 f_0(x) \;-\; (\,k - \frac{1}{2}\,)\, f_0''(x)\,\}\,. \tag{21}$$

The eigenvalues are given by the common formula

$$E_k(n) = s_0 n + \frac{1}{2}\, sn(n - 1 + 2k)\;, \tag{22}$$

referring to (20) and (4). In the following, we list the resulting superpotentials, while the potentials can be written down explicitly by using $2V_k(x) \;=\; \alpha_k^2(x) - \alpha_k'(x)$ in (7). With constant parameters b, d, and putting

$$\eta \;=\; \sqrt{|s|}\;, \qquad a_k \;=\; \frac{2(\,s_0 + ks\,) - s}{2\eta}\;,$$

(1) $s > 0\;;\; f_0'(x) \;=\; sin\,\eta(x + d)$

$$\alpha_k(x) \;=\; \begin{cases} -a_k\, cot\,\eta(x + d) \;+\; \frac{b}{sin\,\eta(x+d)} & \text{Eckart potential} \\ \frac{a_k + b}{2}\, tan\,\frac{\eta}{2}\,(x + d) \;-\; \frac{a_k - b}{2}\, cot\,\frac{\eta}{2}\,(x + d) & \text{Pöschl-Teller potential I} \end{cases} \tag{23}$$

(2) $s < 0$; $f_0'(x) = c_1\, sinh\, \eta(x+d) + c_2\, cosh\, \eta(x+d)$ with constant parameters c_1, c_2 . Thus, the superpotential in (21) is

$$\alpha_k(x) = \frac{a_k\,(\,c_1 + c_2\, tanh\, \eta(x+d)\,)}{c_2 + c_1\, tanh\, \eta(x+d)} + \frac{b}{c_1\, sinh\, \eta(x+d) + c_2\, cosh\, \eta(x+d)} \,. \tag{24}$$

Some well-known potentials in the literature come out :

a. $f_0'(x) = sinh\, \eta(x+d)$ $(\,c_1 = 1\,, c_2 = 0\,)$

$$\alpha_k(x) = \begin{cases} a_k\, coth\, \eta(x+d) + \frac{b}{sinh\, \eta(x+d)} & \text{Rosen-Morse potential} \\ \frac{a_k - b}{2}\, tanh\, \frac{\eta}{2}\,(x+d) + \frac{a_k + b}{2}\, coth\, \frac{\eta}{2}\,(x+d) & \text{Pöschl-Teller potential II} \end{cases} \tag{25}$$

b. Morse potential ; $f_0'(x) = cosh\, \eta(x+d)$ $(\,c_1 = 0\,, c_2 = 1\,)$

$$\alpha_k(x) = a_k\, tanh\, \eta(x+d) + \frac{b}{cosh\, \eta(x+d)} \,. \tag{26}$$

c. Morse potential ; $f_0'(x) = exp\, \eta(x+d)$, $(\,c_1 = c_2 = \frac{1}{2}\,)$ [$c_1 = -c_2 = -\frac{1}{2}$ also gives a similar potential.]

$$\alpha_k(x) = a_k + b\, exp\,(\,-\eta(x+d)\,) \,. \tag{27}$$

(3) $s = 0$; two cases arise as following.

a. Harmonic oscillator potential ; $f_0'(x) = 1$

$$\alpha_k(x) = s_0\,(\,x+d\,) \,. \tag{28}$$

b. Harmonic plus inverse-square potential ; $f_0'(x) = x + d$

$$\alpha_k(x) = \frac{s_0}{2}\,(x+d) - \frac{\lambda_k}{x+d}\,, \qquad \lambda_k = k - \frac{1}{2} + b\,. \tag{29}$$

It should be noted that k has a maximum allowed value when $s < 0$ since $s_k = s_0 + ks$ is assumed to be positive. It is remarkable that a large part of the known exactly solvable potentials given in refs. [4], [5] are reproduced here within a single *ansatz*. Further details of the potentials and the wavefunctions can be found in refs. [2], [4], [5], [7] where a notion of shape-invariance of potentials is utilized in the framework of supersymmetric quantum mechanics.

Let us solve (17) with an *ansatz* which is slightly more general than assuming $h_k(x) = 0$ for all k ,

$$h_k(x) \; = \; \begin{cases} 0 & \text{for k = even} \\ h(x) & \text{for k = odd} \; . \end{cases} \tag{30}$$

Using for $k =$ even, odd integers respectively,

$$\Delta h_k(x) = \pm h(x) \; , \; \Delta p_k(x) = h(x) \, \{1 \pm (k + \frac{1}{2}) \} \; , \; \Delta q_k(x) = \frac{h'(x)}{2} + \frac{h^2}{4} \{1 \pm (2k+1) \} \; ,$$

the equation (17) reduces to

$$\begin{aligned} f_0'''(x) \; + \; \frac{3}{2} h(x) f_0''(x) \; + \; \{ \frac{h'(x)}{2} + \frac{h^2(x)}{2} + \Delta s_0 \} \, f_0'(x) \; - \; s_0 h(x) f_0(x) = 0 \; , \\ f_0'''(x) \; - \; \frac{1}{2} h(x) f_0''(x) \; + \; \{ \frac{h'(x)}{2} - \frac{h^2(x)}{2} + \Delta s_1 \} \, f_0'(x) \; + \; s_0 h(x) f_0(x) = 0 \; , \end{aligned} \tag{31}$$

for $k = 0, 1$ and

$$f_0''(x) \; + \; \{ \frac{h(x)}{2} + \frac{w}{2h(x)} \} \, f_0'(x) \; = \; 0 \; , \tag{32}$$

which results from the consistency of (17) for all $k = 2, 3, 4, 5, \cdots$, together with Δs_k determined with a constant w as

$$\Delta s_{2k} \; = \; \Delta s_0 \; + \; kw \; , \qquad \Delta s_{2k+1} \; = \; \Delta s_1 \; - \; kw \; . \tag{33}$$

Combining equations (31) and (32), one finally arrives at

$$h'(x) \; + \; \frac{1}{2} h^2(x) \, (\frac{1}{\sigma} + 1) \; + \; \lambda \sigma \; = \; 0 \; , \qquad 2\sigma f_0'(x) \; = \; h(x) f_0(x) \; , \qquad w \; = \; 2\lambda\sigma \; , \tag{34}$$

where λ, σ are

$$\lambda \; = \; \Delta s_0 + \Delta s_1 \; , \qquad \sigma \; = \; \frac{\Delta s_0 - \Delta s_1 - w}{4 s_0} \; . \tag{35}$$

We consider $\sigma \neq 0$, since otherwise $h(x) = 0$ in (34), which has been already covered. The energy level spacings are now given as

$$s_{2k} \;=\; s_0 + k\lambda \,, \qquad s_{2k+1} \;=\; (\,1 + 2\sigma\,)\,\{\,s_0 + (k + \frac{1}{2})\,\lambda\,\}\,. \tag{36}$$

Here, $\sigma > -1/2$ for positive level spacings. From this and (4), the energy levels are given as

$$E_k(n) \;=\; (1+\sigma)\,n\,\{\,s_0 \;+\; \tfrac{\lambda}{4}\,(n-1+2k)\,\}$$
$$-\,(-1)^k\,\sigma\,\{\,\frac{1-(-1)^n}{2}\,(\,s_0 \;+\; \frac{\lambda}{4}\,(2k-1)\,) \;-\; \frac{\lambda\,n}{4}\,(-1)^n\,\}\,. \tag{37}$$

This formula reduces to (22) as $\sigma\mathring{0}$, by noting in (36), $\Delta s_k \;=\; \lambda/2$ in this case. The superpotential in (18) is given finally using (34)

$$\alpha_k(x) \;=\; \frac{2\sigma}{h(x)}\,(\,s_0 - \frac{\lambda}{4} + \frac{\lambda k}{2}\,) \;-\; (-1)^k\,\frac{h(x)}{4}\,. \tag{38}$$

Now, we only need to find $h(x)$ satisfying (34). The resulting superpotentials are listed in the following with a constant d, and putting

$$\eta \;=\; \sqrt{\frac{|\lambda|\,(\,1+\sigma\,)}{2}}\,, \qquad b_k \;=\; (\,1+\sigma\,)\,(\,s_0 \;-\; \frac{\lambda}{4} \;+\; \frac{\lambda k}{2}\,)\,, \qquad \mu_k \;=\; \frac{\sigma\,(-1)^k}{2(\,1+\sigma\,)}\,.$$

(1) $\lambda > 0$; $h(x) \;=\; \frac{2\sigma\eta}{1+\sigma}\,cot\,\eta\,(x+d)$

$$\alpha_k(x) \;=\; b_k\,\frac{tan\,\eta\,(x+d)}{\eta} \;-\; \mu_k\,\eta\,cot\,\eta\,(x+d)\,. \tag{39}$$

(2) $\lambda < 0$; $h(x) \;=\; \frac{2\sigma\eta}{1+\sigma}\,coth\,\eta\,(x+d)$

$$\alpha_k(x) \;=\; b_k\,\frac{tanh\,\eta\,(x+d)}{\eta} \;-\; \mu_k\,\eta\,coth\,\eta\,(x+d)\,. \tag{40}$$

(3) $\lambda = 0$; $h(x) \;=\; \frac{2\sigma}{1+\sigma}\,\frac{1}{x+d}$

$$\alpha_k(x) \;=\; s_0\,(1+\sigma)\,(x+d) \;-\; \frac{\mu_k}{x+d}\,. \tag{41}$$

These potentials are similar but different from the ones obtained by our first *ansatz*, that is, Pöschl-Teller potentials I, II given in (23), (25) respectively, and (29). This fact is clear in view of their different energy-level structures and the wavefunctions. For example, the potentials corresponding to the superpotentials in (29), (41) are given respectively (putting $d = 0$ for simplicity)

$$V_k(x) \;=\; \frac{s_0^2}{8}\,x^2 \;+\; \frac{\lambda_k^2 - \lambda_k}{2x^2} \;-\; \frac{s_0}{2}(\lambda_k + \frac{1}{2})\,, \tag{42}$$

$$V_k(x) \;=\; \frac{1}{2}s_0^2(1+\sigma)^2\,x^2 \;+\; \frac{\mu_k^2 - \mu_k}{2x^2} \;-\; \frac{s_0}{2}\,(\,1+\sigma+(-1)^k\sigma\,)\,. \tag{43}$$

The energy level spacings are $s_k = s_0$ for all k, for the potentials (42), while $s_k = s_0\,, s_0\,(\,1+2\sigma\,)$ for $k =$ even, odd integers respectively for (43). The difference between the two systems can be seen more clearly in their wavefunctions. The unnormalized wavefunctions for $V_0(x)$ given in (42) are

$$\psi_{0(k)}(x) \;=\; x^{\lambda_0}\, exp(\,-\frac{1}{4}s_0x^2\,)\, L_k^{\lambda_0-\frac{1}{2}}(\,\frac{s_0x^2}{2}\,)\,, \tag{44}$$

while for $V_0(x)$ given in (43)

$$\begin{aligned} \psi_{0(2k)}(x) &= x^{\mu_0} exp(\,-\frac{1}{2}s_0(1+\sigma)x^2\,)\, L_k^{\mu_0-\frac{1}{2}}(\,s_0(1+\sigma)x^2\,)\,, \\ \psi_{0(2k+1)}(x) &= x^{1-\mu_0}\, exp(\,-\frac{1}{2}s_0(1+\sigma)x^2\,)\, L_k^{\frac{1}{2}-\mu_0}(\,s_0(1+\sigma)x^2\,)\,, \end{aligned} \tag{45}$$

where $L_n^a(z)$ is a generalized Laguerre polynomial. Note that as $\sigma\mathring{0}$, the potentials in (43), the wavefunctions in (45) and the energy levels all reduce to those of the simple harmonic oscillator. In contrast, if the parameter is chosen such that the potentials in (42) become quadratic, only 'half' of the energy levels and wavefunctions of a corresponding harmonic oscillator are obtained as can be seen in (44). We remark that the wavefunctions in (45) are normalizable as long as $\sigma > -1/2$, although their first derivatives are not continuous at $x = 0$. The point is that another class of potentials like in (43) is possible as well as the well-known ones like in (42) if we seek supersymmetry-related potentials possessing normalizable wavefunctions.

How can the (radial) Coulomb potential consisting of $1/x$ and $1/x^2$ terms be reproduced in our approach? Since we need $h_k(x)$ as inputs for solving (17), we assume $h_k(x)$ to be that of Coulomb potential

$$h_k(x) \;=\; g\,(\frac{1}{k+a} \;-\; \frac{1}{k+1+a}\,) \;-\; \frac{g}{a(a+1)}\;, \tag{46}$$

where g, a are constant parameters. The derivation of this will be described later together with other general idea of getting superpotentials. Then (17) becomes for $k=0$,

$$f_0'''(x) \;+\; \frac{3h_1}{2}\,f_0''(x) \;+\; \{\,(\,\lambda + \frac{g(2a+1)}{a(a+1)}\,)\,h_1 \;+\; \frac{g^2}{a^2(a+1)^2} \;+\; b\,\}\,f_0'(x) \;-\; s_0 h_1\,f_0(x) \;=\; 0\;, \tag{47}$$

where $h_1 \;=\; -2g/\{\,a(a+1)(a+2)\,\}$, and

$$(2a+1)\,f_0''(x) \;-\; \{\,2\lambda \;+\; \frac{2g(2a+1)}{a(a+1)}\,\}\,f_0'(x) \;+\; 2s_0\,f_0(x) \;=\; 0\;, \tag{48}$$

which results from the consistency of (17) for $k = 1, 2, 3, \cdots$. Here, s_k is determined as

$$s_k \;=\; \frac{g^2}{2}\,(\frac{1}{(k+a)^2} \;-\; \frac{1}{(k+1+a)^2}\,) \;+\; \lambda h_k \;+\; bk \;+\; c\;, \tag{49}$$

with another constants λ, b, c. Combining (47) and (48), interesting cases arise only when

$$\lambda \;=\; 0\;, \qquad c \;=\; (\,a + \frac{1}{2}\,)\,b\;, \qquad a \neq -\frac{1}{2}\;, \tag{50}$$

and the final equation is

$$f_0''(x) \;-\; 2B\,f_0'(x) \;+\; (\,B^2 + b\,)\,f_0(x) \;=\; 0\;, \tag{51}$$

where

$$B \;=\; \frac{g}{a(a+1)}\;.$$

Now, energy levels are obtained from above and (4) as

$$E_k(n) \;=\; \frac{g^2}{2}\,\{\frac{1}{(k+a)^2} \;-\; \frac{1}{(k+n+a)^2}\} \;+\; b\,\{\frac{n^2}{2} \;+\; (k+a)\,n\,\}\;. \tag{52}$$

The superpotentials are obtained using (18), (51)

$$\alpha_k(x) \;=\; (\,k\,+\,a\,)\,F(x)\,+\,\frac{g}{k+a}\,, \qquad F(x) \;=\; (B^2+b)\,\frac{f_0(x)}{f_0'(x)}\;-\;B\,, \tag{53}$$

where $f_0(x)$ is easily obtained in (51). We list the resulting superpotentials in the following with a constant d (to be chosen suitably if necessary), and putting

$$\eta \;=\; \sqrt{|b|}\,.$$

(1) $b>0$; $f_0(x) \;=\; sin\,\eta(x+d)\,exp\,(\,Bx\,)$

$$\alpha_k(x) \;=\; -(k+a)\,\eta\,cot\,\eta x\;+\;\frac{g}{k+a}\,. \tag{54}$$

(2) $b\;<\;0$; $f_0(x) \;=\; (\,c_1\,sinh\,\eta x + c_2\,cosh\,\eta x\,)\,exp\,(\,Bx\,)$; with appropriate choices of c_1, c_2 , there result two well-known potentials

a. Eckart potential

$$\alpha_k(x) \;=\; -(k+a)\,\eta\,coth\,\eta x\;+\;\frac{g}{k+a}\,. \tag{55}$$

b. Rosen-Morse potential

$$\alpha_k(x) \;=\; -(k+a)\,\eta\,tanh\,\eta x\;+\;\frac{g}{k+a}\,. \tag{56}$$

(3) $b=0$; Coulomb potential ; $f_0(x) \;=\; (x+d)\,exp\,(\,Bx\,)$

$$\alpha_k(x) \;=\; -\frac{k+a}{x}\;+\;\frac{g}{k+a}\,. \tag{57}$$

It is satisfying to see that a single *ansatz* reflecting the common structure of superpotentials as in (53) produces a class of potentials in addition to the starting one, the Coulomb potential in this case. The above potentials and the results of our first *ansatz* given in (23)-(29), constitute the whole shape-invariant potentials given in refs. [4], [5]. (There are some discrepancies compared with the tables presented in these references.)

Finally, we want to mention another way of getting superpotentials as following. Since it is generally expected that the ground state wavefunctions are products of two factors, or equivalently $\alpha_k(x)$ is a sum of two parts, one may try to find new solutions to the difference differential equation (7) by adding an appropriately chosen second part to an old one which is assumed to satisfy it. As a simple *ansatz* to implement this idea, we take

$$\alpha_k(x) \;=\; a_k\, F(x) \;+\; b_k\, G(x)\,. \tag{58}$$

Here, simple forms are taken for the two parts with constants a_k, b_k and k-independent functions $F(x), G(x)$, and $a_k F(x)$ is assumed to solve (7). Then inserting (58) into (7) results in

$$(a_k + a_{k+1})\; F'(x) \;+\; (a_k^2 - a_{k+1}^2)\; F^2(x) \;=\; 2 s_k^{(a)}\;, \tag{59}$$

$$(b_k + b_{k+1})\; G'(x) \;+\; 2\; (a_k b_k - a_{k+1} b_{k+1})\; F(x) G(x) \;+\; (b_k^2 - b_{k+1}^2)\; G^2(x) \;=\; 2 s_k^{(b)}\;, \tag{60}$$

together with a condition

$$s_k \;=\; s_k^{(a)} \;+\; s_k^{(b)}\;. \tag{61}$$

Recall that s_k should be positive definite, while $s_k^{(a)}, s_k^{(b)}$ need not be so. In fact, all the results obtained so far can also be obtained within this simple *ansatz* as may be clear from the forms of the obtained superpotentials, (21), (38), (53). We illustrate the procedure by obtaining the Coulomb potential : A solution to (59) corresponding to $s_k^{(a)} = 0$ is directly obtained to be proportional to $1/x$, a_k being determined correspondingly. Next, note that (60) is consistently solved by taking $a_k b_k$ to be independent of k and $G(x)$ to be a constant, say g . Then one may get

$$\alpha_k(x) \;=\; -\frac{k+a}{x} \;+\; \frac{g}{k+a}\;, \qquad V_k(x) \;=\; -\frac{g}{x} \;+\; \frac{(k+a)(k+a-1)}{2x^2} \;+\; \frac{g^2}{2(k+a)^2}\;.$$

One can find $h_k(x)$ by taking the difference of (18),

$$\Delta\alpha_k(x) \;=\; -\frac{f_0''(x)}{f_0'(x)} \;-\; h_k(x)\;, \tag{62}$$

which can be used to get the $f_0(x)$ and $h_k(x)$ for the known superpotentials by recalling $h_0(x) = 0$.

In summary, we have investigated the recurrence relation of Riccati-type differential equations in supersymmetric quantum mechanics. Recasting the problem into (17) and taking some simple *ansätze* on $h_k(x)$, we reproduced the known shape-invariant potentials in refs. [4], [5], and obtained another classes of potentials which look similar but are different from the known ones, as illustrated in (42)-(45). It should be stressed that different types of potentials can be obtained altogether within a common *ansatz* on the superpotential as shown in the text. By using more intricate *ansätze* or introducing some new ideas into the formalism, we expect that the method presented in this paper will be helpful for finding out more solvable potentials, as well as reproducing the known ones, e.g., Natanzon potentials([9], [7], [2]).

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