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A connection between lattice and surgery constructions of three-dimensional topological field theories

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ABSTRACT

We study the relation between lattice construction and surgery construction of three-dimensional topological field theories. We show that a class of the Chung-Fukuma-Shapere theory on the lattice has representation theoretic reformulation which is closely related to the Altschuler-Coste theory constructed by surgery. There is a similar relation between the Turaev-Viro theory and the Reshetikhin-Turaev theory.

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Various three-dimensional topological field theories which satisfy Atiyah's axiom [1] have been constructed by now [2–4]. Relations among them are, however, still not clear. It is important to investigate them more with a view to systematic classifications of three-dimensional topological field theories [3]. In the following, we study several topological field theories and establish a connection among them.

There are two principal methods to construct three-dimensional topological field theories with mathematical rigor. One is the lattice or 'state-sum' method in which we represent a manifold by a simplicial complex and consider a statistical model on it. The other method employs the surgery representation of three-manifolds. One can construct arbitrary closed three-manifold $M_{(L,f)}$ by a surgery of S^3 along a framed link (L,f) in it. Therefore a class of framed link invariants, actually those which are invariant under Kirby moves, give rise to invariants of three-manifolds. Sometimes this construction is lifted to that of a topological field theory whose partition function is the three-manifold invariant.

Though these two methods differ a lot in nature, surprisingly topological field theories defined by the two methods sometimes reveal close relationship. The most famous example is the relation between the lattice theory by Turaev and Viro [3] (TV) and the Reshetikhin-Turaev (or Chern-Simons [7]) theory [4] (RT) defined by surgery: $Z^{TV} = |Z^{RT}|^2$ [8]. There is also a suggested relation [9, 10] between the Dijkgraaf-Witten [2] or Chung-Fukuma-Shapere [6] theory on the lattice, and the Altschuler-Coste theory [9] constructed using the surgery representation. It is desirable to have a more general statement which relates the lattice construction and the surgery construction.

In this note, we study whether a functor used in the surgery construction of topological field theories can induce a topological field theory on lattice. We take the Altschuler-Coste functor F^{AC} employed in the surgery construction in ref. [9] as an example and show that it is possible.

We are motivated by a remark in ref. [11] on the relation between the Turaev-Viro theory and the Reshetikhin-Turaev theory. In ref. [11], an attempt was made to rewrite the Turaev-Viro invariant in the form of the partition function of a 'three-dimensional q-deformed lattice gauge theory,' defined making use of the functor $F^{RT}_{\mathcal{U}_q(sl(2,\mathbf{C}))}$. It is a functor from the category of colored ribbon graphs to the category of representations of a ribbon Hopf algebra and was originally used to define the Reshetikhin-Turaev theory.

[3] Note that two-dimensional unitary topological field theories are classified completely [5].

We first review the properties of the functor F^{AC} and then show how a three-manifold invariant on the lattice is derived from it and that it is equivalent with the partition function of the Chung-Fukuma-Shapere theory. We see, however, that the Turaev-Viro theory is not reproduced from the Reshetikhin-Turaev functor except in the limit $q \to 1$ if we apply the same prescription in this case.

The functor F^{AC} is a functor from the category of colored ribbon graphs to the category of representations of the quasi-Hopf algebra $D^\omega(G)$. The algebra $D^\omega(G)$ is defined for a finite group G and its 3-cocycle $\omega : G \times G \times G \to \mathrm{U}(1)$, which satisfies

$$\omega(g,x,y)\omega(gx,y,z)^{-1}\omega(g,xy,z)\omega(g,x,yz)^{-1}\omega(x,y,z) = 1. \tag{1}$$

The condition $\omega(x,y,z) = 1$ should also be met if at least one of the arguments x, y, z is equal to the unit element $e \in G$. The bialgebra $D^\omega(G)$ is spanned by a formal basis $\{\chi_{g,x} | g, x \in G\}$ as a $\mathbf{C}$-module. Multiplication, unit, comultiplication and counit on $D^\omega(G)$ is defined with respect to the basis $\{\chi_{g,x}\}$ as follows :

$$\chi_{g,x} \cdot \chi_{h,y} = \delta_{g,xhx^{-1}} \theta_g(x,y)\, \chi_{g,xy} \qquad \text{(multiplication)} \tag{2}$$

$$u(1 \in \mathbf{C}) = \chi_{1,e} \equiv \sum_{g \in G} \chi_{g,e} \qquad \text{(unit)} \tag{3}$$

$$\Delta(\chi_{g,h}) = \sum_{xy=g} \gamma_h(x,y)\, \chi_{x,h} \otimes \chi_{y,h} \qquad \text{(comultiplication)} \tag{4}$$

$$\epsilon(\chi_{g,e}) = \delta_{g,h} \in \mathbf{C} \qquad \text{(counit)} \tag{5}$$

where

$$\theta_g(x,y) \equiv \omega(g,x,y)\, \omega(x,y,(xy)^{-1}gxy)\, \omega(x,x^{-1}gx,y)^{-1}, \tag{6}$$

$$\gamma_x(g,h) \equiv \omega(g,h,x)\, \omega(x,x^{-1}gx,x^{-1}hx)\, \omega(g,x,x^{-1}hx)^{-1}. \tag{7}$$

Furnished with the antipode S, the R-matrix, and an element ϕ, the bialgebra $D^\omega(G)$ becomes quasi-triangular quasi-Hopf algebra :

$$S(\chi_{g,h}) = \theta_{g^{-1}}(h,h^{-1})^{-1}\gamma_h(g,g^{-1})^{-1}\, \chi_{h^{-1}g^{-1}h,h^{-1}}, \tag{8}$$

$$R = \sum_{g,h \in G} \chi_{g,e} \otimes \chi_{h,g}, \tag{9}$$

$$\phi = \sum_{g,h,k \in G} \omega(g,h,k)^{-1} \chi_{g,e} \otimes \chi_{h,e} \otimes \chi_{k,e}. \tag{10}$$

Now to define F^{AC}, we explain the notion of colored ribbon graph Γ. We provide a ribbon and an annulus which are a square $[0,1] \times [0,1]$ and a cylinder $[0,1] \times S^1$ embedded in $\mathbf{R}^2 \times [0,1]$. For an annulus C, the linking number $lk(C^+, C^-)$ is called the framing of the annulus C where C^+ and C^- denote the images of the circles $0 \times S^1$ and $1 \times S^1$, respectively.

Each ribbon (or annulus) is equipped with an arrow along the image of $(1/2) \times [0,1]$ (or $(1/2) \times S^1$) and we assume the right side and the wrong side are defined on it. Given two non-negative integers k and l, a ribbon (k,l)-graph Γ is defined as a disjoint union of finite number of ribbons and annuli. Two ends $[0,1] \times 0$ and $[0,1] \times 1$ of the ribbons are mapped onto

$$\begin{aligned} \Gamma \cap (\mathbf{R}^2 \times 0) &= \cup_{i=1}^{k} \left(\{\, 0 \times [i - 1/4, i + 1/4] \times 0\}\right), \\ \Gamma \cap (\mathbf{R}^2 \times 1) &= \cup_{i=1}^{l} \left(\{\, 0 \times [i - 1/4, i + 1/4] \times 1\}\right). \end{aligned}$$

Near the two ends of each ribbon, the right side is faced to the plus direction of the x-axis.

Furthermore, we introduce 'ribbon graphs colored by regular representations of $D^\omega(G)$,' or 'regular c-graphs.' A regular c-graph is a ribbon (k,l)-graph together with a choice of formal symbols, or *words*, $w_k^{(b)}$ and $w_l^{(t)}$ associated with the bottom end and the top end of Γ. An example of w_7 is

$$((V^{\epsilon_1} \Box ((V^{\epsilon_2} \Box V^{\epsilon_3}) \Box V^{\epsilon_4})) \Box (V^{\epsilon_5} \Box (V^{\epsilon_6} \Box V^{\epsilon_7}))), \tag{11}$$

where $\Box$ is a formal non-associative binary operator. The symbols ϵ_i should take values $+1$ or -1 when the i-th ribbon is directed downward or upward, respectively. The symbol V^1 stands for the $|G|^2$-dimensional $D^\omega(G)$-module acted by the regular representation of $D^\omega(G)$ and V^{-1} its dual. In general the word associated with the top (bottom) of a c-graph is not unique because there are several ways of putting parentheses (i.e. the order of the operation of $\Box$). An example of regular c-graphs is depicted in Fig. 1.

The functor F^{AC} maps a regular colored ribbon (k,l)-graph Γ to a linear map

$$F^{AC}(\Gamma) : V^{\epsilon_1} \otimes V^{\epsilon_2} \otimes \cdots \otimes V^{\epsilon_k} \longrightarrow V^{\epsilon_1} \otimes V^{\epsilon_2} \otimes \cdots \otimes V^{\epsilon_l}. \tag{12}$$

The explicit form of the map $F^{AC}(\Gamma)$ is determined by the composition of $F^{AC}(\Gamma')$ for elementary graphs Γ' given as follows : [4]

$$F^{AC}\left(\begin{array}{c} V^\epsilon \\ \Big| \\ V^\epsilon \end{array} \right) = \mathrm{id}_{V^\epsilon} \ : \ V^\epsilon \to V^\epsilon \tag{13}$$

[4] A single line represents a ribbon whose right side is faced to us.

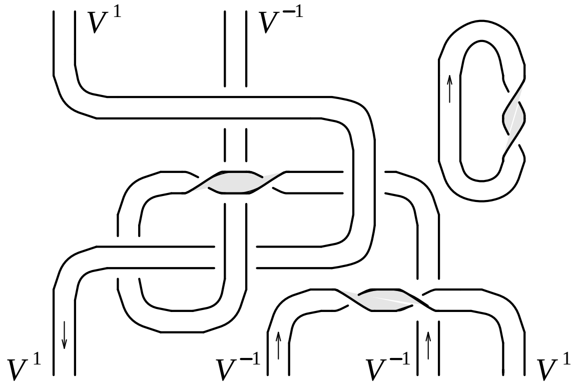


Figure 1: A ribbon $(4,2)$-graph colored by, for example, $w_4^{(b)} = (V^1\Box((V^{-1}\Box V^{-1})\Box V^1))$ and $w_2^{(t)} = (V^1\Box V^{-1})$.

$$F^{AC}\left(\overset{V^1\Box\, V^{-1}}{\cup}\right)(1) \quad = \quad \sum_{g,h} \omega(g, g^{-1}, g)\chi_{g,h} \otimes \psi_{g,h} \ : \ \mathbf{C} \to V \otimes V^{-1} \tag{14}$$

$$F^{AC}\left(\overset{V^{-1}\Box V^1}{\cup}\right)(1) \quad = \quad \sum_{g,h} \psi_{g,h} \otimes \chi_{g,h} \ : \ \mathbf{C} \to V^{-1} \otimes V \tag{15}$$

$$F^{AC}\left(\underset{V^{-1}\Box V^1}{\cap}\right)(\psi_{g,h} \otimes \chi_{x,y}) \quad = \quad \delta_{g,x}\, \delta_{h,y} \ : \ V^{-1} \otimes V \to \mathbf{C} \tag{16}$$

$$F^{AC}\left(\underset{V^1\Box\, V^{-1}}{\cap}\right)(\chi_{g,h} \otimes \psi_{x,y}) \quad = \quad \omega(g^{-1}, g, g^{-1})\, \delta_{g,x}\, \delta_{h,y} \ : \ V \otimes V^{-1} \to \mathbf{C} \tag{17}$$

$$\begin{aligned} F^{AC}\left(\overset{V^{\epsilon_2}\Box\, V^{\epsilon_1}}{\underset{V^{\epsilon_1}\Box\, V^{\epsilon_2}}{\times}}\right) \quad &= \quad P_{12} \circ (\pi^{\epsilon_1} \otimes \pi^{\epsilon_2})\,(R) \\ &\qquad\qquad : \ V^{\epsilon_1} \otimes V^{\epsilon_2} \to V^{\epsilon_2} \otimes V^{\epsilon_1} \end{aligned} \tag{18}$$

$$F^{AC}\left(\overset{V^{\epsilon_1}\Box\, V^{\epsilon_2}}{\underset{V^{\epsilon_2}\Box\, V^{\epsilon_1}}{\times}}\right) \quad = \quad (\pi^{\epsilon_1} \otimes \pi^{\epsilon_2})\,(R^{-1}) \circ P_{21}$$

$$: \; V^{\epsilon_2} \otimes V^{\epsilon_1} \to V^{\epsilon_1} \otimes V^{\epsilon_2} \tag{19}$$

$$\begin{aligned} F^{AC}\left(\begin{array}{c} (V^{\epsilon_1} \square V^{\epsilon_2}) \square V^{\epsilon_3} \\ \Big| \qquad \Big| \qquad \Big| \\ V^{\epsilon_1} \square (V^{\epsilon_2} \square V^{\epsilon_3}) \end{array} \right) &= (\pi_1^{\epsilon_1} \otimes \pi_2^{\epsilon_2} \otimes \pi_3^{\epsilon_3})\,(\phi) \\ &: \; V^{\epsilon_1} \otimes V^{\epsilon_2} \otimes V^{\epsilon_3} \to V^{\epsilon_1} \otimes V^{\epsilon_2} \otimes V^{\epsilon_3} \end{aligned} \tag{20}$$

where P_{12} denotes the permutation operator :

$$P_{12} : a_1 \otimes b_2 \mapsto b_2 \otimes a_1 : \qquad V^{\epsilon_1} \otimes V^{\epsilon_2} \to V^{\epsilon_2} \otimes V^{\epsilon_1}.$$

There appear the regular representation $\pi^1 (= \pi)$ and its dual $\pi^{-1} (= \pi^*)$. Explicitly, they are

$$\pi(a)\,x \equiv a \cdot x \tag{21}$$

$$\pi^*(a)\,x^* \equiv x^*\,(\pi \circ S(a)) \tag{22}$$

where $a, x \in D^\omega(G)$ and $x^* \in D^\omega(G)^*$.

We proceed to defining a three-manifold invariant using a lattice and the functor F^{AC}. Our basic strategy is as follows. We provide a triangulation T of a closed three-manifold M and associate a framed link L_T in M with T. Then we try to operate the functor F^{AC} on L_T 'interpreted' as a colored $(0,0)$-graph Γ_T.

We associate a framed link L_T with a triangulation T in the following way:

1. With each 2-simplex f_i in T, we associate a trivial knot C_i along the boundary ∂f_i, which yields a trivial link with n_2 components (We denote by n_i the number of i-simplices in T).

2. To the link $\cup_{i=1}^{n_2} C_i$, we add components C_{n_2+j} $(j = 1, \ldots, n_1)$ so as to encircle each bundle of segments of knots corresponding to each 1-simplex j (Fig.2 (b)). We obtain a link L_T with $n_1 + n_2$ components altogether: $L_T = \cup_{i=1}^{n_2+n_1} C_i$.

3. We thicken each component in L_T to be an annulus of 0 framing [5] and associate an arrow along each annulus arbitrarily to have a framed link.

[5] This makes sense because each link component is contained in a 3-ball in M.

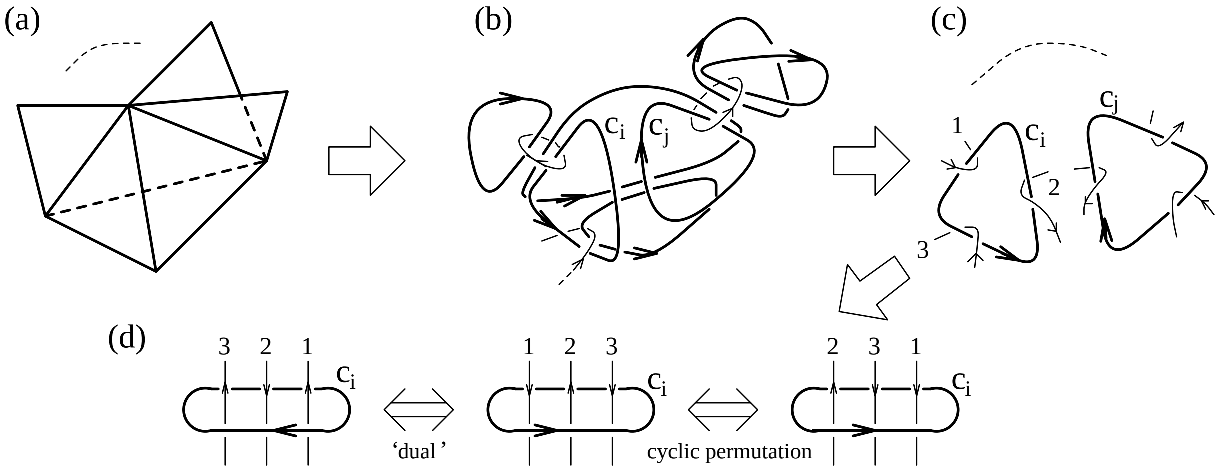


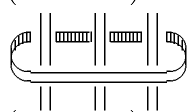
Figure 2: The link representation of T : (a) A part of T. (b) A part of L_T with arrows. (c) The decomposition into building blocks. Two building blocks are drawn. (d) Interpretation as ribbon $(3,3)$-graphs.

Let us pause for a moment here to explain why we construct framed links in this way. Eventually we would hope to interpret L_T as a ribbon $(0,0)$-graph Γ_T and to prove that $F^{AC}(\Gamma_T)$ is invariant under the Alexander moves [13] of T. Since we have a map from triangulations to framed links, we can determine a set of local modifications of links that corresponds to Alexander moves. In fact, such a set of local modification is generated by the Kirby moves, under which $F^{AC}(\Gamma_T)$ is invariant. Thus we would expect that the correspondence between triangulations and framed links is a key relation which connects lattice and surgery constructions of topological field theories.

However here we face the fact that $F^{AC}(\Gamma_T)$ is not well-defined. This is because Γ_T should be in $\mathbf{R}^3$ while L_T is contained in the manifold M. So our task is not so straightforward.

Nevertheless, it is possible to generalize the action of F^{AC} to yield a number for the framed link obtained above. We first decompose the framed link L_T into n_2 building blocks each of which corresponds to a 2-simplex as depicted in Fig.2 (c). Then we decorate them to make them $(3,3)$ c-graphs (Fig.2(d)). The framings are derived from those of framed links. Arrows are already associated with the components. We decide which end is the top of the graph arbitrarily. This fixes the order of the three vertical ribbons up to a cyclic permutation. We also associate words with the bottom end and the top end of a $(3,3)$-graph arbitrarily under the restriction that the ϵ's match the direction of the arrows. It is possible to associate two different words with the bottom (top) end of the graph. Namely it can be $(V^{\epsilon_1}\Box V^{\epsilon_2})\Box V^{\epsilon_3}$ or $V^{\epsilon_1}\Box(V^{\epsilon_2}\Box V^{\epsilon_3})$. If we set the position of two parentheses in ${w_3}^{(t)}$ and ${w_3}^{(b)}$ to be identical, we have only to deal with two kinds of c-graphs:

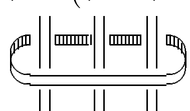
$(V^{\epsilon_1}\Box V^{\epsilon_2})\Box V^{\epsilon_3}$

$\Gamma_{(\epsilon_1,\epsilon_2),\epsilon_3} \equiv$  (23)

$(V^{\epsilon_1}\Box V^{\epsilon_2})\Box V^{\epsilon_3}$

and

$V^{\epsilon_1}\Box(V^{\epsilon_2}\Box V^{\epsilon_3})$

$\Gamma_{\epsilon_1,(\epsilon_2,\epsilon_3)} \equiv$  . (24)

$V^{\epsilon_1}\Box(V^{\epsilon_2}\Box V^{\epsilon_3})$

Here V^{ϵ_i} denotes the vector space V^{ϵ_1} associated with a ribbon i. The relation between the two maps corresponding to these two c-graphs is determined by (20) as

$$\begin{aligned} & F^{AC}(\Gamma_{\epsilon_1,(\epsilon_2,\epsilon_3)}) \\ = \ & (\pi_1^{\epsilon_1}\otimes\pi_2^{\epsilon_2}\otimes\pi_3^{\epsilon_3})(\phi^{-1})\circ F^{AC}(\Gamma_{(\epsilon_1,\epsilon_2),\epsilon_3})\circ(\pi_1^{\epsilon_1}\otimes\pi_2^{\epsilon_2}\otimes\pi_3^{\epsilon_3})(\phi) \\ & \quad : V^{\epsilon_1}\otimes V^{\epsilon_2}\otimes V^{\epsilon_3}\longrightarrow V^{\epsilon_1}\otimes V^{\epsilon_2}\otimes V^{\epsilon_3}. \end{aligned} \tag{25}$$

Now, let us demand that

$$F^{AC}\left(\Gamma_{\epsilon_1,(\epsilon_2,\epsilon_3)}\right) = F^{AC}\left(\Gamma_{(\epsilon_1,\epsilon_2),\epsilon_3}\right). \tag{26}$$

One sees that it is necessary and sufficient to assume that $\omega\equiv 1$ or G is a commutative group. In this case, there occurs much simplification in, e.g. eqs (6) and (7), and it can be shown shown that eq.(26) is satisfied by explicitly checking for every choice of ϵ_i $(i=1,2,3)$.

From now on, we assume that $\omega\equiv 1$ or G is a commutative group. Then the linear transformation $F^{AC}(\Gamma)$ acquires more symmetries. It commutes with cyclic permutations

$$F^{AC}\left(\Gamma_{(\epsilon_1,\epsilon_2,\epsilon_3)}\right) = P_{231}^{-1}\circ F^{AC}\left(\Gamma_{(\epsilon_2,\epsilon_3,\epsilon_1)}\right)\circ P_{231}. \tag{27}$$

Furthermore, it satisfies

$$P_{321} \circ \left(F^{AC}(\Gamma_{(\epsilon_1,\epsilon_2,\epsilon_3)})\right)^* \circ P_{321}^{-1} = F^{AC}(\Gamma_{(-\epsilon_3,-\epsilon_2,-\epsilon_1)}) \\ : V^{-\epsilon_3} \otimes V^{-\epsilon_2} \otimes V^{-\epsilon_1} \longrightarrow V^{-\epsilon_3} \otimes V^{-\epsilon_2} \otimes V^{-\epsilon_1} \tag{28}$$

where $*$ means dual. This relation shows that π rotation of the graph $\Gamma_{(\epsilon_1,\epsilon_2,\epsilon_3)}$ within the plane where Γ lies corresponds to taking the dual map of $F^{AC}(\Gamma_{(\epsilon_1,\epsilon_2,\epsilon_3)})$ (Fig.2(d)).

Because of the symmetries mentioned above, there are only two independent c-graphs $\Gamma_{(\epsilon_1,\epsilon_2,\epsilon_3)}$, which are for $(\epsilon_1,\epsilon_2,\epsilon_3) = (1,1,1)$ and $(-1,1,1)$. The linear transformations they induce are explicitly given by

$$\begin{aligned} & F^{AC}(\Gamma_{(+1,+1,+1)})\,(\chi_{g_1,x_1} \otimes \chi_{g_2,x_2} \otimes \chi_{g_3,x_3}) \\ &= |G| \sum_{h\in G} \delta_{g_1g_2g_3,e}\theta_h(g_1,g_2)\,\theta_h({g_3}^{-1},g_3)\,\theta_{g_1}(h,x_1)\,\theta_{g_2}(h,x_2)\,\theta_{g_3}(h,x_3) \\ &\qquad \times\chi_{g_1,hx_1} \otimes \chi_{g_2,hx_2} \otimes \chi_{g_3,hx_3} \end{aligned} \tag{29}$$

and

$$\begin{aligned} & F^{AC}(\Gamma_{(-1,+1,+1)})\,(\psi_{g_1,x_1} \otimes \chi_{g_2,x_2} \otimes \chi_{g_3,x_3}) \\ &= |G| \sum_{h\in G} \delta_{{g_1}^{-1}g_2g_3,e}\theta_h(g_2,g_3)\,\theta_{g_1}(h,x_1)^{-1}\,\theta_{g_2}(h,x_2)\,\theta_{g_3}(h,x_3) \\ &\qquad \times\psi_{g_1,hx_1} \otimes \chi_{g_2,hx_2} \otimes \chi_{g_3,hx_3}. \end{aligned} \tag{30}$$

Now we 'compose' these n_2 linear maps reflecting the connectivity of $3n_2$ vertical ribbons. Let us imagine a situation such that a top end of a building block corresponding to the linear map $F^{AC}(\Gamma_{(\epsilon_1,\epsilon_2,\epsilon_3)})$ is connected to a bottom end of that corresponding to $F^{AC}(\Gamma_{(\epsilon_1,\epsilon_4,\epsilon_5)})$ (Fig. 3(a)). To recombine these building blocks, we consider a linear transformation

$$\Phi(e^{\epsilon_1}_{i_1} \otimes e^{\epsilon_2}_{i_2} \otimes e^{\epsilon_3}_{i_3} \otimes e^{\epsilon_4}_{i_4} \otimes e^{\epsilon_5}_{i_5}) = \sum_{j,k} f^{j_1j_4j_5}_{k_1i_4i_5} f^{k_1j_2j_3}_{i_1i_2i_3} e^{\epsilon_1}_{j_1} \otimes e^{\epsilon_2}_{j_2} \otimes e^{\epsilon_3}_{j_3} \otimes e^{\epsilon_4}_{j_4} \otimes e^{\epsilon_5}_{j_5} \tag{31}$$

on $V^{\epsilon_1} \otimes V^{\epsilon_2} \otimes V^{\epsilon_3} \otimes V^{\epsilon_4} \otimes V^{\epsilon_5}$ following the spirit of the definition of the action of F^{AC} on c-graphs. The basis $\{e^{\epsilon}_k\}_{k=1,\ldots,|G|^2}$ span V^ϵ (i.e. $e^1 \sim \chi, e^{-1} \sim \psi$) and

$$F^{AC}(\Gamma_{(\epsilon_1,\epsilon_2,\epsilon_3)})(e^{\epsilon_a}_{i_a} \otimes e^{\epsilon_b}_{i_b} \otimes e^{\epsilon_c}_{i_c}) = \sum_j f^{j_aj_bj_c}_{i_ai_bi_c} e^{\epsilon_a}_{j_a} \otimes e^{\epsilon_b}_{j_b} \otimes e^{\epsilon_c}_{j_c}. \tag{32}$$

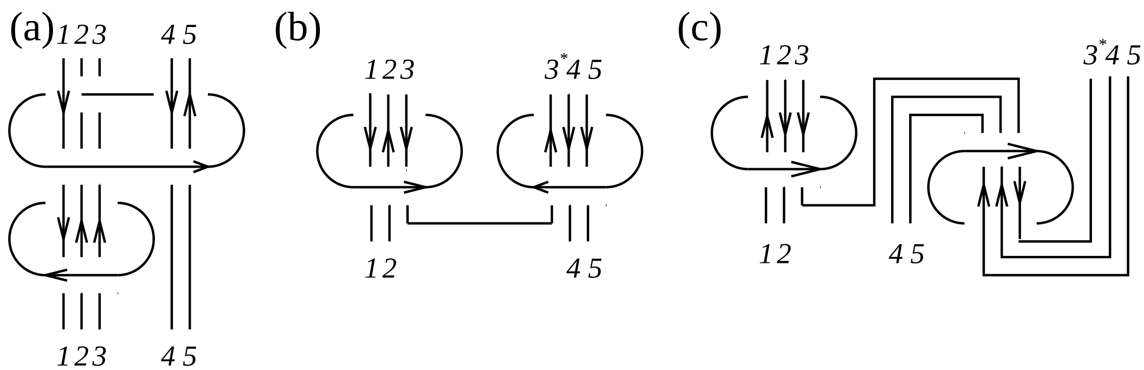


Figure 3: (a) Connecting a bottom end and a top end. (b) Connecting two bottom ends. (c) Operation (b) is equivalent with first taking the dual (rotating the graph) and then connect a bottom end and a top end.

If two building blocks are connected through the bottom ends of their vertical ribbons (Fig.3(b)), we consider a linear map

$$\Psi(e_{i_1}^{\epsilon_1} \otimes e_{i_2}^{\epsilon_2} \otimes e_{i_4}^{\epsilon_4} \otimes e_{i_5}^{\epsilon_5}) = \sum_{j,j_3',i_3,i_3'} f_{i_1 i_2 i_3}^{j_1 j_2 j_3} f_{i_3' i_4 i_5}^{j_3' j_4 j_5} \delta_{i_3 i_3'} e_{j_1}^{\epsilon_1} \otimes e_{j_2}^{\epsilon_2} \otimes e_{j_3}^{\epsilon_3} \otimes e_{j_3'}^{-\epsilon_3} \otimes e_{j_4}^{\epsilon_4} \otimes e_{j_5}^{\epsilon_5}$$
$$: \quad V^{\epsilon_1} \otimes V^{\epsilon_2} \otimes V^{\epsilon_4} \otimes V^{\epsilon_5} \to V^{\epsilon_1} \otimes V^{\epsilon_2} \otimes V^{\epsilon_3} \otimes V^{-\epsilon_3} \otimes V^{\epsilon_4} \otimes V^{\epsilon_5} \tag{33}$$

by a reason we are going to explain shortly.

We continue these operations until $6n_2$ free ends of vertical ribbons are contracted. Finally we are led to the definition of a number $F^{AC}(L_T)$:

$$F^{AC}(L_T) = \sum_{i_1,\ldots,i_{6n_2}} f_{i_1,i_2,i_3}^{i_{3n_2+1},i_{3n_2+2},i_{3n_2+3}} f_{i_4,i_5,i_6}^{i_{3n_2+4},i_{3n_2+5},i_{3n_2+6}} \cdots f_{i_{3n_2-2},i_{3n_2-1},i_{3n_2}}^{i_{6n_2-2},i_{6n_2-1},i_{6n_2}} \prod_{n=1}^{3n_2} \delta_{i_{a_n},i_{b_n}}, \tag{34}$$

where (a_n, b_n) $(n = 1, \ldots, 3n_2)$ are the pairs of ends of ribbons that are connected in L_T. We see that $F^{AC}(L_T)$ does not change when we change the order of the tensor product in eqs.(31),(33).

It might seem more natural to use the action of F^{AC} on various graphs, e.g. eqs.(14) or (15) (eqs.(16) or (17)) when we connect two bottom (top) ends. However, because we have decided which end is the top of a graph by hand, it is expected that the final result (34) is invariant under interchanging the top end and the bottom end of a graph (i.e. taking the dual of one of the linear maps by eq.(28). See Fig. 3(c).) It is necessary to define the linear map by eq.(33) in order to have the invariance.

Furthermore, we verify that this quantity $F^{AC}(L_T)$ is a topological invariant of three-manifold when multiplied by a simple numerical factor[6] depending on n_i and that it has a state-sum representation. In fact, this is exactly the same quantity as the Chung-Fukuma-Shapere invariant Z_A^{CFS} for $A = D^\omega(G)$. We carry out the proof of the invariance using the equivalence with the Chung-Fukuma-Shapere invariant $Z^{CFS}_{D^\omega(G)}$. We comment that the proof can be performed following the idea of generating the Alexander moves from the Kirby moves, without mentioning the Chung-Fukuma-Shapere invariant.

Before doing this we briefly explain the Chung-Fukuma-Shapere invariant $Z_A^{CFS}(M)$. It is defined for each involutory Hopf algebra A, where involutory means that the square of the antipode operator $S : A \to A$ is the identity [6, 14]. We provide a lattice of M and decompose it into the set of polygonal faces $F = \{f\}$ and 'hinges' $H = \{h\}$ which have a role of connecting faces. We put an arrow on each edge of each face f by a cyclic order around a face, which also induces arrows on hinges. By assigning $C_{f_1 f_2 \cdots f_n}$ and $\Delta^{h_1 \cdots h'_j \cdots h_n} \prod_j S^{h_j}{}_{h'_j}$ to each n-face and n-hinge, the partition function $Z_A^{CFS}(M)$ is defined as [6, 14]

$$Z_A^{CFS}(M) \equiv |\dim A|^{-n_3-n_1} \prod_{f\in F} C_{x_{f[1]}\cdots x_{f[n_f]}} \prod_{h\in H} \left[\Delta^{x_{h[1]}\cdots x'_{h[j]} \cdots x_{h[n_h]}} \prod_{j\in R_h} S^{x_{h[j]}}{}_{x'_{h[j]}} \right]. \quad (35)$$

For $A = D^\omega(G)$, the theory is defined if $\omega \equiv 1$ or G is commutative. It is characterized by the data C, Δ and S given as

$$\begin{aligned} C_{(g_1,h_1)(g_2,h_2)\cdots(g_k,h_k)} &= |G| \delta_{g_1,g_2} \delta_{g_1,g_3} \cdots \delta_{g_1,g_k} \delta_{h_1\cdots h_k,e} \times \\ &\quad \times \prod_{j=1}^{k-1} \theta_{g_1}\left(\prod_{\ell=1}^{j} h_\ell, h_{j+1} \right) \qquad (\text{for } k \geq 3)\,, &(36)\\ \Delta^{(h_1,g_1)(h_2,g_2)\cdots(h_k,g_k)} &= C_{(g_1,h_1)(g_2,h_2)\cdots(g_k,h_k)}\,, &(37)\\ S^{(h,y)}{}_{(g,x)} &= \delta_{gh,e}\delta_{xy,e}\theta_h(y^{-1},y)^{-1}\theta_{y^{-1}}(h,h^{-1})^{-1}. &(38) \end{aligned}$$

By comparing the eqs. (29) and (30) to these data C, Δ and S, they are rewritten as

$$\begin{aligned} &F^{AC}(\Gamma_{(+1,+1,+1)})\ (\chi_{g_1,x_1} \otimes \chi_{g_2,x_2} \otimes \chi_{g_3,x_3}) \\ &\quad = \Delta^{(a_1,b_1)(a_2,b_2)(a_3,b_3)} C^{(h_1,y_1)}{}_{(a_1,b_1)(g_1,x_1)} C^{(h_2,y_2)}{}_{(a_2,b_2)(g_2,x_2)} C^{(h_3,y_3)}{}_{(a_3,b_3)(g_3,x_3)} \end{aligned}$$

[6] This factor has a simple interpretation in terms of the Kirby moves. When we generate Alexander moves from Kirby moves, we neglect trivial (disconnected) link components produced by the latter. This factor is to compensate contributions of such link components.

$$\times\ \chi_{h_1,y_1} \otimes \chi_{h_2,y_2} \otimes \chi_{h_3,y_3} \tag{39}$$

$$\begin{aligned}
&F^{AC}(\Gamma_{(-1,+1+1)})\ (\psi_{g_1,x_1} \otimes \chi_{g_2,x_2} \otimes \chi_{g_3,x_3}) \\
&= \ \Delta^{(a_1,b_1)(a_2,b_2)(a_3,b_3)} S^{(a_1',b_1')}{}_{(a_1,b_2)} C^{(g_1,x_1)}{}_{(a_1',b_1')(h_1,y_1)} C^{(h_2,y_2)}{}_{(a_2,b_2)(g_2,x_2)} C^{(h_3,y_3)}{}_{(a_3,b_3)(g_3,x_3)} \\
&\qquad \times\ \psi_{h_1,y_1} \otimes \chi_{h_2,y_2} \otimes \chi_{h_3,y_3}.
\end{aligned} \tag{40}$$

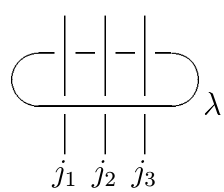
Here we see that the number $F^{AC}(L_T)$ defined by eq.(34) is identical with the partition function $Z^{CFS}_{A=D^\omega(G)}$ (35) for the dual lattice T^* of T up to a normalization factor, i.e.,

$$|G|^{-2(n_0+n_2)} \times F^{AC}(L_T) = Z^{CFS}_{A=D^\omega(G)}(T^*). \tag{41}$$

We started with the Altschuler-Coste functor F^{AC} which gave an invariant for framed links. We 'operated' the functor on a framed link L_T associated with the triangulation T and ended up with the Chung-Fukuma-Shapere theory for $D^\omega(G)$. Originally, the functor F^{AC} was applied to framed links along which surgeries were performed and Altschuler-Coste theory was obtained. So we have two distinct topological field theories via the lattice construction and the surgery construction starting with a single functor F^{AC}. It is known that the Chung-Fukuma-Shapere theory for $D^\omega(G)$ is related to the Altschuler-Coste theory for $D^\omega(G)$ in the sense that the former is a tensor product of two latter theories for $G = \mathbf{Z}_{2N+1}$ [10]. Therefore in some cases the lattice and surgery constructions produce the identical topological field theory (up to the square).

In ref. [11] it is suggested that a link representation of a manifold L_T and the Reshetikhin-Turaev functor $F^{RT}_{\mathcal{U}_q(sl(2,\mathbf{C}))}$ for $q = \exp(2\pi i/r)$ induces the Turaev-Viro invariant of three-manifolds. Roughly speaking, the idea is as follows. Define

$$F^{RT}_q(j_1,j_2,j_3) \ \equiv \ \sum_{\lambda\in I} [2\lambda+1]_q \ F^{RT}_{\mathcal{U}_q(sl(2,\mathbf{C}))}\Bigg(\qquad \Bigg) \tag{42}$$



$$: V^{j_1} \otimes V^{j_2} \otimes V^{j_3} \to V^{j_1} \otimes V^{j_2} \otimes V^{j_3}$$

for a ribbon (3,3)-graph colored by $j \in \{0, 1/2, 1, \cdots, r/2-1\}$. The vector space V^j is $2j+1$ dimensional and is spanned by b^j_m $(m = j, j-1, \cdots, -j)$ [15]. It is calculated as

$$F^{RT}_q(j_1,j_2,j_3)(b^{j_1}_{m_1} \otimes b^{j_2}_{m_2} \otimes b^{j_3}_{m_3})$$

$$= \delta(j_1 j_2 j_3) \sum_{n_1,n_2,n_3} \begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{bmatrix}_q \begin{bmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & -n_3 \end{bmatrix}_q$$
$$\times [2j_3+1]_q^{-1} q^{\frac{m_3+n_3}{2}} (-1)^{2j_3+m_3+n_3}\, b^{j_1}_{n_1} \otimes b^{j_2}_{n_2} \otimes b^{j_3}_{n_3} \qquad (43)$$

where $[:::]_q$ is the q-$3j$-symbol and $\delta(j_1j_2j_3)$ takes 1 if $j_1 \le j_2 + j_3, j_2 \le j_1 + j_3, j_3 \le j_1 + j_2, j_1 + j_2 + j_3 \le r-2$ and $j_1+j_2+j_3$ is integer, and 0 otherwise. The claim is that the combination of four q-$3j$-symbols from (43) gives rise to q-$6j$-symbol, which yields the Turaev-Viro invariant.

However, (43) shows us that the cyclic symmetry $F^{RT}_q(j_1, j_2, j_3) = F^{RT}_q(j_2, j_3, j_1)$ holds only in the case of $q = 1$. It means that in the case of $q \neq 1$ there is no well-defined way of interpreting each building block of L_T as a ribbon (3,3)-graph. It follows that we cannot construct the Turaev-Viro theory from the Reshetikhin-Turaev functor $F^{RT}_{\mathcal{U}_q(sl(2,\mathbf{C}))}$ by this method, though there are connections between them [16].

On the other hand, it can be shown that the set of $F^{RT}_q(j_1, j_2, j_3)$ produce the Turaev-Viro invariant in the limit $q \to 1$, i.e., the Ponzano-Regge partition function [17], by the similar equation as (34) defined by (43).

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