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(26th July 1995)

We examine string (vortex) formation at a quench for a weakly-coupled global U(1) theory when the excitation spectrum is non-relativistic. It is so similar to vortex production in the corresponding relativistic plasma as to reinforce arguments for the similarity of vortex production in the early universe and in low-temperature many-body physics.

PACS Numbers : 2400

I. INTRODUCTION

For many years it has been argued by Kibble [1] and others [2] that the large-scale structure of the universe can be attributed to cosmic strings formed during a symmetry-breaking transition at the Grand Unification scale. Unfortunately, given the unlikely event of observing a cosmic string (vortex) directly, chains of inference are sufficiently long that it is difficult, if not impossible, to make the case compelling.

However, the formation of topological defects like vortices during symmetry-breaking phase transitions is not unique to the early universe but generic to many physical systems. In particular, recent experiments on the production of vortices in superfluid ^4He [3] and ^3He [4] have excited considerable interest. The suggestion that these, and other experiments [5], may provide an insight on the early universe have been made by the experimental groups concerned. In this they have been championed by some theoretical astroparticle physicists, most notably Zurek [6] whose most recent review article [7] was, indeed, titled *Cosmological Experiments in Superfluids and Superconductors*. Nonetheless, despite the hard thinking that has already taken place, there is no doubt that a fuller understanding of the *nonequilibrium* dynamics of vortex production is required if comparisons are to be more than superficial analogies.

What encourages us in the hope that vortex production in the early universe and the laboratory has close parallels is that, in the first instance, the most plausible production mechanism refers to neither. We recapitulate it now for the simplest theory permitting vortices, that of a complex scalar field $\phi(\mathbf{x}, t)$. The complex order parameter of the theory is $\langle\phi\rangle = \eta e^{i\alpha}$ and the theory possesses a global $O(2)$ (or $U(1)$) symmetry that we take to be spontaneously broken at a phase transition. The field ϕ could be either a complex non-relativistic order field appropriate to a superfluid or a relativistic field in the early universe. The transition is continuous in both cases but, as we shall see, the details of the transition order are largely irrelevant to our conclusions, as long as it is not strongly first-order [1].

Initially, we take the system to be in the symmetry-unbroken (disordered) phase. We have no reason to choose any particular initial field configuration, beyond the requirement that the field is distributed about $\phi = 0$ with zero mean $\eta = |\langle\phi\rangle| = 0$. The simplest assumption is that, beginning at some time $t = t_0$, the $O(2)$ symmetry of the ground-state (vacuum) is broken by a quench, a rapid change in the environment inducing an explicit time-dependence in the field parameters. Once the quench is completed the ϕ-field potential $V(\phi) = -a|\phi|^2 + b|\phi|^4$ is taken to have the symmetry-broken form $a > 0,\ b > 0$ of the familiar 'wine-bottle' bottom. The ground-state manifold (the circle S^1, labelled by the phase α of $\langle\phi\rangle$) is infinitely connected and the theory possesses *global* strings or vortices, labelled by a winding number $n \in Z$. Specific straight-string solutions to the classical field equations are well-known, both for the non-relativistic [8] and the relativistic cases [2]. The only property of these solutions that we need is that, as tubes of false vacuum, their thickness is the Compton wavelength of the massive (Higgs) excitations of the theory.

These strings cost considerable energy to produce. That they should appear at all follows from a general argument, due to Kibble [1], which goes as follows. During the transition, the complex scalar field begins to fall from the false

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[1]In that case the mechanism for the transition, bubble nucleation, would lead to very different consequences from those outlined below.

ground-state into the true ground-state, choosing a point on the ground-state manifold at each point in space, subject to the constraint that it is continuous and single-valued. For continuous transitions, for which this collapse to the true ground-state occurs by spinodal decomposition or phase separation, the resulting field configuration is expected to be one of domains within each of which the scalar field has relaxed to a constant ground-state value (i.e. constant field magnitude and phase).

If this is so, then the requirements of continuity and single valuedness will sometimes force the field to remain in the false ground-state between some of the domains. For example, the phase of the field may change by an integer multiple of 2π on going round a loop in space. This requires at least one *zero* of the field within the loop, each of which has topological stability and characterises a vortex passing through the loop. The density of strings is then closely linked to the number of effective domains and the evolution of this density is, correspondingly, linked to the nature of the domain growth. When the phase transition is complete and there is no longer sufficient thermal energy available for the field to fluctuate into the false ground-state, the topological defects are frozen into the field. Simple counting arguments suggest defect densities at this time of the order of one string passing through each correlation area. From then on, the defect density alters almost entirely by interactions of defects amongst themselves, rather than by fluctuations in the fields [9], but such calculations go beyond the scope of this paper.

In an earlier paper [10], henceforth known as I, (but see also [11]), two of us (A.G and R.R) showed how defects were produced in a very simple model of a *relativistic* $O(2)$ scalar field theory in which the long wavelength fluctuations that drove the transition were Gaussian. The fluctuations were taken to be set in train by the implementation of a quench from a high temperature, above the transition, to a low temperature. Despite its oversimplifications, its main interest is that it provides a concrete example in which the scenario outlined above is, indeed, true and quantifiable.

With low temperature many-body theory in mind, in this paper we show, in the same $O(2)$ model of Gaussian fluctuations, how a density quench of a *nonrelativistic* medium also leads to vortices. A priori, we might expect significant differences in the vortex production. The relativistic regime considered in I was characterised by initial temperatures $T \gg m$ for all particle masses m (in units $k_B = c = 1$). On the other hand, non-relativistic media are characterised by $T \ll m$, to freeze out antiparticles, but with $T \simeq \mu_{\rm nr}$, the non-relativistic chemical potential. Surprisingly, in our simple model we find no difference once a proper identification of mass parameters has been made.

As it stands our model, being weakly coupled (in effect, with zero-coupling for short times), is far too simple to mimic superfluid helium. Equally, unless we are considering extremely weak couplings of the strength invoked in inflationary models, our approximation will only have limited applicability to hot quantum fields. However, in principle, if not yet in practice, we know how to incorporate stronger interactions within our approximation scheme. The identity of the weak-coupling models should be reflected in similarities in the more realistic stronger coupling counterparts. As a result, our main conclusion about the similarity of vortex production in the two energy extremes may have greater generality, even if the detailed results given here are too limited.

The organisation of this paper is as follows. In I we considered the production of both $O(3)$ monopoles and $O(2)$ vortices. Since monopoles are somewhat easier to manage we concentrated on them at the expense of vortices, for which we quoted only such properties as were needed. In the next section we repair this omission by showing the way *vortex* densities, and density correlations, can be inferred from field configuration probabilities through field correlation functions. As in I, this summary is largely a recapitulation of some early work of Halperin [12], modified to our purposes.

One of the difficulties of making straightforward comparisons between early universe physics and many-body physics is the difference in formalism, a reflection of the freezing out of antiparticles in the non-relativistic regime. We evade this problem by showing how to incorporate both regimes in a common relativistic expression, in which the non-relativistic limit is obtained by tuning the chemical potential. This permits the relevant field correlation functions to be calculated in the same approximations as for the relativistic case.

In the final sections we use these field correlation functions as input for the vortex density functions as shown in section 2. We conclude with a discussion of some of the results, and consider their implications for numerical simulations. In particular, the Kibble mechanism as presented above says nothing about the fraction of strings that is produced in small loops against the fraction of 'infinite' string (i.e. string that does not self-intersect). Infinite string is necessary for the creation of large-scale structure in the early universe. We shall suggest that the fraction of string in small loops is substantial.

II. VORTEX DISTRIBUTIONS FROM FIELD DISTRIBUTIONS

Consider an ensemble of systems evolving from a given disordered state or, more realistically, from one of a set of disordered states whose relative probabilities are known, to an ordered state as indicated above, producing $O(2)$ vortices. If the phase change begins at time t_0 then, for $t > t_0$, it is possible in principle to calculate the probability

$p_t[\Phi]$ that the complex field $\phi(\mathbf{x},t)$ takes the value $\Phi(\mathbf{x})$ at time t. Throughout, it will be convenient to decompose Φ into real and imaginary parts as $\Phi = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2)$ (and ϕ accordingly). This is because we wish to track the field as it falls from the unstable ground-state hump at the centre of the potential to the ground-state manifold in Cartesian field space. It is trivial to reconvert the real fields into complex fields (and similarly for conjugate fields, as we need them).

The calculation of $p_t[\Phi]$ will be performed later in our simple model. For the moment, consider it given. The question is, how can we infer the string densities and the density correlations from $p_t[\Phi]$? That we can calculate them at all is a consequence of the fact, noted earlier, that the string core is a line of field zeroes. This is equally true for both relativistic and non-relativistic $O(2)$ theories [2]. The zeroes of Φ_a (a=1,2) which define the vortex positions form either closed loops or the 'infinite' string mentioned in the introduction. Following Halperin [12] we define the *topological line density* $\vec{\rho}(\mathbf{r})$ by

$$\vec{\rho}(\mathbf{r}) = \sum_n \int ds \frac{d\mathbf{R}_n}{ds} \delta^3[\mathbf{r} - \mathbf{R}_n(s)]. \tag{1}$$

In (2.1) ds is the incremental length along the line of zeroes $\mathbf{R}_n(s)$ (n=1,2,.. .) and $\frac{d\mathbf{R}_n}{ds}$ is a unit vector pointing in the direction which corresponds to positive winding number. Only winding numbers $n = \pm 1$ are considered. Higher winding numbers are understood as describing multiple zeroes. If dA_j is an incremental two-dimensional surface containing the point $\mathbf{r}$, whose normal is in the jth direction, then $\rho_j(\mathbf{r})$ is the net density of strings (i.e the density of strings *minus* the density of antistrings on dA_j).

Ensemble averaging $\langle F[\Phi]\rangle_t$ at time t is understood as averaging over the field probabilities $p_t[\Phi]$ as

$$\langle F[\Phi]\rangle_t = \int \mathcal{D}\Phi \; p_t[\Phi] \; F[\Phi]. \tag{2}$$

We stress that, in general, this ensemble averaging is not thermal averaging since, out of equilibrium, we have no Boltzmann distribution. We shall only consider situations in which

$$\langle \rho_j(\mathbf{r})\rangle_t = 0, \tag{3}$$

That is, we assume equal likelihood of a string or an antistring passing through an infinitesimal area. However, the line density correlation functions

$$C_{ij}(\mathbf{r};t) = \langle \rho_i(\mathbf{r})\rho_j(\mathbf{0})\rangle_t \tag{4}$$

will be non-zero, and give information on the persistence length of strings.

It follows that, in terms of the zeroes of $\Phi(\mathbf{r})$, $\rho_i(\mathbf{r})$ can be written as

$$\rho_i(\mathbf{r}) = \delta^2[\Phi(\mathbf{r})]\epsilon_{ijk}\partial_j\Phi_1(\mathbf{r})\partial_k\Phi_2(\mathbf{r}), \tag{5}$$

where $\delta^2[\Phi(\mathbf{r})] = \delta[\Phi_1(\mathbf{r})]\delta[\Phi_2(\mathbf{r})]$. The coefficient of the δ-function in (5) is the Jacobian of the transformation from line zeroes to field zeroes. It permits us to define a further line density that we shall also find useful, the *total line density* $\bar{\rho}(\mathbf{r})$

$$\bar{\rho}_i(\mathbf{r}) = \delta^2[\Phi(\mathbf{r})]|\epsilon_{ijk}\partial_j\Phi_1(\mathbf{r})\partial_k\Phi_2(\mathbf{r})|. \tag{6}$$

Unlike the case for $\rho_i(\mathbf{r})$

$$n(t) = \; \langle \bar{\rho}_i(\mathbf{r})\rangle_t \; > 0 \tag{7}$$

and measures the *total* string density in the direction i, without regard to string orientation. The isotropy of the initial state guarantees that $n(t)$ is independent of the direction i. We note that the Jacobian factor multiplying the field δ-functions in (5) and (6) guarantees that random field zeroes with no vorticity will not be counted.

In general, the best that we can do is write $\langle \bar{\rho}_i(\mathbf{r})\rangle_t$ as

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[2] However, this is not necessarily the case for the more sophisticated vortices of ^3He, for example, for which different methods would be required.

$$\langle \bar{\rho_i}(\mathbf{r})\rangle_t = \int \mathcal{D}\Phi\ p_t[\Phi]\ \delta^2[\Phi(\mathbf{r})]|\epsilon_{ijk}\partial_j\Phi_1(\mathbf{r})\partial_k\Phi_2(\mathbf{r})|$$
$$= \int \mathcal{D}\alpha\ \langle|\epsilon_{ijk}\partial_j\Phi_1(\mathbf{r})\partial_k\Phi_2(\mathbf{r})|\, e^{i\int d\mathbf{x}\,\alpha_a\Phi_a}\rangle_t \tag{8}$$

Our simple model assumes that $p_t[\Phi(\mathbf{r})]$ is Gaussian. Details as to why this could be will be given later, but if it is so then n and C_{ij} are easily calculable. Given that

$$\langle\Phi_a(\mathbf{r})\rangle_t = 0, \tag{9}$$

suppose that

$$\langle\Phi_a(\mathbf{r})\partial_j\Phi_b(\mathbf{r})\rangle_t = 0, \tag{10}$$

that

$$\langle\Phi_a(\mathbf{r})\Phi_b(\mathbf{r}')\rangle_t = W_{ab}(|\mathbf{r}-\mathbf{r}'|;t)$$
$$= \delta_{ab}W(|\mathbf{r}-\mathbf{r}'|;t), \tag{11}$$

and all other connected correlation functions are zero.

Then all ensemble averages are given in terms of $W(r;t)$, where $r=|\mathbf{r}|$. In particular, $\langle\bar{\rho_i}(\mathbf{r})\rangle_t$ separates as

$$\langle\bar{\rho_i}(\mathbf{r})\rangle_t = \langle\delta^2[\Phi(\mathbf{r})]\rangle_t\ \langle|\epsilon_{ijk}\partial_j\Phi_1(\mathbf{r})\partial_k\Phi_2(\mathbf{r})|\rangle_t \tag{12}$$

or, equivalently, from (8)

$$\langle\bar{\rho_i}(\mathbf{r})\rangle_t = \int \mathcal{D}\alpha\ \langle|\epsilon_{ijk}\partial_j\Phi_1(\mathbf{r})\partial_k\Phi_2(\mathbf{r})|\rangle_t\ \langle e^{i\int d\mathbf{x}\,\alpha_a\Phi_a}\rangle_t. \tag{13}$$

It follows [12], on first performing the α integration in the second factor, that

$$n(t) = \frac{1}{2\pi}\left|\frac{W''(0;t)}{W(0;t)}\right|, \tag{14}$$

where primes on W denote differentiation with respect to r. Thus

$$n(t) = O\left(\frac{1}{\xi^2}\right), \tag{15}$$

where ξ is the length at time t that sets the scale in $W(r;t)$.

On decomposing the density-density correlation functions as

$$C_{ij}(\mathbf{r};t) = A(r;t)\delta_{ij} + B(r;t)\left(\frac{r_ir_j}{r^2} - \delta_{ij}\right), \tag{16}$$

then, in the same Gaussian approximation, A and B can also be calculated in terms of W and its derivatives, with rather more difficulty, as

$$A = \frac{W'}{2\pi^2 r\Delta^2}(W''\Delta + (W')^2W) \tag{17}$$

and

$$B = \frac{(W')^2}{2\pi^2r^2\Delta} \tag{18}$$

in which

$$\Delta(r;t) = W(0;t)^2 - W(r;t)^2. \tag{19}$$

We note that both the density $n(t)$ and the correlation functions $C_{ij}(\mathbf{r};t)$ are independent of the scale of W. The reader is referred to [12] for further details.

To understand what $A(r;t)$ and $B(r;t)$ measure, suppose $\mathbf{x} = (0,0,z)$. Then C_{ij} is diagonal, with non-zero elements

$$C_{33} = A,\ C_{11} = C_{22} = A - B \tag{20}$$

For the sake of argument, consider the idealised situation in which there is exactly *one* string passing through each area $n(t)^{-1}$ in the $1-2$ plane and that these areas form a regular lattice of cell-length ξ. Normalise C_{ij}, A, B with respect to n^2, as

$$c_{ij} = \frac{C_{ij}}{n^2},\ a = \frac{A}{n^2},\ b = \frac{B}{n^2}, \tag{21}$$

whence $c_{11} = a - b$, etc.. The situation in which strings passing through adjacent faces of this lattice are oppositely oriented (i.e. string-antistring) then essentially corresponds to $c_{11}(\xi;t) = c_{22}(\xi;t) = -1$, a case of maximum anti-correlation. On the other hand, if there is equal likelihood of the next face containing a string or an antistring then $c_{11}(\xi;t) = c_{22}(\xi;t) = 0$. As for $C_{33}/n = c_{33}$, this takes the value $c_{33}(\xi;t) = 1$ if the string is guaranteed to continue in the same 3-direction through the next lattice cell, and takes the value $c_{33}(\xi;t) = -1$ if it changes direction at intervals ξ. B itself can be isolated by observing that, if $\mathbf{x} = \frac{1}{\sqrt{2}}(r,r,0)$, then $C_{12} = \frac{1}{2}B$. Thus a value $b(\xi;t) = 1$ is also a guarantee that the string changes direction every ξ. As we shall see, the situation is more complicated, even in our simple model. Nonetheless, we shall interpret strong *anticorrelation* in the diagonal c_{ii} (and positive b) on scales ξ as signifying a persistence length (the typical length along the string before it has completely changed direction) of ξ.

Nothing that we have said so far discriminates between vortices in a relativistic or a non-relativistic medium. This distinction appears in the definition of $\langle ... \rangle_t$, to which we now turn.

III. FIELD DYNAMICS WITH A CHEMICAL POTENTIAL

In order to compare vortex formation in relativistic and non-relativistic media it is convenient to develop a single formalism that encompasses both. For simplicity we assume in each case that the system is initially in *equilibrium* in the disordered phase and that the transition to the ordered phase occurs as the result of a 'quench', a rapid change in the environment of the system. The interpolation between relativistic and non-relativistic regimes is then effected by introducing the chemical potential μ, coupled to the conserved charge Q arising from the $O(2)$ symmetry. Provided μ is small in comparison to a particle mass the introduction of such a potential will have little effect on a phase transition for the relativistic theory initiated by quenching the system from a high temperature T_0 to, effectively, zero temperature. However, on increasing μ prior to the transition it becomes more costly to produce antiparticles and, if the initial temperature T_0 is decreased to a value much less than μ, antiparticles are frozen out. The system is then one of non-relativistic particles at a temperature much less than the particle rest masses. In this nonrelativistic regime the transition is induced by a quench in μ itself or equivalently, since μ determines the density, by a density (pressure) quench.

The situation is summarised in Fig.1, which shows the (equilibrium) phase structure of the global $O(2)$ theory.

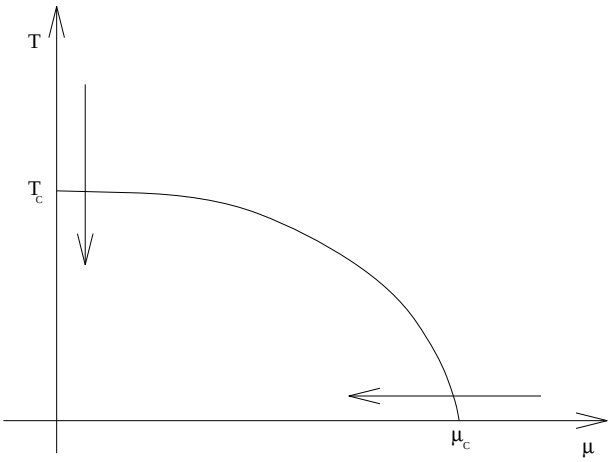

FIG. 1. Equilibrium phase structure of the global $O(2)$ theory. The arrows show the directions in which the quenches are performed.

The inner sector in the $T-\mu$ quadrant is the ordered phase, the outer region the disordered phase, and the line separating them the phase boundary. T_c is the critical temperature at zero chemical potential and μ_c is the critical chemical potential at zero temperature. The relativistic quench discussed in I corresponds to a transition at zero (small) μ across the phase boundary at T_c. The non-relativistic quench with which we shall be comparing it corresponds to a transition at small T across the phase boundary near μ_c.

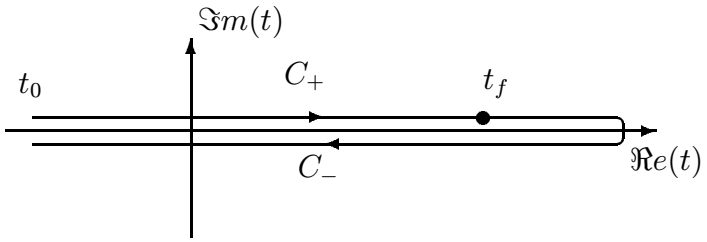

FIG. 2. The closed timepath contour $C_+ \oplus C_-$.

From this viewpoint the chemical potential is seen as determining the initial conditions for the subsequent dynamics, for which we adopt the closed time path method (Schwinger-Keldysh formalism) [13–16], generalising the analysis begun in I for the relativistic theory. As a starting point suppose that, at the initial time t_0, we are in a disordered state with $\langle\phi\rangle = 0$. Our ignorance is parametrised by the probability distribution $p_{t_0}[\Phi]$ that, at time t_0, $\phi(t_0, \mathbf{x}) = \Phi(\mathbf{x})$. For the moment we take it as given. Whether we are in a relativistic or non-relativistic regime is largely encoded in $p_{t_0}[\Phi]$. The subsequent, essentially generic, non-equilibrium field evolution is driven by a change in the environment. Specifically, for $t > t_0$ the action for the field is taken to be

$$S[\phi] = \int d^4x \left(\frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a - \frac{1}{2}m^2(t)\phi_a^2 - \frac{1}{4}\lambda(t)(\phi_a^2)^2 \right). \tag{22}$$

where $m(t)$, $\lambda(t)$ describe the evolution of the parameters of the theory under external influences, to which the field responds. As with Φ, it is convenient to decompose ϕ in terms of two massive real scalar fields ϕ_a, $a = 1, 2$ as $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, in terms of which $S[\phi]$ shows a global $O(2)$ invariance, broken by the mass term if $m^2(t)$ is negative.

The change of phase that begins at time t_0 will, by the mechanism indicated earlier, lead to the appearence of vortices. We saw in the previous section that the vortex distributions at later times $t_f > t_0$ can be read off from the probability $p_{t_f}[\Phi_f]$ that the measurement of ϕ will give the value Φ_f. The evolution of $p_t[\Phi]$ from t_0 to t_f is most simply written as a closed time-path integral in which the field ϕ is integrated along the closed path $C_+ \oplus C_-$ of Fig.2, where $\phi = \phi_+$ on C_+ and $\phi = \phi_-$ on C_-.

If $\mathcal{D}\phi_\pm = \prod_{a=1}^2 \mathcal{D}\phi_{\pm,a}$ and spatial labels are suppressed then

$$p_{t_f}[\Phi_f] = \int \mathcal{D}\Phi\, p_{t_0}[\Phi] \int_{\phi_\pm(t_0)=\Phi} \mathcal{D}\phi_+\mathcal{D}\phi_-\, \delta[\phi_+(t_f) - \Phi_f]\, \exp\left\{ i\Big(S[\phi_+] - S[\phi_-] \Big) \right\}. \tag{23}$$

where $\delta[\phi_+(t) - \Phi_f]$ is a delta functional, imposing the constraint $\phi_+(\mathbf{x}, t) = \Phi_f(\mathbf{x})$ for each $\mathbf{x}$. This is no more than the statement that, for a given initial state, the probability amplitude is given by the integration along C_+, and its complex conjugate (which, when multiplied with it, gives the probability) is given by the integration back along C_-. The $\pm$ two-field notation is misleading in that it suggests that the ϕ_+ $(= \phi_{a,+})$ and ϕ_- fields are decoupled. That this is not so follows immediately from the fact that $\phi_+(t_f) = \phi_-(t_f)$.

To return to the initial conditions, we said that we could only achieve the simple analytic results of the previous section if $p_t[\Phi]$ were Gaussian and, therefore, that $p_{t_0}[\Phi]$ itself be Gaussian. The simplest such distribution has Φ Boltzmann distributed at time t_0 at a temperature of $T_0 = \beta_0^{-1}$ and chemical potential μ according to a *free*-field Hamiltonian H_0, which we take as

$$H_0 = \int d^3x \left[\frac{1}{2}\pi_a^2 + \frac{1}{2}(\nabla\phi_a)^2 + \frac{1}{2}m^2\phi_a^2 \right]. \tag{24}$$

where $\pi_a = \dot{\phi}_a$.

We shall motivate the use of this free-field Hamiltonian in specifying the initial field distribution later, when we implement our Gaussian approximations. However, since our comments on extracting the non-relativistic limit from

a relativistic field theory have greater applicability than our particular model we would like to be more general and extend H_0 to include interactions as

$$H_0 = \int d^3x \left[\frac{1}{2}\pi_a^2 + \frac{1}{2}(\nabla\phi_a)^2 + \frac{1}{2}m^2\phi_a^2 + \frac{1}{4}\lambda(\phi_a^2)^2\right], \tag{25}$$

without enforcing a Gaussian straightjacket. We stress that Gaussian does not necessarily mean *free*. Most simply, particles need to interact before they can equilibriate. More importantly, the free-field Gaussian approximation adopted in I and here can be extended to include interactions self-consistently in a Hartree approximation [22]. This will be considered elsewhere.

Since chemical potentials are not usually relevant to relativistic bosons a few words are in order. The $O(2)$ invariance of H_0 leads to a conserved Noether current with conserved charge

$$\begin{aligned} Q &= \int d^3x\ (\phi_2\pi_1 - \pi_2\phi_1) \\ &= \int d^3x\ (\phi_2\dot{\phi_1} - \dot{\phi_2}\phi_1). \end{aligned} \tag{26}$$

(in Minkowski space). The numerical value of Q is the number of particles minus the number of antiparticles [3]. The thermal probability distribution $p_{t_0}[\Phi]$ is thus taken to be

$$p_{t_0}[\Phi] = \langle\Phi, t_0|e^{-\beta_0(H_0-\mu Q)}|\Phi, t_0\rangle. \tag{27}$$

There are two ways to proceed. The first, which to us is the most natural, accepts H_0 as determining the temporal evolution of the physical fields, and relegates μ to the boundary conditions on the fields. From this viewpoint, $p_{t_0}[\Phi]$ can be written as the imaginary-time path integral

$$p_{t_0}[\Phi] = \int_{B_\mu[\Phi]} \mathcal{D}\phi \exp\Big\{iS_0[\phi]\Big\}, \tag{28}$$

for a corresponding action

$$S_0[\phi] = \int d^4x \left[\frac{1}{2}(\partial_\nu\phi_a)(\partial^\nu\phi_a) - \frac{1}{2}m^2\phi_a^2 - \frac{1}{4}\lambda(\phi_a^2)^2\right]. \tag{29}$$

The label *zero* on $S_0[\phi]$ (and H_0 previously) is a reminder that all are defined at time t_0 (and not that the theory is free). The boundary condition $B_\mu[\Phi]$ incorporates the chemical potential. In terms of the eigenstates of Q, the complex field $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ and its adjoint, B_μ becomes

$$B_\mu[\Phi] :\ \phi(t_0) = \Phi = e^{-\beta_0\mu q}\phi(t_0 - i\beta_0), \tag{30}$$

where $q = 1$ is the ϕ-field eigenvalue of Q and the time-integral in (29) is taken in imaginary time from t_0 to $t_0 - i\beta_0$. For ϕ^*, $q = -1$. In terms of the same complex fields we have

$$S_0[\phi] = \int d^3x[(\partial_\nu\phi^*)(\partial^\nu\phi) - m^2\phi^*\phi - \lambda(\phi^*\phi)^2] \tag{31}$$

and, with $\pi = (\pi_1 - i\pi_2)/\sqrt{2}$,

$$H_0 = \int d^3x[\pi^*\pi + (\nabla\phi^*)(\nabla\phi) + m^2\phi^*\phi + \lambda(\phi^*\phi)^2] \tag{32}$$

However, for calculational purposes a decomposition in terms of ϕ_a, π_a is usually preferable.

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[3]For a relativistic theory the natural next step would be to gauge the $O(2)$ or $U(1)$ symmetry by the introduction of the electromagnetic field, whereby Q becomes proportional to the electric charge of the system. With non-relativistic media in mind, our constituents are neutral (e.g. He atoms) and we shall not do this.

Although we are just setting an initial condition the effect is, inevitably, to give an action $S_0[\phi]$ of the form of $S[\phi]$ of (22). This permits the interpretation that the action $S[\phi]$ is valid for all times t, with the proviso that the system is in thermal equilibrium for $t < t_0$, during which period the mass $m(t)$ takes the constant value m and $\lambda(t) = \lambda$, also constant.

On relabelling the integration variable ϕ of (28) by ϕ_3, we now have the explicit form for $p_{t_f}[\Phi_f]$:-

$$\begin{aligned} p_{t_f}[\Phi_f] &= \int \mathcal{D}\Phi \int_{B_\mu[\Phi]} \mathcal{D}\phi_3\, e^{iS_0[\phi_3]} \int_{\phi_\pm(t_0)=\Phi} \mathcal{D}\phi_+\Phi\mathcal{D}\phi_-\, e^{i(S[\phi_+]-S[\phi_-])}\delta[\phi_+(t_f)-\Phi_f] \\ &= \int_{B_\mu} \mathcal{D}\phi_3\mathcal{D}\phi_+\mathcal{D}\phi_-\, \exp\Big\{iS_0[\phi_3]+i(S[\phi_+]-S[\phi_-])\Big\}\,\delta[\phi_+(t_f)-\Phi_f], \end{aligned}$$

where the boundary condition B_μ is now (in terms of the field combinations $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$)

$$B_\mu :\ \phi_+(t_0) = e^{-\beta_0\mu q}\phi_3(t_0 - i\beta_0). \tag{33}$$

More succinctly, $p_{t_f}[\Phi_f]$ can be written as the time ordering of a single field doublet:-

$$p_{t_f}[\Phi_f] = \int_{B_\mu} \mathcal{D}\phi\, e^{iS_C[\phi]}\,\delta[\phi_+(t_f)-\Phi_f], \tag{34}$$

along the contour $C = C_+ \oplus C_- \oplus C_3$ of Fig.3, extended to include a third imaginary leg, where ϕ takes the values ϕ_+, ϕ_- and ϕ_3 on C_+, C_- and C_3 respectively, for which S_C is $S[\phi_+]$, $S[\phi_-]$ and $S[\phi_3]$, for which last case $m(t) = m, \lambda(t) = \lambda$, $t \in C_3$.

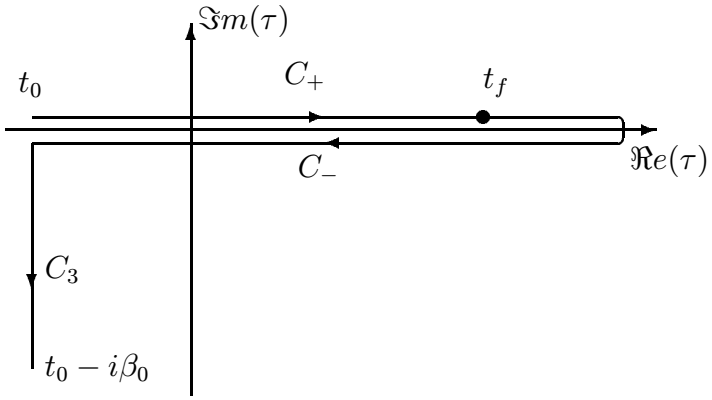

FIG. 3. A third imaginary leg

As a final step in these formal manipulations we see that expression (34) enables us to write the Φ-field ensemble averages $\langle ...\rangle_t$ in terms of the ϕ-field thermal Wightman functions. Consider the generating functional:-

$$Z_\mu[j_+, j_-, j_3] = \int_{B_\mu} \mathcal{D}\phi\, \exp\Big\{iS_C[\phi] + i\int j\phi\Big\}, \tag{35}$$

where $\int j\phi$ is a short notation for:-

$$\int j\phi \equiv \int_0^\infty dt\, [\, j_+(t)\phi_+(t) - j_-\phi_-(t)\,] + \int_0^{-i\beta} j_3(t)\phi_3(t)\, dt, \tag{36}$$

omitting spatial arguments. Then introducing $\alpha_a(\mathbf{x})$ where $a = 1, 2$, we find:-

$$\begin{aligned} p_{t_f}[\Phi] &= \int \mathcal{D}\alpha \int_{B_\mu} \mathcal{D}\phi\ \exp\Big\{iS_C[\phi]\Big\}\ \exp\Big\{i\int d^3x\alpha_a(\mathbf{x})[\phi_+(t_f,\mathbf{x}) - \Phi(\mathbf{x})]_a\Big\} \\ &= \int \mathcal{D}\alpha\ \exp\Big\{-i\int \alpha_a\Phi_a\Big\}\, Z_\mu[\overline{\alpha}, 0, 0], \end{aligned} \tag{37}$$

where $\overline{\alpha}$ is the source $\overline{\alpha}(\mathbf{x},t)=\alpha(\mathbf{x})\delta(t-t_f)$. As with $\mathcal{D}\phi$, $\mathcal{D}\alpha$ denotes $\prod_1^N \mathcal{D}\alpha_a$.

Ensemble averages are now expressible in terms of Z_μ. Of particular relevance, $W_{ab}(|\mathbf{r}-\mathbf{r}'|;t)=\langle\Phi_a(\mathbf{r})\Phi_b(\mathbf{r}')\rangle_t$ is given by

$$\begin{aligned} W_{ab}(|\mathbf{r}-\mathbf{r}'|;t) &= \int \mathcal{D}\Phi\ \Phi_a(\mathbf{r})\Phi_b(\mathbf{r}')\int \mathcal{D}\alpha\ \exp\Big\{-i\int \alpha_a\Phi_a\Big\}\ Z_\mu[\overline{\alpha},0,0] \\ &= -\int \mathcal{D}\alpha\ \frac{\delta^2}{\delta\alpha_a(\mathbf{r})\delta\alpha_b(\mathbf{r}')}\int \mathcal{D}\Phi\ \exp\Big\{-i\int \alpha_a\Phi_a\Big\}\ Z_\mu[\overline{\alpha},0,0] \\ &= -\int \mathcal{D}\alpha\ \frac{\delta^2}{\delta\alpha_a(\mathbf{r})\delta\alpha_b(\mathbf{r}')}\Big\{\delta^2[\alpha]\ Z_\mu[\overline{\alpha},0,0]\Big\} \end{aligned} \tag{38}$$

On integrating by parts

$$\begin{aligned} W_{ab}(|\mathbf{r}-\mathbf{r}'|;t) &= -\frac{\delta^2}{\delta\alpha_a(\mathbf{r})\delta\alpha_b(\mathbf{r}')}Z_\mu[\overline{\alpha},0,0]\Big|_{\alpha=0} \\ &= \langle\phi_a(\mathbf{r},t)\phi_b(\mathbf{r}',t)\rangle, \end{aligned} \tag{39}$$

the equal-time thermal Wightman function with the given thermal boundary conditions. Because of the time evolution there is no time translation invariance in the double time label.

Not surprisingly, $p_t[\Phi]$ can only be calculated explicitly in very simple circumstances, like our Gaussian approximation, but before we do that we still have to extract the relativistic limit. Since the chemical potential is embedded in the equilibrium boundary conditions and, like temperature, can only be defined in equilibrium, this is essentially an equilibrium problem and, for the moment,we forget the dynamics.

IV. MANIPULATING THE CHEMICAL POTENTIAL

The expression (34) is valid for all μ and all T_0 but, as it stands, is not sympathetic to the isolation of a non-relativistic limit. Nonetheless, the extraction of a non-relativistic regime from it is not difficult. Instead of working with $p_{t_0}[\Phi]$ directly, we can work with the partition function Z_μ, as we saw in the previous section. The partition function for this equilibrium theory is (for doublet sources j_a restricted to the C_3-contour) $Z_\mu[0,0,j_3]$ of (35), written

$$Z_\mu[j]\equiv Z_\mu[0,0,j]=\int_{B_\mu}\mathcal{D}\phi\ \exp\Big\{iS_0[\phi]+i\int j_a\phi_a\Big\}, \tag{40}$$

where $S_0[\phi]$ is given in (29) and we integrate only along the contour C_3. As before, $\mathcal{D}\phi\equiv\mathcal{D}\phi_1\mathcal{D}\phi_2$. On rotating to Euclidean time $\tau=it$, $Z_\mu[j]$ becomes

$$Z_\mu[j]=\langle\exp\Big\{\int j_a\phi_a\Big\}\rangle=\int_{B_\mu}\mathcal{D}\phi\ \exp\Big\{-S_{0,E}[\phi]-\int j_a\phi_a\Big\}, \tag{41}$$

where $S_{0,E}[\phi]$ is the relativistic Euclidean action

$$S_{0,E}=\int_0^{\beta_0}d\tau\int d^3x\Big[\frac{1}{2}\dot{\phi}_a^2+\frac{1}{2}(\nabla\phi_a)^2+\frac{1}{2}m^2\phi_a^2-\frac{1}{4}\lambda(\phi_a^2)^2\Big]. \tag{42}$$

and the sum is taken over fields $\phi=(\phi_1+i\phi_2)/\sqrt{2}$ satisfying the boundary condition B_μ: $\phi(\mathbf{x},\tau)=e^{-\beta_0\mu}\phi(\mathbf{x},\tau-\beta_0)$ in imaginary time. The dot now means differentiation with respect to τ, $\dot{\phi_a}=\partial_\tau\phi_a$.

The bracket $\langle ...\rangle$ here denotes the thermal average

$$\langle F[\phi]\rangle=Tr\{e^{-\beta_0(H_0-\mu Q)}\ F[\phi]\}. \tag{43}$$

where H_0 is given in (25). Because of its time-independence we have dropped the t-suffix. The net charge $\bar{Q}$ (excess of particles over antiparticles) in the system is obtained by differentiating $Z_\mu[j]=\langle e^{\int j\phi}\rangle$ as

$$\bar{Q}=\langle Q\rangle=\frac{1}{\beta_0}\frac{\partial}{\partial\mu}\ \ln Z_\mu[j]\Big|_{j=0}. \tag{44}$$

This determines μ in terms of the chosen $\bar{Q}$ and T_0. In a non-relativistic environment in which antiparticles are not present, $\bar{Q}$ becomes the mean particle number (and hence is proportional to the density).

$Z_\mu[j]$ suffers from the presence of the chemical potential in the boundary conditions, which makes it difficult to identify the phase of the system readily. This is clarified by adopting a second approach to chemical potentials, in which we transfer the chemical potential term μQ to the Hamiltonian, to create an effective Hamiltonian $H_0 - \mu Q$. $H_0 - \mu Q$ no longer generates time translations of the physical fields ϕ_a or, equivalently, $\phi(t)$ and $\phi^*(t)$. However, it does generate time translations of the effective fields $\tilde{\phi}(t) = e^{i\mu q t}\phi(t) = e^{i\mu t}\phi(t)$ and $\tilde{\phi}^*(t)$, with $q = -1$. It follows that $\tilde{\phi}(t)$ and $\tilde{\phi}^*(t)$ integrations, and hence $\tilde{\phi}_a$ integrations, are now taken over *periodic* configurations with boundary conditions $B_0 : \tilde{\phi}_a(\mathbf{x}, \tau) = \tilde{\phi}_a(\mathbf{x}, \tau - \beta_0)$ in imaginary time, with no μ-dependence Details are given in the Appendix.

The advantages of removing μ from the boundary conditions become apparent when we express Z_μ of (40) not as a sum over ϕ-field histories but as a sum over histories in the $\tilde{\phi}$-fields. Although not usually posed this way, the details are well-understood. For example, see Kapusta [23], Haber & Weldon [24], and Bernstein et al. [25]. On direct substitution, $Z_\mu[j]$ takes the form

$$Z_\mu[j] = \int_{B_0} \mathcal{D}\tilde{\phi}\, \exp\left\{-S_0[\tilde{\phi}; \mu] - \int j_a \tilde{\phi}_a\right\}, \tag{45}$$

where $S_0[\phi; \mu]$ is now the effective action

$$S_0[\phi; \mu] = \int_0^{\beta_0} d\tau \int d^3x \left[\frac{1}{2}\dot{\phi}_a^2 + \frac{1}{2}(\nabla\phi_a)^2 + \frac{1}{2}m_0^2\phi_a^2 + i\mu(\phi_2\dot{\phi}_1 - \dot{\phi}_2\phi_1) - \frac{1}{4}\lambda(\phi_a^2)^2\right]. \tag{46}$$

where

$$m_0^2 = m^2 - \mu^2 \tag{47}$$

and, as above, the sum is taken over periodic field configurations B_0, rather than the boundary condition B_μ. Details are given in Kapusta [23] and [25]. We note that ϕ_a and $\tilde{\phi}_a$ have the same zeroes and hence are equally good for the calculation of vortices. It is a separate exercise to rewrite $Z_\mu[j]$ in terms of the conventional actions for a non-relativistic order field $\psi(t)$. Details will be given elsewhere [17] but, in the interim, the basic idea is given in recent conference proceedings [18,19] by one of us (R.J.R).

This displacement of μ from the boundary conditions to the action has enabled us to replace the classical potential by an effective potential in which the state of the system is more transparent. The relevant quantity is m_0^2, rather than m^2 and μ^2 separately [4]. Semiclassically, when $m_0^2 > 0$ (i.e. $\mu^2 < m^2$) the $O(2)$ symmetry is unbroken. This describes our disordered initial state. However, once $m_0^2 < 0$ (i.e. $\mu^2 > m^2$) the free theory is unstable. This is a signal that the $O(2)$ symmetry is broken and a transition to an ordered phase has occurred. Thus, effectively, it is m_0^2, rather than m^2 that carries the time dependence, changing from positive to negative at $t = t_0$.

Let us first consider the relativistic regime in which the initial environment is very hot, with $T \gg m \gg \mu$. In this case, with μ irrelevant, the symmetry is broken by a change in m^2. Suppose that, on the completion of the transition, $m_0^2 \simeq m^2$ takes the value $m_0^2 = -M^2 < 0$. If, in this relativistic case, the final temperature is very low, there are then no thermal effects and M is a physical parameter, determining the Higgs mass, $m_H = \sqrt{2}M$. In order to recover the phase boundary in Fig.1 in this case it is necessary to include thermal radiative effects. As it stands $m_0(t \leq 0) = m_0$ is not a physical parameter, but there is no loss in taking it as the effective scalar field mass at temperature $T_0 = \beta_0^{-1}$. That is, in the mean-field approximation we take

$$m_0^2 = -M^2\left(1 - \frac{T_0^2}{T_c^2}\right) \tag{48}$$

where T_0 is greater than the transition temperature T_c, given by $T_c^2 = 3M^2/\lambda$ in the same approximation. It might be objected that this is inconsistent in that this mass is defined in terms of fluctuations at scales much larger than the typical domain size. Since results will turn out to be largely independent of m_0 provided it is comparable to M this is not really a problem. With the parametrisation above this would not be true only if we quenched from very close to the transition, and we do not consider this possibility yet.

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[4] We shall see later that, in our approximations, the other term depending on μ, $i\mu(\phi_2\dot{\phi}_1 - \dot{\phi}_2\phi_1)$, plays no role.

On the other hand, for the non-relativistic theory, for which $T/m \ll 1$, the parameter m can always be identified with the boson mass. We change the sign of m_0^2 in the action by a change in μ from $\mu^2 < m^2$ to $\mu^2 > m^2$, a density quench. Equivalently, in terms of the non-relativistic chemical potential

$$\mu_{nr} = \mu - m \tag{49}$$

we go from negative μ_{nr} to positive μ_{nr}. To preserve uniformity of notation, we take $m_0^2(t) = m_0^2$ when $t \leq t_0$ and $m_0^2(t) = -M^2 < 0$ once the transition is complete.

V. VORTEX FORMATION

Having established the role of the chemical potential in letting us interpolate between relativistic and non-relativistic regimes in the initial conditions, we are now in a position to determine the effect of these initial conditions on vortex production in a simple model of Gaussian field fluctuations.

We begin by recapitulating the methods adopted in *I* for a relativistic regime, for which $\mu = 0$. We have already assumed that the initial conditions correspond to a disordered state in equilibrium for $t < t_0$. Specifically, in *I* we adopted an initial Gaussian distribution of fields arising from the *quadratic* Hamiltonian H_0 of (24). Although quadratic, the choice of $m_0 > 0$ of (48) has thermal interactions encoded within it. We assume that interactions are sufficiently weak that, having established equilibrium, their effect on the Gaussian distribution is small.

For $t > t_0$, the system is forced to change. The simplest assumption, made in *I*, was that, for $t > t_0$, $m^2(t)$ and $\lambda(t)$ could be taken as *constant*, an idealised quench in which $m^2(t)$ changes sign to take the *negative* value $m^2(t) = -M^2 < 0$ *immediately*. That is, the potential at the origin has been instantaneously inverted everywhere, breaking the global $O(2)$ symmetry. Given the initial high temperature T_0, this can be thought of as a temperature quench. If $\lambda(t) = \lambda$ is very *weak* then, for times $Mt < \ln(1/\lambda)$, the ϕ-field, falling down the hill away from the metastable vacuum, will not yet have experienced the upturn of the potential, before the point of inflection at $\langle|\phi|\rangle = \sqrt{2M^2/3\lambda}$. Thus, for these small times, $\lambda(t)$ can also be set to zero, and $p_t[\Phi]$ is Gaussian, as required.

The onset of the phase transition at time $t = t_0$ is characterised by the instabilities of long wavelength fluctuations permitting the growth of correlations. Although the initial value of $\langle\phi\rangle$ over any volume is zero, we anticipate that the resulting phase separation or spinodal decomposition will lead to domains of constant $\langle\phi\rangle$ phase, whose boundaries will trap vortices.

We now turn to the case of non-relativistic vortex formation, cast in as similar a way as possible. The situation we have in mind is the following. Consider a gas of non-relativistic bosons of mass m in thermal equilibrium at temperature $T_0 \ll m$. The state is disordered.

By changing the density we then force $\mu_{\rm nr}$ to change sign, from $\mu_0 < 0$ to $\mu_{\rm nr} = \mu_f > 0$, as rapidly as possible. Yet again, in this idealised model we suppose that this change is implemented instantaneously everywhere at time $t = t_0$. This sets in motion a change to a superfluid phase, in which the $O(2)$ particle symmetry is broken by the condensate. Order is again established by the growth of long wavelength fluctuations in which domains form. These domains will trap vortices on their boundaries.

So as to give a solvable Gaussian theory, for which we can use the results of Section 2, we maintain our Procrustean approach of *I* and ignore interactions before and after the change in chemical potential. Our initial condition is now equivalent to the statement that, for $t \leq t_0$, the particles are distributed according to the Boltzmann distribution

$$n(\epsilon) = \frac{1}{e^{\beta_0(\epsilon-\mu_0)} - 1}, \tag{50}$$

where $\epsilon = k^2/2m$ and $\mu_0 < 0$ is the non-relativistic chemical potential $\mu - m$.

In the previous section we showed how the model is that of the relativistic theory of the field ϕ with effective action $S_0[\phi;\mu]$ of (46), in which $|\mu_{nr}| \ll m$. In the notation of that section, when $\mu \simeq m$, m_0^2 changes from

$$m_0^2(t) = m_0^2 = -2m\mu_0 \ > 0, \tag{51}$$

when $t \leq t_0$, to

$$m_0^2(t) = -M^2 = -2m\mu_f \ < 0, \tag{52}$$

when $t > t_0$. Henceforth, we take $t_0 = 0$.

Of course, instantaneous change is physically impossible. Consider small amplitude fluctuations of ϕ_a, at the top of the parabolic potential hill. Long wavelength fluctuations, for which $|\mathbf{k}|^2 < M^2$, begin to grow exponentially. If

their growth rate $\Omega(k) = \sqrt{M^2 - |\mathbf{k}|^2}$ is much slower than the rate of change of the environment which is causing the quench, then those long wavelength modes are unable to track the quench. It will turn out that the time-scale at which domains appear in this instantaneous quench is $t_d = O(M^{-1})$. As long as the time taken to implement the quench is comparable to t_d and much less than $t_f = O(M^{-1} ln(1/\lambda)$ the approximation is relevant.

We note that, in the non-relativistic regime, $\Omega(k)$ has the same definition, but can be rewritten as

$$\begin{aligned}\Omega^2(k) &= M^2 - |\mathbf{k}|^2 = 2m\left(\mu_f - \frac{k^2}{2m}\right) \\ &= 2m(\mu_f - \epsilon(k)).\end{aligned} \tag{53}$$

Thus the momentum restriction $|\mathbf{k}| < M$ is just $\epsilon(k) < \mu_f$.

We are now in a position to evaluate $p_t[\Phi]$, or rather $W_{ab}(r;t)$, for $t > 0$, and calculate the defect density accordingly. Details are given in the Appendix. Before we quote the result we note that the $i\mu(\phi_2\dot{\phi}_1 - \dot{\phi}_2\phi_1)$ term in $S_0[\phi;\mu]$ of (46) couples the $a = 1$ and $a = 2$ fields ϕ_a together and, in general, $G_{ab}(\mathbf{r} - \mathbf{r}';t,t') = \langle\phi_a(\mathbf{r},t)\phi_a(\mathbf{r}',t')\rangle$ is *not* diagonal in the $O(2)$ labels. However, for *equal* times diagonal behaviour is restored as $G_{ab}(\mathbf{r};t,t) = \delta_{ab}G(\mathbf{r};t,t)$ and the $i\mu(\phi_2\dot{\phi}_1 - \dot{\phi}_2\phi_1)$ term can effectively be discarded. This leaves us in a situation very like the original relativistic case of I and Section 3 for which the results of Halperin are directly applicable. That is, W_{ab} is diagonal,

$$W_{ab}(|\mathbf{r} - \mathbf{r}'|;t,t) = \delta_{ab}W(|\mathbf{r} - \mathbf{r}'|;t,t), \tag{54}$$

whence $W(|\mathbf{r} - \mathbf{r}'|;t) = \langle\phi_a(\mathbf{r},t)\phi_a(\mathbf{r}',t)\rangle$ (no summation), the thermal Wightman function for either ϕ_1 or ϕ_2.

For all the utility of bringing the effective potential into the action from the boundary conditions, $W(r;t)$ is still most easily built from the modes $\mathcal{U}^{\pm}_{a,k}$, satisfying the equations of motion

$$\left[\frac{d^2}{dt^2} + \mathbf{k}^2 + m^2(t)\right]\mathcal{U}^{\pm}_{a,k} = 0, \tag{55}$$

for $m^2(t)$ above, subject to the initial condition of a thermal distribution with chemical potential μ See the Appendix for greater detail.

But for the chemical potential, this situation of inverted harmonic oscillators was studied many years ago by Guth and Pi [20] and Weinberg and Wu [21]. In the context of domain formation, we refer to the recent work of Boyanovsky et al., [22]. For the case in hand, if we make a separation into the unstable long wavelength modes, for which $|\mathbf{k}| < M$, and the short wavelength modes $|\mathbf{k}| > M$, then $W(r;t)$ is the real quantity

$$\begin{aligned}W(r;t) = &\int_{|\mathbf{k}|<M} d\!\!\!^{-3}k\, e^{i\mathbf{k}.\mathbf{x}}C(k;\mu)\Big[1 + A(k)(\cosh(2\Omega(k)t) - 1)\Big] \\ &+ \int_{|\mathbf{k}|>M} d\!\!\!^{-3}k\, e^{i\mathbf{k}.\mathbf{x}}C(k;\mu)\Big[1 + a(k)(\cos(2w(k)t) - 1)\Big]\end{aligned} \tag{56}$$

with $r = |\mathbf{x}|$ and

$$\begin{aligned}\Omega^2(k) &= M^2 - |\mathbf{k}|^2 \\ w^2(k) &= -M^2 + |\mathbf{k}|^2 \\ A(k) &= \frac{1}{2}\left(1 + \frac{\omega^2(k)}{\Omega^2(k)}\right) \\ a(k) &= \frac{1}{2}\left(1 - \frac{\omega^2(k)}{w^2(k)}\right)\end{aligned} \tag{57}$$

All the μ-dependence is contained in the factor

$$\mathcal{C}(k,\mu_0) = \frac{1}{2\omega(k)}\left[\frac{e^{-\beta_0\omega} - e^{\beta_0\omega}}{e^{\beta_0(m+\mu_0)} - e^{-\beta_0\omega} - e^{\beta_0\omega} + e^{-\beta_0(m+\mu_0)}}.\right] \tag{58}$$

where

$$\omega^2(k) = |\mathbf{k}|^2 + m_0^2 \tag{59}$$

In the zero chemical potential limit $\mu = m_0 + \mu_0 = 0$, $\mathcal{C}(k,\mu_0)$ takes the familiar form

$$\mathcal{C}(k,\mu_0) = \frac{1}{2\omega(k)} coth(\beta_0\omega(k)/2) \tag{60}$$

On the other hand, in the non-relativistic limit $\epsilon(k) = \mathbf{k}^2/2m \simeq \omega - m$ and $\mu_0 \ll m$ $\mathcal{C}(k,\mu_0)$ is the equally familiar Bose distribution

$$\mathcal{C}(k,\mu_0) \simeq \frac{1}{2\omega(k)}\left(\frac{1}{1-e^{-\beta_0(\epsilon(k)-\mu_0)}}\right) \tag{61}$$

Further, in the high temperature relativistic limit $T_0 \gg m_0$, $\mathcal{C}(k,\mu_0)$ of (60) simplifies as

$$\frac{1}{2\omega(k)} coth(\beta_0\omega(k)/2) \simeq \frac{T_0}{\mathbf{k}^2+m_0^2}. \tag{62}$$

Equally, in the non-relativistic limit when $\epsilon(k) - \mu_0 \ll T_0$,

$$\mathcal{C}(k,\mu_0) \simeq \frac{T_0}{\epsilon(k)+|\mu_0|} \tag{63}$$

from (61). This reproduces (62),

$$\mathcal{C}(k,\mu_0) \simeq \frac{T_0}{\mathbf{k}^2+m_0^2} \tag{64}$$

up to an irrelavant coefficient of proportionality, on using the definition of m_0 given earler in (52). That is, in these regimes the relativistic and non-relativistic $W(r,t)$ are *identical*, once we take the identifications of (52) into account.

We observe that, if we were to use $W(r;t)$ of (56) as it stands, then both $W(0;t)$ and $W''(0;t)$ necessarily suffer from ultraviolet divergences. However, the string thickness at the end of the quench will be $O(M^{-1})$. It is the zeroes coarse-grained to this scale that will provide the subsequent network. Thus, if we do not probe the field zeroes within a string we need consider only the first term in (56). There is another point. Even before the quench begins there is a high density of line zeroes coarse-grained to this same scale $O(M^{-1})$ in the initial equilibrium phase. From (15)) their density is [5] $n(t) = O(M^2)$ if $M \geq m$, which we assume. However, these modes are entirely transient due to the uncertainty principle. If we were to calculate the correlations of $\rho_i(\mathbf{x})$ at *different* times t and t', we would find rapidly oscillating behaviuor with period $\Delta t = O(m^{-1})$. On the other hand, a calculation of the density correlations at different times from the unstable modes in (56) does not give oscillatory, but damped, behaviour. It is the residue of the strings produced by the unstable modes that survives to produce the network, and the transient strings can be ignored. Henceforth, we retain only the first term

$$W(r;t) = \int_{|\mathbf{k}|<M} \not{d}^3k\, e^{i\mathbf{k}.\mathbf{x}} \mathcal{C}(k,\mu_0)\Big[1 + A(k)(\cosh(2\Omega(k)t) - 1)\Big] \tag{65}$$

of (56). In the two critical regimes where (62) and (64) are valid, $W(r;t)$ is again the same for both cases.

Even though the approximation is only valid for small times, there is a regime $Mt \geq 1$, for small couplings, in which t is large enough for $\cosh(2Mt) \approx \frac{1}{2}\exp(2Mt)$ and yet Mt is still smaller than the time $O(\ln 1/\lambda)$ at which the fluctuations begin to sample the ground-state manifold. In this regime

$$W(r;t) \simeq \int_{|\mathbf{k}|<M} \not{d}^3k\, \mathcal{C}(k,\mu) e^{i\mathbf{k}.\mathbf{x}} A(k)\; e^{2\Omega(k)t} \tag{66}$$

In these circumstances the integral at time t is dominated by a peak in the integrand $k^2 e^{2\Omega(k)t}$ at k around k_c, where

$$tk_c^2 = M\left(1 + O\left(\frac{1}{Mt}\right)\right). \tag{67}$$

and we have assumed M and m_0 to be comparable. The effect of changing β_0 is only visible in the $O(1/Mt)$ term. In fact, we are being unnecessarily restrictive in wanting to preserve the idnetical behaviour of (62) and (64). From (14)

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[5] This was essentially the basis for Halperin's results on string densities in [12].

onwards it follows that the overall scale of W is immaterial to the vortex density. All that is required for identical *leading* behaviour in relativistic and non-relativistic regimes is that $\mathcal{C}(k,\mu_0)$ varies slowly in the vicinity of the peak of the integrand at $tk_c^2 \simeq M$. [$A(k)$ is already slowly varying]. This is the case when, allowing for coefficients $O(1)$,

$$1 < \frac{\mu_f}{|\mu_0|} = \frac{M^2}{m_0^2} \ll tM < \ln\left(\frac{1}{\lambda}\right) \tag{68}$$

where, in the same spirit, we have taken $|\mu_0| < \mu_f$. The upper bound on tM is a reminder that interactions are always present, and the Gaussian approximation must fail as soon as the field fluctuations have extended to the true ground-states at the minima of the potential. The lower bound is necessary for the integrand to be peaked strongly so that the saddle-point approximation is valid. As long as (68) is basically correct, any difference between the relativistic and non-relativistic regimes will be non-leading.

Assuming these limits, we recover what would have been our first naive guess for a correlation function based on the growth of the unstable modes,

$$W(r;t) \simeq \int_{|\mathbf{k}|<M} d^3k\, e^{i\mathbf{k}.\mathbf{r}}\, e^{2\Omega(k)t}, \tag{69}$$

We understand the dominance of wavevectors at $k_c^2 = M/t$ in the integrand as indicating the formation of domains of mean size

$$\xi(t) = O(\sqrt{t/M}) \tag{70}$$

once $Mt > 1$. Specifically, we take [6] $\xi(t) = 2\sqrt{t/M}$. Once $Mt > 1$ then $\xi(t) > M^{-1}$, where M^{-1} characterises the cold vortex radius. In the weak coupling approximation individual domains become large enough to accomodate many vortices before the approximation breaks down. There is no difficulty with causality since domains increase in size as $\dot{\xi} = \frac{1}{\sqrt{Mt}} < 1$. On neglecting terms exponentially small in Mt, $W(r;t)$ of (69) can be further rewritten as

$$W(r;t) \simeq e^{2Mt} \int_0^\infty dk\ \mathrm{sinc}(kr)\, k^2 e^{-tk^2/M} \tag{71}$$

$$= W(0;t)\, exp\left(-\frac{r^2}{\xi^2(t)}\right) \tag{72}$$

where

$$W(0;t) \approx C\frac{e^{2Mt}}{(Mt)^{3/2}}, \tag{73}$$

for some C. The exponential growth of $W(0;t)$ in t reflects the way the field amplitudes fall off the hill centred at $\Phi = 0$. With the peaking in wavelength $l = k^{-1}$ understood as indicating the appearence of domains of characteristic linear dimension $\xi(t)$, the Gaussian in r is a reflection of the rms variation $\Delta\xi$ in domain size ξ. This variation is large. If we isolate the Gaussian saddle-point in (69) as

$$W(r;t) \simeq e^{2Mt} \int_0^M dk\ \mathrm{sinc}(kr)\, k_c^2 e^{-(k-k_c)^2/2(\Delta k)^2} \tag{74}$$

then

$$\frac{\Delta\xi}{\xi} = \frac{\Delta k}{k_c} = \frac{1}{2}. \tag{75}$$

To calculate the number density of vortices at early times we insert the expression (71) for W into the equations derived earlier, to find

$$n(t) = \frac{1}{\pi}\frac{1}{\xi(t)^2} \tag{76}$$

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[6] This somewhat arbitrary choice differs from that of [22] and I by a factor of $\sqrt{2}$.

for an $O(2)$ theory with strings in three dimensions. We note that the dependence on time t of both the density and density correlations is only through the correlation length $\xi(t)$. We have a *scaling* solution in which, as the domains of coherent field form and expand, the interstring distance grows accordingly. Since the only way the defect density can decrease without the background space-time expanding is by defect-antidefect annihilation, we deduce that the coalescence of domains proceeds by the annihilation of small loops of string. However, because the density of vortices only depends on $\xi(t)$ in this early stage, the fraction of string in 'infinite' string remains constant. Thus, at the same time as small loops disappear, other loops must rearrange themselves so that the length of 'infinite string decreases accordingly. Finally, there is roughly one string zero per coherence area, a long held belief for whatever mechanism.

There is one final concern. A necessary condition for this rolling down of the field to be valid is that the initial field distribution at $t \leq 0$ should not overhang the point of inflection at $\langle|\phi|\rangle = \sqrt{2M^2/3\lambda} = O(T_c)$. That is, the initial field fluctuations about $\phi_a = 0$ should be small enough that there is no significant probability that the field is already in the true vacuum. To check this for the relativistic regime, we anticipate that, when a domain structure forms, the smallest domains that can be identified will be of the size of the cold vortex radius, the Higgs field Compton wavelength, $\xi_0 = O(M^{-1})$. For temperature $T > T_c$ (but not too close), the rms field fluctuations on this scale are [27] $\Delta\phi = O((TM)^{\frac{1}{2}})$. The condition that $(\Delta\phi)^2 < T_c^2$ is guaranteed when T_c is much larger than M, as happens for small coupling. In fact, since there are small prefactors, the coupling does not have to be very small for this to happen. Details are given in I. For the non-relativistic case there are no reasons, a priori, why the situation should be different.

VI. VORTEX DENSITY CORRELATIONS

In addition to the gross vortex density (76) we can calculate the density-density correlation functions $C_{ij}(r;t)$ of (16), identical for both the relativistic plasma and the non-relativistic medium. These were not considered in I.

Yet again, as in the case of the density $n(t)$, the t-dependence of C_{ij} only occurs implicitly through $\xi(t)$. The simple analytic form of (71) enables us to calculate A and B, up to exponentially small terms in Mt, as

$$A(r;t) = \frac{2}{\pi^2\xi^4(t)}\frac{e^{-2r^2/\xi^2(t)}}{(1-e^{-2r^2/\xi^2(t)})^2}\left[(1-e^{-2r^2/\xi^2(t)}) - 2\frac{r^2}{\xi^2(t)}\right] < 0,$$
$$B(r;t) = \frac{2}{\pi^2\xi^4(t)}\frac{e^{-2r^2/\xi^2(t)}}{(1-e^{-2r^2/\xi^2(t)})} > 0, \tag{77}$$

whence the diagonal elements C_{ii} (no summation) are all *negative*. Specifically,

$$A(r;t) - B(r;t) = -\frac{2}{\pi^2\xi^4(t)}\left(\frac{2r^2}{\xi^2(t)}\right)\frac{e^{-2r^2/\xi^2(t)}}{(1-e^{-2r^2/\xi^2(t)})^2}. \tag{78}$$

That is, we have *anticorrelation* of densities for parallel directions and positive B for orthogonal ones. This is just the situation discussed in Section 2.

It is useful to expand (77) and (78) for $r < \xi$. On normalising by a factor of n^2, a and b of (21) are given by

$$a(r;t) = -1 + O\left(\frac{r^2}{\xi^2}\right) < 0,$$
$$b(r;t) = \left(\frac{\xi^2(t)}{r^2}\right) + O(1) > 0. \tag{79}$$

For $r > \xi$ there is exponential falloff but, from (79) we see that, in units of $(\pi\xi^2)^{-2}$, the anticorrelation is large. The discussion of Section 2 is directly relevant. Since strings with a long persistence length would imply *positive* parallel correlations, and weak orthogonal correlations, we can interpret these anticorrelations as a reflection of an increased string bendiness. Although it is difficult to be precise, this suggests a significant amount of string in small loops. This is an important issue since, as we noted, early universe cosmology requires infinite string if string is to be the source of large-scale structure. In practice, some is enough.

There is another, indirect, way in which we can see the tendency for more string to be in small loops than we might have thought. In numerical simulations of string networks based in throwing down field phases at random, the rule of thumb for a *regular* domain structure in field phase is that a significant fraction of string, if not most, is infinite string [9,28].

However, as we saw earlier in (75), we do not have a regular domain structure in our model but have domains with a large variance in their size $\Delta\xi/\xi = \frac{1}{2}$. Unfortunately, the domain structure that we have here is not yet appropriate for a direct comparison since, as well as the domain size, the field magnitude has a variance about $\langle|\phi|\rangle = O(M\,e^{Mt})$. The work of Guth and Pi [20] shows that this can be parametrised by an effective dispersion in t in $\langle|\phi|\rangle$ of $\Delta t = O(M^{-1})$. Nonetheless, consider an idealised case in which domain growth stops instantaneously because of back-reaction at some time t_f. The distribution of strings will then be as above for $\xi = \xi(t_f)$, while the field adjusts to the vacuum manifold, without changing phase, in each domain. We would expect some string-antistring annihilation to continue while this adjustment occurs, so that the $n(t)$ calculated previously is an overestimate of the string density at the end of the transition. However, the domains will still be of varying size with variance plausibly given approximately by (75). Although it has not been introduced along the lines above, the inclusion of domain variance in numerical simulation of string networks shows [29] that, the greater the variance, the more string is in small loops. Beyond observing that there seems to be some infinite string, we will say no more.

All our results are for quenches that go from significantly above the transition to significantly below it. These give the *smallest* value of $\Delta\xi/\xi$ possible and, plausibly, the best chance of producing infinite string. Suppose the *initial* state characterised by S_0 of (46) is very close to the transition. For relativistic theories this means starting from a temperature only just above the critical temperature. For a non-relativistic theory it corresponds to beginning from a density very close to the critical density. In either case it corresponds to taking m_0^2 suffficiently small that the inequalities (68) cannot be satisfied. Some caution is required, since fluctuations are large near the transition and the semiclassical action S_0 of(46) is unreliable. Despite that, let us take it seriously and assume that it is possible to neglect m_0^2 in comparision to k_c^2 within the time interval $1 < Mt < \ln(1/\lambda)$.

The overhang calculation of I shows that there is some leeway. Then, with

$$\frac{T_0}{\mathbf{k}^2 + m_0^2} \simeq \frac{T_0}{\mathbf{k}^2} \tag{80}$$

in (64) and (62) no longer slowly varying, instead of (69), we have

$$W(r;t) \simeq \int_{|\mathbf{k}|<M} \frac{d^3k}{\mathbf{k}^2}\, e^{i\mathbf{k}.\mathbf{x}}\, e^{2\Omega(k)t} \tag{81}$$

$$\simeq \int dk\ \mathrm{sinc}(kr)\ e^{2\Omega(k)t} \tag{82}$$

up to irrelevant factors.

There is now no peaking of the integrand in (82) at any preferential wave number, and hence no preferred length scale that can be identified as a typical domain size. This lack of domain structure, and the inapplicability of the Kibble mechanism, is because of the enhanced long-wavelength fluctuations that come from being too close to the transition initially. We stress that this does not mean that there are no vortices, or even a small vortex density. $W(r;t)$ of (82) can be integrated as

$$W(r;t) = W(0;t)\,{}_1F_1\left(\frac{1}{2}, \frac{3}{2}, -\frac{r^2}{\xi^2(t)}\right) \tag{83}$$

for $\xi(t)$ as before. A little algebra shows that the vortex density $n(t)$ is

$$n(t) = \frac{1}{3\pi}\frac{1}{\xi^2(t)}, \tag{84}$$

one-third of its previous value in (76). For distances $r > \xi$, $W(r;t)$ now has power behaviour

$$W(r;t) = O\left(\frac{\xi(t)}{r}\right) \tag{85}$$

instead of the Gaussian behaviour of (72). As before, there is anticorrelation with $A < 0, B > 0$ but, at large distances, the density-density correlation functions are very different, only showing power-behaviour in their fall-off. However, for $r \simeq \xi$ their behaviour is very like that of (79). Nonetheless, if we interpret (82) as the limiting case of domain structure with maximum variance in size (compatible with causality), the same numerical data quoted earlier [29], with due caveats, suggest even more string in small loops.

Finally, with superfluid films in mind, we can repeat the analysis in *two* space dimensions. Instead of (72) $W(r;t)$ is given, when $Mt \geq 1$, by

$$W(r;t) \simeq e^{2Mt} \int dk\ \mathrm{sinc}(kr)\, k e^{-tk^2/M} \tag{86}$$

The integrand is strongly peaked at $k_c^2 = M/2t$ which, from our previous analysis, we interpret as the existence of domains, trapping $O(2)$ monopoles (vortex cross-sections) on their boundaries. The domains are of varying size. If we isolate the Gaussian saddle-point in (85) as

$$W(r;t) \simeq e^{2Mt} \int_0^M dk\ \mathrm{sinc}(kr)\, k_c e^{-(k-k_c)^2/2(\Delta k)^2} \tag{87}$$

then the rms variation $\Delta\xi$ in domain size ξ is larger than in three dimensions, as

$$\frac{\Delta\xi}{\xi} = \frac{\Delta k}{k_c} = \frac{1}{\sqrt{2}}. \tag{88}$$

This is independent of the definition of ξ, but if we take $\xi^2(t) = 4/k_c^2$ as before, then

$$W(r;t) = W(0;t)\, {}_1F_1\left(1, \frac{3}{2}, -\frac{2r^2}{\xi^2(t)}\right) \tag{89}$$

$$= O\left(\frac{\xi^2(t)}{r^2}\right) \tag{90}$$

for $r > \xi$. This is intermediate between (85) and the Gaussian behaviour of (72). Yet again there is anticorrelation in the density-density correlation function $\langle\rho(\mathbf{r})\rho(\mathbf{0})\rangle = A(r;t)$, where $A(r;t)$ is given in (17).

VII. CONCLUSIONS

In this paper we have shown how global O(2) vortices appear, at a quench from the ordered to disordered state, as a consequence of the growth of unstable Gaussian long wavelength fluctuations. Most importantly, in the light of discussions about the extent to which vortex production in low-temperature many-body systems simulates vortex production in the early universe, our model supports the analogy. We have shown how, with our simple assumptions, vortex production is *identical* in both a relativistic high-temperature quench, in which the initial state is characterised by $T \gg m$, and in a non-relativistic density quench in which the initial state is described by $T \ll m$ (to freeze out antiparticles) with a chemical potential $\mu_{\rm nr} \simeq T$. All that is required is an appropriate translation of the parameters from the one case to the other.

In our simple model of Gaussian fluctuations the resulting string configurations scale as a function of the correlation length $\xi(t) = O(t^{\frac{1}{2}})$, at about one vortex/correlation area. This is compatible with the Kibble mechanism for vortex production on domain boundaries upon phase separation. For a weak coupling theory the domain cross-sections are significantly larger than a vortex cross-section at the largest times for which the approximations are valid. However, there is a large variance in their size, with $\Delta\xi/\xi = \frac{1}{2}$. Because of this there is more string in small loops than we might have anticipated. Nonetheless, we expect some infinite string.

We stress that our model can, at best, describe weak coupling systems for the short times while the domains are growing before the defects freeze out. This is unsatisfactory for most early universe applications and for low-temperature many-body systems. However, we know in principle [22] how to include back-reaction (still within the context of a Gaussian approximation) to slow down domain growth as the field fluctuations spread to the ground-state manifold. The identity of the weak-coupling results of the two regimes should survive to this case also, although it will probably lead to different conclusions from those above. Further, there is no difficulty in principle of embedding these results in a FRW metric, along the lines of [30].

ACKNOWLEDGEMENTS

The authors are grateful to T.W.B. Kibble for useful conversations on defect formation. Two of us (R.J.R. and A.J.G.) would like to thank Prof. M. Krusius and Prof. G. Volovik for their hospitality at the Low Temperature Laboratory, Helsinki University of Technology, where some of this work was performed. T.S.E. wishes to thank the Royal Society for its support and A.J.G. would like to thank PPARC.

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APPENDIX

We choose to work with 'physical' fields, that is fields which evolve according to the original Hamiltonian, H_0, which has not had the chemical potential absorbed into it. The chemical potential then appears in the boundary conditions and so is placed on the same footing as the temperature. For this reason we work with the eigenstates of Q, $\phi(x)$ and $\phi^\dagger(x)$, rather than the Cartesian components $\phi_1(x)$ and $\phi_2(x)$. The conversion to the Cartesian thermal Wightman functions $W_{ab}(r,t)$ of the text is trivial.

For the complex scalar field we are interested in, the Hamiltonian has the usual form as given in (32) in which, for the reasons given in text, $\lambda = 0$. That is, we are interested in 'free' thermal Wightman functions, albeit with changing mass. The Wightman functions in question are defined to be

$$G^{>}(x,x') = \mathrm{Tr}\{e^{-\beta(H_0-\mu Q)}\phi(x)\phi^\dagger(x')\} \tag{A1}$$

$$G^{<}(x,x') = \mathrm{Tr}\{e^{-\beta(H_0-\mu Q)}\phi^\dagger(x')\phi(x)\}. \tag{A2}$$

Other types of propagator are defined in a similar way, retarded, advanced, time-ordered etc. but will not be needed here and in any case they can be built from $G^{>(<)}$ relatively easily.

The free propagators satisfy differential equations which look like the usual Klein-Gordon equation, the only difference in our case is that the mass term is time dependent - a simple step function for the quench at time $t = 0$ that we have considered in the text. Otherwise our dicsussion is general. Adopting it to our specific model is straightforward.

We start by solving the homogeneous equation

$$(\partial_t^2 + \omega^2(t))U^{\pm}(t) = 0 \tag{A3}$$

where

$$\omega(t) = \theta(-t)\omega_1 + \theta(t)\omega_2. \tag{A4}$$

As it stands both ω_1 and ω_2 are assumed to be real. If we have an unstable mode at late times then we just replace $\omega_2 = \pm i\Omega$. The boundary conditions are

$$\lim_{t\to-\infty} U^{\pm}(t) = \exp\{\pm i\omega_1 t\} \tag{A5}$$

$$U^{\pm}(t = 0^+) = U^{\pm}(t = 0^-) \quad \partial_t U^{\pm}(t = 0^+) = \partial_t U^{\pm}(t = 0^-). \tag{A6}$$

This gives

$$U^{\pm}(t) = \theta(-t)e^{\pm i\omega_1 t} + \theta(t)[A_{\pm}e^{\pm i\omega_1 t} + B_{\pm}e^{\mp i\omega_1 t}] \tag{A7}$$

$$A_{\pm} = \frac{1}{2}\left(1 + \frac{\omega_1}{\omega_2}\right), \quad B_{\pm} = \frac{1}{2}\left(1 - \frac{\omega_1}{\omega_2}\right). \tag{A8}$$

Now we can construct the thermal Wightman functions. As they satisfy

$$(\partial_t^2 + \omega^2(t))G^{>(<)}(t, t') = 0 \tag{A9}$$

$$(\partial_{t'}^2 + \omega^2(t'))G^{>(<)}(t, t') = 0 \tag{A10}$$

we try

$$G^{>(<)}(t, t') = a^{>(<)}U^-(t')U^+(t) + d^{>(<)}U^+(t')U^-(t) \tag{A11}$$

where terms such as $U^-(t')U^-(t)$ and $U^+(t')U^+(t)$ are eliminated by demanding time translation at early times.

We start by noting that it is common, when considering the thermal Wightman functions of real fields in thermal equilibrium, to use their definitions to deduce a trivial identity relating the two thermal Wightman functions at equal times. In the case of the complex fields we see that we have

$$[G^{>(<)}(t, t')]^* = G^{>(<)}(t'^*, t^*)G^{>(<)}(t, t) = G^{>(<)}(t^*, t^*). \tag{A12}$$

We have been very careful to ensure that the definition of $\phi^\dagger(t)$ is such that it evolves like $\phi(t)$, namely $\phi^\dagger(t) = \exp\{-iH_0t\}\phi^\dagger(0)\exp\{iH_0t\}$ (where we have dropped spatial labels). Problems occur with complex time shifts, such as are encountered in thermal field theory, as roughly speaking we require $(\phi(t))^* = \phi^\dagger(t^*)$. The U^+ and U^- functions below are related in a similar way.

The ETCR (equal time commutation relations) can be used to provide boundary conditions,

$$[\phi(t), \Pi(t)] = [\phi^\dagger(t), \Pi^\dagger(t)] = i. \tag{A13}$$

We have $\Pi(t) = \partial_t\phi^\dagger(t)$ and $\Pi^\dagger(t) = \partial_t\phi(t)$ since, for the physical fields, there is no chemical potential in the expressions for Π and $\Pi^\dagger$. Looking at these equations for any one negative time gives

$$a^< - d^< - a^> + d^> = \frac{1}{\omega_1} \tag{A14}$$

and we find the same relation if the ETCR is imposed at any positive time. This confirms that the ETCR are indeed maintained at all times if they are true at any one time.

There is a second ETCR which we can use;

$$[\phi(t), \phi^\dagger(t)] = 0 \tag{A15}$$

This requires

$$G^{>}(t,t) = G^{<}(t,t) \tag{A16}$$

which gives

$$a^{>} + d^{>} = a^{<} + d^{<}. \tag{A17}$$

Once this is satisfied it ensures that this ETCR also holds at all times.

The temperature and chemical potential appear only in the last boundary condition. This is the well known KMS condition [31]. Here we enforce this at any negative time as then the system is in equilibrium. Thus

$$G^{>}(t,t') = e^{-\beta\mu} G^{<}(t + i\beta, t') \tag{A18}$$

This holds for all $t, t' < 0$, but not for later times, so we find

$$a^{>} = e^{-\beta(\omega_1+\mu)} a^{<} \tag{A19}$$

$$d^{>} = e^{\beta(\omega_1-\mu)} d^{<} \tag{A20}$$

Putting all this together gives

$$a^{<} = \frac{1}{2\omega_1} \frac{1}{1 - e^{-\beta(\omega_1+\mu)}} \tag{A21}$$

$$a^{>} = \frac{1}{2\omega_1} \frac{-1}{1 - e^{\beta(\omega_1+\mu)}} \tag{A22}$$

$$d^{<} = \frac{1}{2\omega_1} \frac{-1}{1 - e^{\beta(\omega_1-\mu)}} \tag{A23}$$

$$d^{>} = \frac{1}{2\omega_1} \frac{1}{1 - e^{-\beta(\omega_1-\mu)}} \tag{A24}$$

For our purposes we just need the equal time propagator which can be obtained from either of the thermal Wightman functions because of the second ETCR (A16). From the above we find that

$$G^{>}(t,t) = G^{<}(t,t) = \frac{1}{2\omega_1} \frac{e^{-\beta\omega_1} - e^{\beta\omega_1}}{e^{\beta\mu} - e^{-\beta\omega_1} - e^{\beta\omega_1} + e^{-\beta\mu}} . U^{-}(t) U^{+}(t) \tag{A25}$$

from which our results of section 5 follow.

We might ask whether we need worry about *off-diagonal* terms in the Cartesian propagator or, equivalently, $\langle \phi(t)\phi(t') \rangle$ and $\langle \phi^{\dagger}(t)\phi^{\dagger}(t') \rangle$. In some approaches using different definitions of the fields, e.g. [25], these are not zero. Trying the same ansatz as before, we see that time-translation invariance and the KMS condition, applied at negative times, ensure that these are zero for all times in our case. This is a big advantage of our definitions.