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Post-Inflation Reheating in an Expanding Universe

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Abstract An analytic means of studying the resonant decay of the inflaton field is developed for the case of background expansion, $H \neq 0$. It is shown that the parametric resonance in the inflaton's decay need not disappear when the expansion of the universe is taken into account, although the total number of particles produced is fewer than in the $H \simeq 0$ case.

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1 Introduction

Soon after the first models of inflation were published in 1981-1982 [1] [2], a number of papers were written to study how the universe would reheat following the supercooled inflationary state. [3] The process appeared to be straightforward: in models like new inflation (and chaotic inflation [4]) which incorporate a second order phase transition to end inflation, the inflaton field would wind up oscillating around the minimum of its potential near the end of inflation. These oscillations would produce a sea of relativistic particles, if one added (by hand) interaction terms between the inflaton and these lighter species. The decay rates $\Gamma_{\phi\rightarrow\chi,\psi}$ could then be calculated, where ϕ is the (decaying) inflaton field, χ is some light boson field, and ψ is some light fermion field. From these decay rates, the energy density (ρ_p) of the produced particles could be calculated, and related to the final reheat temperature ($T_{\rm rh}$): $\rho_p \simeq \pi^2 N_{\rm eff}\, T_{\rm rh}^4/30$, where $N_{\rm eff}$ counts the effective number of massless spin degrees of freedom. This method was thought to estimate $T_{\rm rh}$ to within an order of magnitude. In particular, the process of thermalization was ignored: it is unclear how efficiently the newly-produced particles will attain thermal equilibrium, to allow use of this relation between ρ_p and $T_{\rm rh}$.

In 1990, Traschen and Brandenberger [5] , and independently Dolgov and Kirilova [6], noted an overlooked but dramatic feature of the production-via-oscillation model: the inflaton's oscillations should be *unstable*, and should exhibit *parametric resonance*. Certain modes χ_k within small bands $\epsilon \ll k$ should grow *exponentially*, swamping the production in other modes. This analysis has been expanded recently in [7] [8] [9]. The crucial feature of the new reheating scenario is that the equation of motion for the light boson field will not obey the (previously-assumed) simple harmonic oscillator equation. Instead, for an interaction of the form $\mathcal{L}' = -\frac{1}{2}g^2\phi^2\chi^2$, the mode functions of the boson field will obey:

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \left(\frac{k^2}{a^2} + g^2\phi^2\right)\chi_k = 0, \tag{1}$$

where overdots denote derivatives with respect to cosmic time t, $a(t)$ is the scale factor for the (assumed) flat Friedmann-Robertson-Walker spacetime, and $H \equiv \dot{a}/a$ is the Hubble parameter.

Since the inflaton field ($\phi(t)$) is rapidly oscillating, for frequencies $\omega_I \gg H$ the $3H\dot{\chi}_k$ term may be neglected and $a(t)$ treated as constant. Then this equation for the boson field mode functions will look like the well-known Mathieu equation (see, *e.g.*, [10] [11]):

$$\ddot{\chi}_k + \omega_k^2 (1 + h \cos \omega_I t) \chi_k = 0. \tag{2}$$

As demonstrated in [5] [7] [8] and treated below, for $\omega_I = 2\omega_k + \epsilon$, with $\epsilon \ll \omega_k$, solutions grow like $\chi_k(t) \propto \exp(\pm\mu_\pm t)$, instead of like $\chi_k(t) \propto \cos(\omega_k t)$. This means that the number of χ bosons produced goes like $N_\chi^{\rm res} \propto \exp(2\mu_+ t)$, so that the decay rate for $\phi \to 2\chi$ is much larger than originally calculated: $\Gamma_\chi^{\rm res} \gg \Gamma_\chi$. (Note that this parametric resonance is only effective for inflaton decay into bosons; the exponential increase in fermion modes is forbidden because of Fermi-Dirac statistics.) This large decay rate increases ρ_p, ultimately giving $T_{\rm rh}^{\rm res} \gg T_{\rm rh}$.

Others contend, however, that this large resonance effect *only* appears when the expansion of the universe is completely ignored. Dolgov and Freese rewrote the equations of motion, similar to equation (1), and numerically integrated. [12] They found no signs of the parametric resonance reported in [7] [8]. In particular, they reasoned that when $H \neq 0$, the physical wavenumbers k/a quickly redshift out of the narrow resonance band. Similar conclusions were reached by Yoshimura in [13].

It is thus important to develop an *analytic* means of studying the inflaton's decay into other particles for the case of $H \neq 0$, in order better to determine the effectiveness of resonant decays for raising the reheat temperature $T_{\rm rh}$. Such an analytic study is presented below. We find that the parametric resonance effect does *not* necessarily disappear when $H \neq 0$, although the total number of particles produced does shrink. Rather than expansion of the universe, the true threat to resonant inflaton decays appears to be back-reaction of the produced particles on the amplitude of the decaying inflaton field. This point is discussed further in the Appendix.

In this paper, we concentrate on the case of the inflaton field ϕ decaying into further inflaton bosons, via the self-interaction potential $V(\phi)$. In section 2, the number of particles produced per mode k, N_k, is calculated for non-zero expansion, assuming that the decay is not resonant. Section 3 presents the calculation of $N_k^{\rm res}$ analogous to [7] [8], when the expansion of the universe is completely neglected. In section 4 we calculate $N_k^{\rm res}$ taking into account the expansion of the universe, and demonstrate that the parametric resonance need not vanish. From these values of N_k, the decay rates and reheating temperatures may be calculated using methods similar to those developed in [8], and will not be repeated here. Concluding remarks follow in section 5.

2 Non-Resonant Inflaton Decay in an Expanding Background

We will study a very simple model of inflation: chaotic inflation with the potential $V(\phi) = \frac{1}{4}\lambda\phi^4$. We will assume that the metric may be written in the form of a flat Friedmann-Robertson-Walker line element, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)d\vec{x}^2$. Then, given the lagrangian density

$$\mathcal{L} = \sqrt{-g}\left[\frac{1}{16\pi G}R - \frac{1}{2}\phi_{;\,\mu}\phi^{;\,\mu} - V(\phi)\right], \tag{3}$$

the equations of motion take the familiar form:

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\left[V(\phi) + \frac{1}{2}\dot{\phi}^2\right], \\ \ddot{\phi} &+ 3H\dot{\phi} - \frac{1}{a^2}\nabla^2\phi + \frac{dV}{d\phi} = 0. \end{aligned} \tag{4}$$

Next we decompose the inflaton field into the sum of a classical background field and a quantum fluctuation: $\phi(\vec{x},t) = \varphi(t) + \delta\phi(\vec{x},t)$. (We will neglect the metric fluctuations which couple to these inflaton fluctuations for the present analysis; for more on these metric fluctuations, see, *e.g.*, [14] [15].) In order to study the production of ϕ particles when the inflaton field is oscillating near the minimum of its potential, we will need to calculate mode solutions for the quantum fluctuations $\delta\phi$ before and after the phase transition, matching the solutions by means of a Bogolyubov transformation.

We will first concentrate on the modes for the fluctuations near the end of the inflationary period. In terms of conformal time, $d\eta \equiv a^{-1}dt$, the fluctuations are quantized as follows:

$$\delta\hat{\phi}(x) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}}\left[\delta\phi_k(\eta)\,\hat{a}_{\vec{k}}\,e^{i\vec{k}\cdot\vec{x}} + \delta\phi_k^*(\eta)\,\hat{a}_{\vec{k}}^\dagger\,e^{-i\vec{k}\cdot\vec{x}}\right], \tag{5}$$

where hats denote quantum operators, and the creation and annihilation operators obey the canonical commutation relations:

$$\begin{aligned}\left[\hat{a}_{\vec{k}}\,,\,\hat{a}_{\vec{\ell}}\right] &= \left[\hat{a}^{\dagger}_{\vec{k}}\,,\,\hat{a}^{\dagger}_{\vec{\ell}}\right] = 0\ , \\ \left[\hat{a}_{\vec{k}}\,,\,\hat{a}^{\dagger}_{\vec{\ell}}\right] &= \delta^3(\vec{k}-\vec{\ell})\ ,\ \text{ with } \hat{a}_{\vec{k}}|0>=0. \end{aligned} \tag{6}$$

From equation (4), and denoting $d/d\eta$ by a prime, the equation of motion for the mode functions $\delta\phi_k(\eta)$ becomes:

$$\delta\phi_k'' + 2\frac{a'}{a}\delta\phi_k' + \left(k^2 + 3\lambda\varphi^2 a^2\right)\delta\phi_k = 0. \tag{7}$$

Next we introduce a "conformal field" $\psi \equiv a\delta\phi$, whose mode functions obey the equation of motion:

$$\psi_k'' + \left[k^2 - \frac{a''}{a} + 3\lambda\varphi^2 a^2\right]\psi_k = 0. \tag{8}$$

Near the end of inflation, the scale factor $a(t)$ will not in general retain its de Sitter (exponential) form. Instead we may write $a(t) \propto t^p$, with $1/2 \leq p \leq \infty$, where $p = 1/2$ corresponds to the radiation-dominated epoch, and $p \to \infty$ recovers the de Sitter epoch. This scale factor corresponds to a conformal-time scale factor of $a(\eta) \propto \eta^{p/(1-p)}$. Because we are ignoring the parametric resonance from the classical φ-field's oscillations in this section, we may simply assume that near the minimum of its potential, $\varphi^2 \ll k^2$. If we lastly define a new field $\chi \equiv \eta^{-1/2}\psi$, and define a new variable $z \equiv k\eta$, then equation (8) becomes

$$\frac{d^2\chi_k}{dz^2} + \frac{1}{z}\frac{d\chi_k}{dz} + \left(1 - \frac{1}{z^2}\left[\frac{(3p-1)^2}{4(p-1)^2}\right]\right)\chi_k \simeq 0. \tag{9}$$

This is now in the form of Bessel's equation. Mode functions for the original field $\delta\phi$ may then be written in terms of Hankel functions:

$$\begin{aligned}\delta\phi_k(\eta) &= \frac{\eta^{1/2}}{a(\eta)}\left[A_k H_\nu^{(1)}(k\eta) + B_k H_\nu^{(2)}(k\eta)\right], \\ \nu &= \frac{3p-1}{2(p-1)}. \end{aligned} \tag{10}$$

We assume that during inflation, the inflaton field is in its vacuum state (see section 5.2 of [17] for further discussion of this point). The coefficients A_k and B_k thus may be fixed by choosing an appropriate quantum vacuum state. The Bunch-Davies, or "adiabatic", vacuum requires that the fluctuations $\psi_k(\eta)$ behave as Minkowski-spacetime mode functions far inside the horizon, that is, $\psi_k \to (2k)^{1/2} \exp(-ik\eta)$ for $k\eta \gg 1$. [15] [16] [18] From the asymptotic properties of the Hankel functions, this gives

$$A_k = 0 \ , \ \ B_k = \frac{\sqrt{\pi}}{2} \exp\left[-i\frac{\pi}{2}\left(\nu + \frac{1}{2}\right)\right] . \tag{11}$$

For the problem of (non-resonant) reheating, we will be interested in the opposite asymptotic limit: $k\eta \ll 1$, corresponding to long-wavelength modes. In this limit, the fluctuations $\psi_k(\eta)$ take the form

$$\psi_k(\eta) \to \frac{2^{\nu-1}}{\sqrt{\pi}} \exp\left[-i\frac{\pi}{2}\left(\nu - \frac{1}{2}\right)\right] \Gamma(\nu)\, \eta^{1/2}\, (k\eta)^{-\nu}. \tag{12}$$

In order to calculate the number (N_k) of ϕ-particles produced per mode k, we match this solution to long-wavelength mode solutions following the phase transition into the radiation-dominated era.

To do this, we make use of a Bogolyubov transformation. (See, *e.g.*, [16].) For long-wavelength modes, we may approximate the phase transition as instantaneous, occuring at some time η_* [19]:

$$\begin{aligned} \eta < \eta_* \quad &: \quad a(\eta) = (a_o \eta)^{p/(1-p)} \\ \eta > \eta_* \quad &: \quad a(\eta) = C\,(\eta - \bar{\eta})\,, \end{aligned} \tag{13}$$

where $\bar{\eta} \equiv \eta_* - (a_o^2 \eta_*)^{-1}$. It is convenient to set $a(\eta_*) = 1$, which sets $\eta_* = a_o^{-1}$, and thus $\bar{\eta} = 0$ and $C = a_o$. (Note that in the limit $p \to \infty$, the reference scale a_o becomes $-H_o$, where H_o is the Hubble constant of the de Sitter spacetime.) Next we perform a Bogolyubov transformation to match the long-wavelength mode solutions in the radiation-dominated era:

$$\psi_k(\eta > \eta_*) = \frac{1}{\sqrt{2k}} \left[\alpha_k e^{-ik\eta} + \beta_k e^{+ik\eta}\right] . \tag{14}$$

The Bogolyubov coefficients α_k and β_k may be determined by requiring that both ψ_k and ψ_k' be continuous at $\eta = \eta_*$. Then, using $N_k = |\beta_k|^2$, this gives

$$N_k = |\beta_k|^2 = \frac{4^{\nu-3/2}}{\pi} \Gamma^2(\nu) \left(\nu - \frac{1}{2}\right)^2 \left(\frac{k}{a_o}\right)^{-2\nu-1} . \tag{15}$$

Equation (15) is the main result of this section, and can now be compared with the cases in which the effects of the inflaton's parametric resonance are included.

3 Particle creation from Parametric Resonance, with $H \simeq 0$

We return to equation (8) for the fluctuations $\psi_k(\eta)$. Rather than neglect the $3\lambda\varphi^2(\eta)a^2(\eta)$ term in this section, however, we study the effects of this term as the inflaton field φ oscillates near the minimum of its potential. If the frequency of these oscillations is large enough, then we may expand the fields $\varphi(\eta)$ and $\varphi^2(\eta)$ in conformal-time harmonics, analogous to the cosmic-time harmonic decomposition adopted in [8]. Keeping the lowest term, we may write

$$\varphi^2(\eta) \simeq \overline{\varphi^2}\cos(\gamma\eta), \tag{16}$$

where $\overline{\varphi^2}$ is a slowly-decreasing amplitude. This quasi-periodic approximation for $\varphi^2(\eta)$ may be compared with the numerical results (for Minkowski-spacetime) calculated in [9]. For this section, we will assume that $\overline{\varphi^2} \simeq$ constant. (The chief contributor to the decrease of $\overline{\varphi^2}$ over time is the back-reaction from created particles, and will be further discussed in the Appendix.) Substituting this ansatz into equation (8) yields:

$$\psi_k'' + \omega_k^2\left[1 + ga^2\cos(\gamma\eta) - \frac{a''}{k^2 a}\right]\psi_k = 0, \tag{17}$$

where

$$\omega_k^2 \equiv k^2 \ , \ g \equiv \frac{3\lambda\overline{\varphi^2}}{\omega_k^2}. \tag{18}$$

Note that by working in terms of conformal time and the "conformal field" ψ, the frequency ω_k equals the (constant) comoving wavenumber k, and does not redshift with increasing $a(\eta)$. All of the effects of expansion are included in the explicit factors of $a(\eta)$ and $a''(\eta)$. When expansion is neglected (*i.e.*, when we set $a = 1$), equation (17) reduces to the Mathieu equation, the solutions of which are exponentially unstable when $g \ll 1$ and $\gamma = 2\omega_k + \epsilon$, with $\epsilon \ll \omega_k$. To see this, we may proceed as follows. Our treatment follows the methods outlined in [11], and substantially reproduces the results from the alternative approach adopted in [8].

We introduce the trial solution:

$$\psi_k(\eta) = c(\eta) f\left[a(\eta)\right]\cos\left[\left(\omega_k + \frac{1}{2}\epsilon\right)\eta\right] + d(\eta) f\left[a(\eta)\right]\sin\left[\left(\omega_k + \frac{1}{2}\epsilon\right)\eta\right]. \tag{19}$$

Here we have introduced the 'scaling' function $f(a)$ to absorb effects from the expansion of the universe. We assume that the coefficients $c(\eta)$ and $d(\eta)$ are slowly-varying with time, as compared with the frequency $\gamma/2$, although we make no such assumptions about the behavior of $f(a)$. Then it is self-consistent to put c', $d' \sim \mathcal{O}(\epsilon)$, and to neglect higher derivatives. In general, resonant solutions can be found for frequencies $n\gamma/2 \simeq \omega_k$, but each integer $n \geq 1$ corresponds to keeping terms of $\mathcal{O}(g^n)$ in the perturbation expansion. [11] Since we are only going to keep terms to $\mathcal{O}(g)$, we may make the following approximations:

$$\begin{aligned}\cos(\gamma\eta/2)\cos(\gamma\eta) &= \frac{1}{2}\cos(\gamma\eta/2) + \frac{1}{2}\cos(3\gamma\eta/2) \simeq \frac{1}{2}\cos(\gamma\eta/2), \\ \sin(\gamma\eta/2)\cos(\gamma\eta) &= -\frac{1}{2}\sin(\gamma\eta/2) + \frac{1}{2}\sin(3\gamma\eta/2) \simeq -\frac{1}{2}\sin(\gamma\eta/2).\end{aligned} \tag{20}$$

Then the equation of motion (17) applied to the trial solution of equation (19) yields:

$$\omega_k A(\eta)\ \sin(\gamma\eta/2) + \omega_k B(\eta)\ \cos(\gamma\eta/2) + \mathcal{O}\left(\epsilon^2,\ g^2\right) = 0, \tag{21}$$

with the coefficients A and B given by

$$\begin{aligned}A &= -2c'f - cf'\left(2 + \frac{\epsilon}{\omega_k}\right) + \frac{1}{\omega_k}\left(2d'f' + df''\right) - df\left(\frac{1}{2}g\omega_k a^2 + \epsilon + \frac{1}{\omega_k}\frac{a''}{a}\right), \\ B &= 2d'f + df'\left(2 + \frac{\epsilon}{\omega_k}\right) + \frac{1}{\omega_k}\left(2c'f' + cf''\right) + cf\left(\frac{1}{2}g\omega_k a^2 - \epsilon - \frac{1}{\omega_k}\frac{a''}{a}\right).\end{aligned} \tag{22}$$

For equation (21) to be satisfied, we require that both of the coefficients $(A,\ B)$ of the trigonometric terms vanish identically. This gives a pair of coupled differential equations involving the coefficients $c(\eta)$, $d(\eta)$, and $f(a)$.

For the remainder of this section we will neglect all expansion of the universe, and set $f(a) = a = 1$. Then, if we set $c(\eta) = C\exp(s\eta)$ and $d(\eta) = D\exp(s\eta)$, equations (21) and (22) yield

$$\begin{aligned}sC + \frac{1}{2}\left(\frac{1}{2}g\omega_k + \epsilon\right)D &= 0, \\ sD + \frac{1}{2}\left(\frac{1}{2}g\omega_k - \epsilon\right)C &= 0.\end{aligned} \tag{23}$$

These equations may be solved with

$$\begin{aligned}\frac{D}{C} &= \mp\frac{1}{y_k} \equiv \mp\sqrt{\frac{\frac{1}{2}g\omega_k - \epsilon}{\frac{1}{2}g\omega_k + \epsilon}}, \\ s &= \pm\frac{1}{2}\sqrt{\left(\frac{1}{2}g\omega_k\right)^2 - \epsilon^2}.\end{aligned} \tag{24}$$

Thus there exists a growing solution and a decaying solution:

$$\psi_k^{\pm}(\eta) = \frac{\exp\left[\pm s\eta\right]}{\sqrt{\gamma y_k}} \left[y_k \cos(\gamma\eta/2) \mp \sin(\gamma\eta/2)\right]. \tag{25}$$

This normalization was chosen, following [5], so that $\psi_k^{+\prime}\psi_k^- - \psi_k^{-\prime}\psi_k^+ = -1$.

From equations (24) and (25) it is clear that the solutions ψ_k will be exponentially unstable whenever s is real. Thus, the parametric resonance will occur only within a small frequency band:

$$|\epsilon| < \frac{1}{2} g\omega_k. \tag{26}$$

It will be convenient to introduce the variable ℓ as

$$\ell \equiv \frac{2\epsilon}{g\omega_k}, \tag{27}$$

so that resonance occurs when $-1 < \ell < 1$. In terms of ℓ, s and y_k may be rewritten

$$s = \frac{g\omega_k}{4}\sqrt{1-\ell^2} \ , \ \ y_k = \sqrt{\frac{1+\ell}{1-\ell}}. \tag{28}$$

Note that $y_k \to 1$ as $\ell \to 0$, near the center of the resonance band. We further introduce the function $S_{\pm}(\eta)$ as

$$S_{\pm}(\eta) \equiv y_k \cos(\gamma\eta/2) \mp \sin(\gamma\eta/2), \tag{29}$$

so that near the center of the resonance band, as $\ell \to 0$,

$$|S_{\pm}(\eta)|^2 \to 1 \mp \sin(\gamma\eta), \tag{30}$$

or, averaging over a few oscillations near $\ell \sim 0$,

$$\langle |S_{\pm}(\eta)|^2 \rangle \to 1. \tag{31}$$

Near the center of the resonance band, then, the solutions $\psi_k^{\pm}(\eta)$ become

$$\psi_k^{\pm}(\eta) \to \frac{\exp\left[\pm s\eta\right]}{\sqrt{2k}} S_{\pm}(\eta) \simeq \frac{\exp\left[\pm s\eta\right]}{\sqrt{2k}}, \tag{32}$$

where the irrelevant phase from the $S_{\pm}$ term has been dropped.

We can now make use of a Bogolyuobov transformation to solve for the number of particles per mode, $N_k^{\rm res}$, produced by the decaying inflaton field during its resonant oscillations. In section 2, we made the approximation of an instantaneous phase transition at some time η_*, which can only be appropriate for long-wavelength modes. In the present case, the resonant modes cannot have arbitrarily long wavelengths: from equations (18) and (26), we require $k \geq 2|\epsilon|/g$. Instead, use may be made of the more elaborate time-dependent Bogolyubov transformation developed in Appendix B of [8], the results of which are:

$$N_k = |\beta_k|^2 = \frac{1}{1-\ell^2} \sinh^2(s\eta). \tag{33}$$

As explained in [8], when s is not exactly constant in time, but changes adiabatically (such that $|s'| \ll s^2$), then this equation for N_k may be modified to

$$N_k \simeq \sinh^2 \left(\int_{\rm res.\ band} s\, d\eta \right), \tag{34}$$

where the integral extends over the resonance band (equation (26)). Following [8], the (divergent) coefficient $(1-\ell^2)^{-1}$ has been dropped, because near $|\ell| \simeq 1$, the exponent s no longer changes adiabatically.

Equation (34), with s given by equation (24), is the main result of this section. Clearly, if $\int s d\eta$ is large enough within the resonance band, then the number of particles produced will be exponentially greater than the number produced from the non-resonant decay studied in section 2, given by equation (15). There is an important difference, however, between our result for N_k in equation (34) and the corresponding expressions in both [7] [8]. The analysis in these papers was carried out in terms of cosmic time t and the fluctuations $\delta\phi_k$; their equations of motion thus resemble equations (1) and (2) above, with $\omega_k = k/a$, and with the assumption $\omega_k \gg H$. As studied in these papers, then, the resonant modes exit the resonance band due to the redshift of the physical wavenumbers k/a. This is to be contrasted with the foregoing analysis: In our case, by working in terms of conformal time and the "conformal field" ψ_k, the only time-dependence of the exponent s comes from the slow time-dependence of the decaying amplitude $\overline{\varphi^2}$. That is, in our case it is g which is slowly changing with time, and not ω_k; and it is this change of g which causes various modes ψ_k^+ to slide outside of the resonance band. The time-dependence of g is further addressed in the Appendix. Meanwhile, we will assume for the remainder of this paper that $\overline{\varphi^2}(\eta)$, and hence $g(\eta)$ and $s(\eta)$, may be treated as slowly-varying with time. With this assumption, we may now study what happens to the resonant modes when the expansion of the universe is included.

4 Resonant Inflaton Decay in an Expanding Background

We return to the full equation of motion for the fluctuations ψ_k, equation (17), with ω_k and g as defined in equation (18). We introduce the same trial solution, equation (19), and keep terms to $\mathcal{O}(\epsilon, g)$. This leads to equation (21), with the full coefficients $A(\eta)$ and $B(\eta)$ given by equation (22). We saw in the previous section that a consistent resonant solution may be found when $f(a)$ and a are both set equal to unity. In this section, we keep these terms explicit and general, and show that consistent resonant solutions may be found when the expansion of the universe is included.

We split the coefficients A and B into two terms each: $A = A_1 + A_2$, and $B = B_1 + B_2$, with

$$\begin{aligned}
A_1 &= -2c'f - df\left(\frac{1}{2}g\omega_k + \epsilon\right), \\
A_2 &= -cf'\left(2 + \frac{\epsilon}{\omega_k}\right) + \frac{1}{\omega_k}\left(2d'f' + df''\right) - df\left(\frac{1}{2}g\omega_k\left(a^2 - 1\right) + \frac{1}{\omega_k}\frac{a''}{a}\right), \\
B_1 &= 2d'f + cf\left(\frac{1}{2}g\omega_k - \epsilon\right), \\
B_2 &= df'\left(2 + \frac{\epsilon}{\omega_k}\right) + \frac{1}{\omega_k}\left(2c'f' + cf''\right) + cf\left(\frac{1}{2}g\omega_k\left(a^2 - 1\right) - \frac{1}{\omega_k}\frac{a''}{a}\right).
\end{aligned} \tag{35}$$

In order for equation (21) to be satisfied, we again require that the coefficients vanish: $A = B = 0$. Because we are interested in resonant behavior, we will further look for solutions when we set each of the terms A_1, A_2, B_1, and B_2 separately equal to zero. Then the coefficients A_1 and B_1 reduce to the same pair of linear differential equations as in equation (23), which may be solved with the same y_k and s as given in equation (24). The task now is to use the equations

$$\begin{aligned}
-c\left(\frac{f'}{f}\right)\left(2 + \frac{\epsilon}{\omega_k}\right) + \frac{1}{\omega_k}\left[2d'\left(\frac{f'}{f}\right) + d\left(\frac{f''}{f}\right)\right] - d\left(\frac{1}{2}g\omega_k\left(a^2 - 1\right) + \frac{1}{\omega_k}\frac{a''}{a}\right) &= 0, \\
d\left(\frac{f'}{f}\right)\left(2 + \frac{\epsilon}{\omega_k}\right) + \frac{1}{\omega_k}\left[2c'\left(\frac{f'}{f}\right) + c\left(\frac{f''}{f}\right)\right] + c\left(\frac{1}{2}g\omega_k\left(a^2 - 1\right) - \frac{1}{\omega_k}\frac{a''}{a}\right) &= 0,
\end{aligned} \tag{36}$$

to find a solution for the expansion factor, $f[a(\eta)]$.

The first step is to notice that several terms in equation (36) may be combined:

$$\left[2d'\left(\frac{f'}{f}\right) + d\left(\frac{f''}{f}\right)\right] = \frac{1}{(fd)}\frac{d}{d\eta}\left(d^2 f'\right), \tag{37}$$

and similarly for the corresponding cf-term in the second line of equation (36). Then, because the coefficients $c(\eta)$ and $d(\eta)$ are determined by equation (23), the two equations in (36) may be combined to give

$$\begin{aligned}
\frac{d}{d\eta}\left[e^{2s\eta} f'\right] &= \left[\frac{a''}{a} + \frac{1}{2}\frac{(y_k^2-1)}{(y_k^2+1)} g\omega_k^2\left(1 - a^2(\eta)\right)\right] e^{2s\eta} f(\eta) \\
&= \left[\alpha(p)\,\eta^{-2} + \frac{1}{2} g\ell\omega_k^2\left(1 - a^2(\eta)\right)\right] e^{2s\eta} f(\eta) \qquad (38)\\
&\equiv E^2(\eta) e^{2s\eta} f(\eta). \qquad (39)
\end{aligned}$$

In the second line, we have used the definitions of y_k in equation (24) and ℓ in equation (27), and have introduced the coefficient $\alpha(p)$. For a cosmic-time scale factor $a(t) \propto t^p$, or, equivalently, a conformal-time scale factor $a(\eta) = (a_o\eta)^{p/(1-p)}$, the coefficient $\alpha(p)$ is defined as:

$$\frac{a''}{a} = \frac{p(2p-1)}{(p-1)^2}\,\eta^{-2} \equiv \alpha(p)\,\eta^{-2}. \tag{40}$$

Note that $\alpha(p) \geq 0$ for $p \geq 1/2$, and that for de Sitter expansion, $\alpha(\infty) = 2$.

From equation (39), we may write the second-order differential equation for f:

$$f'' + 2sf' - E^2(\eta) f = 0. \tag{41}$$

If we now use the fact that $s \sim \mathcal{O}(g,\,\epsilon)$, and that $g \ll 1$, then we may approximate[1] $E^2(\eta) \simeq \alpha(p)\,\eta^{-2}$, and

$$f'' - \alpha(p)\,\eta^{-2} f \simeq 0. \tag{42}$$

This equation may be solved with the ansatz $f(\eta) = \beta\,\eta^{\mu_{1,\,2}}$, with

$$\mu_{1,\,2} = \frac{1}{2}\left[1 \pm \sqrt{1 + 4\alpha(p)}\right]. \tag{43}$$

[1]This approximation for $E^2(\eta)$ is further justified far from the edge of the resonance band, when $|\ell| \ll 1$. Also, we may assume, as in section 2, that the phase transition occurs near $a(\eta_*) = 1$.

The two possible solutions for μ, labeled by subscript 1 and 2, correspond to the choice of $\pm$ in equation (43); the choice of whether μ_1 or μ_2 is appropriate will be determined below. With this solution for $f(\eta)$, the resonant solutions ψ_k may be written:

$$\psi_k^{\pm}(\eta) = \frac{f(\eta)\exp[\pm s\eta]}{\sqrt{\gamma y_k}} S_{\pm}(\eta), \tag{44}$$

with $S_{\pm}(\eta)$ given by equation (29).

We may now choose the appropriate exponent $\mu_{1,\,2}$ for the function f by matching the η-dependence of this resonant solution for $\psi_k^{\pm}$ with the non-resonant solutions found in section 2. For fixed η, the (non-resonant) long-wavelength solutions behave as (see equations (10) and (12)):

$$\psi_k^{\mathrm{nr}} \propto \eta^{p/(1-p)}. \tag{45}$$

In the resonant case considered in this section, $\psi_k^{\pm} \propto f \propto \eta^{\mu_{1,\,2}}$. From the definition of $\alpha(p)$ in equation (40), the two exponents may be written

$$\mu_1 = \frac{1-2p}{1-p} \;\; , \;\; \mu_2 = \frac{p}{1-p}. \tag{46}$$

Thus the appropriate exponent for the resonant case is μ_2. Note that $f(\eta)$ has the same η-dependence as the conformal-time scale factor, $a(\eta)$. Because we require that $f[a(\eta)] = 1$ when $a(\eta) = 1$, we may fix the normalization of f as $f = a(\eta) = (a_o\eta)^{\mu_2}$. In addition, given that $\alpha(p) \geq 0$ for $p \geq 1/2$, we will write this exponent as

$$\mu_2 = -\left|\mu_2\right| = -\frac{1}{2}\left[\sqrt{1+\alpha(p)} - 1\right], \tag{47}$$

and thus $f = (a_o\eta)^{-|\mu_2|}$.

With the 'scaling' function f now determined, the full resonant solutions in an expanding background may be written:

$$\psi_k^{\pm}(\eta) = \frac{\exp\left[\pm x_{\pm}\eta\right]}{\sqrt{\gamma y_k}} S_{\pm}(\eta), \tag{48}$$

with the exponent $x_{\pm}(\eta)$ defined by:

$$x_{\pm}(\eta) \equiv \pm s - \frac{|\mu_2|}{\eta}\ln(a_o\eta). \tag{49}$$

The s-term is given by equation (24). As in the previous section, resonance occurs when $2|\epsilon| < g\omega_k$. With the fluctuations $\psi_k^\pm(\eta)$ now in the form of equation (48), we may again use the result of Appendix B of [8], and solve for the number of ϕ-particles produced per mode k:

$$N_k \simeq \sinh^2\left(\int_{\text{res. band}} x_+(\eta)\, d\eta\right). \tag{50}$$

From the form of μ_2 in equation (47), it is clear that the expansion of the universe decreases the exponential production: $\int x_+ d\eta$ over the resonance band is less than $\int s d\eta$, corresponding to the non-expanding case of section 3. Nevertheless, it is possible to find self-consistent resonant solutions for the fluctuations $\psi_k^\pm$ when the expansion of the universe is included. Unlike the preliminary conclusion in [12], the resonance need not disappear when $H \neq 0$.

5 Conclusion

We have demonstrated that the resonance effects pointed to in [5] [6] and further developed in [7] [8] [9] can indeed have dramatic consequences for post-inflationary reheating. Whereas these earlier treatments all exclude expansion of the universe as an opening approximation, it has been shown that the parametric resonance need not disappear when a more general background expansion is taken into account. Although the total number of particles produced in this expanding case, N_k in equation (50), is less than the corresponding expression when $H \simeq 0$ (equation 34), there still remain regions of parameter space for which this resonant production far outweighs the non-resonant production described in section 2. The potential trouble for the new resonant reheating scenario is therefore not expansion of the universe, but rather the decaying amplitude of the classical φ-field. This question is discussed further below, in the Appendix.

As pointed out in [7], the case of inflaton decay into further inflatons (as studied above) may be of interest for dark matter searches. Following their production, the inflatons would decouple from the rest of matter. If the inflatons were given a tiny mass (which has been neglected here), then these bosons could serve as a natural candidate for the missing dark matter. Furthermore, much of the formalism developed in this paper may be taken over, unchanged, for studying the case of inflaton decay into some distinct light species of boson, as studied in [8]. In such decays, it is again expected that the parametric resonance would survive a non-zero expansion of the universe. The difference for these decays would reside solely in the form of $\overline{\varphi^2}(\eta)$.

The foregoing analysis for the simple model of a minimally-coupled scalar field ϕ with an Einstein-Hilbert gravitational action can also be applied directly to many classes of Generalized Einstein Theory inflationary models, such as those studied in [20]. For these models, the conformal transformation factor $\Omega(x) \to 1$ as the inflaton field reaches the minimum of its potential. During the ϕ-field's oscillatory phase, the non-minimal $\phi^2 R$ coupling takes the form of the standard $(16\pi G)^{-1} R$ gravitational action of a minimally-coupled theory. Near the epoch of reheating, then, the effective lagrangian takes the form of equation (3), from which the reheating analysis may proceed as above. It would be interesting to calculate N_k for the new models of open inflation [21], following the methods developed above, which might display qualitatively different reheating scenarios. This is the subject of further study.

6 Appendix

It is important to understand how $\overline{\varphi^2}$, and hence g and s, changes with time. A proper treatment would be to extend the Minkowski-spacetime approach adopted in [9], and to work out the thermal Green's functions for the decaying inflaton field for the general background expansion considered here. (See also [22].) As a first approximation, we may posit the reasonable phenomenological ansatz, which may be compared with figure 1 in [9]:

$$\overline{\varphi^2}(\eta) \simeq \varphi_o^2 e^{-\eta/\tau}, \tag{51}$$

where τ is some damping time-scale, most likely to be determined from numerical integration. From figure 1 in [9], it is clear that we may assume that $\tau^{-1} \ll \omega_k$. If we then write $g(\eta) = g_o \exp(-\eta/\tau)$, and keep terms to first order in g_o, (ϵ/ω_k), and τ^{-1}, we may repeat the above analysis in equations (17)-(25). We will restrict attention to the non-expanding case, since the expanding case examined in section 4 reveals the same dependence of N_k on $g(\eta)$.

We introduce the trial solution

$$\psi_k(\eta) = c(\eta) e^{-\eta/\tau} \cos(\gamma\eta/2) + d(\eta) e^{-\eta/\tau} \sin(\gamma\eta/2). \tag{52}$$

Again taking c', $d' \sim \mathcal{O}(\epsilon)$ leads to the coupled differential equations:

$$\begin{aligned} -2c' + 2c\tau^{-1} - d\epsilon - \frac{1}{2} d g_o \omega_k e^{-\eta/\tau} &= 0, \\ 2d' - 2d\tau^{-1} - c\epsilon + \frac{1}{2} c g_o \omega_k e^{-\eta/\tau} &= 0. \end{aligned} \tag{53}$$

These equations may be solved by writing $c(\eta) = C \exp[\theta\eta]$ and $d(\eta) = D \exp[\theta\eta]$, with

$$\begin{aligned} \frac{D}{C} = \mp \frac{1}{y_k(\eta)} &\equiv \mp \sqrt{\frac{\frac{1}{2} g_o \omega_k \exp[-\eta/\tau] - \epsilon}{\frac{1}{2} g_o \omega_k \exp[-\eta/\tau] + \epsilon}}, \\ \theta(\eta) = \pm \frac{1}{2} \sqrt{\left(\frac{1}{2} g_o \omega_k\right)^2 e^{-2\eta/\tau} - \epsilon^2} + \frac{1}{\tau} &\equiv \pm s(\eta) + \frac{1}{\tau}. \end{aligned} \tag{54}$$

With these solutions, the growing and decaying mode functions take the form:

$$\psi_k^{\pm}(\eta) = \frac{\exp[\pm s(\eta)\,\eta]}{\sqrt{\gamma y_k(\eta)}} \left[y_k(\eta) \cos(\gamma\eta/2) \mp \sin(\gamma\eta/2) \right]. \tag{55}$$

These solutions have the benefit over the corresponding solutions in equation (25) of ending the resonance in a natural way: unlike the solutions from section 3, these mode functions do not continue to grow exponentially forever. Now the resonance band is explicitly time-dependent:

$$|\epsilon| < \frac{g_o \omega_k}{2} e^{-\eta/\tau}. \tag{56}$$

In addition, the slight time-dependence of $y_k(\eta)$ and $s(\eta)$ still allows these mode functions to be normalized as: $\psi_k^{+\prime} \psi_k^- - \psi_k^{-\prime} \psi_k^+ = -1 + \mathcal{O}\left(\tau^{-1}\right)$.

With this simple ansatz for the decaying inflaton field amplitude, the exponent $s(\eta)$ changes in time as

$$\frac{|s'|}{s^2} = \frac{\tau^{-1}}{s(\eta)} \frac{1}{[1 - \ell^2(\eta)]}. \tag{57}$$

In analogy with the situation in section 3, we have defined the variable $\ell(\eta) \equiv (2\epsilon)/(g(\eta)\omega_k)$. In order to satisfy the adiabaticity requirement for $s(\eta)$ (away from the edges of the resonance band), we must require $\tau^{-1} < g_o \omega_k$. Note that this is more stringent than the original assumption, $\tau^{-1} \ll \omega_k$. If this new constraint may be satisfied, then the number of ϕ-particles produced may be calculated from:

$$\int_{\text{res. band}} s(\eta)\, d\eta = \frac{\tau\epsilon}{2} \int_{-1}^{+1} d\ell \frac{\sqrt{1 - \ell^2}}{\ell^2}$$

$$= \frac{\tau\epsilon}{2}\left[-\frac{\sqrt{1-\ell^2}}{\ell} - \arcsin(\ell)\right]_{\ell=-1}^{\ell=+1} \tag{58}$$

$$= -\frac{\pi\tau\epsilon}{2}. \tag{59}$$

This gives $N_k \simeq \sinh^2(\int s d\eta) = \sinh^2(\pi\tau\epsilon/2)$. Given the constraints on both τ and ϵ, we find $N_k \gg 1$. Still, it is clear that a more extended treatment of the effects of back-reaction on the decaying amplitude $\overline{\varphi^2}(\eta)$ is required, and is the subject of further study.

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