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The Decay of Magnetic Fields in Kaluza-Klein Theory

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Abstract

Magnetic fields in five-dimensional Kaluza-Klein theory compactified on a circle correspond to "twisted" identifications of five dimensional Minkowski space. We show that a five dimensional generalisation of the Kerr solution can be analytically continued to construct an instanton that gives rise to two possible decay modes of a magnetic field. One decay mode is the generalisation of the "bubble decay" of the Kaluza-Klein vacuum described by Witten. The other decay mode, rarer for weak fields, corresponds in four dimensions to the creation of monopole-anti-monopole pairs. An instanton for the latter process is already known and is given by the analytic continuation of the Kaluza-Klein Ernst metric, which we show is identical to the five dimensional Kerr solution. We use this fact to illuminate further properties of the decay process. It appears that fundamental fermions can eliminate the bubble decay of the magnetic field, while allowing the pair production of Kaluza-Klein monopoles.

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1. Introduction

The standard Kaluza-Klein vacuum, $M^4 \times S^1$, is known to be unstable. Witten showed [1] that it can semiclassically decay by nucleating a "bubble of nothing" which appears to expand into space. In this paper we will consider another class of "vacua" in Kaluza-Klein theory which correspond to static magnetic flux tubes in four dimensions. Although these solutions are non-trivial four dimensional configurations they are simply obtained from dimensional reduction of five dimensional Minkowski space, M^5, with "twisted" identifications. We will see that these backgrounds are also unstable, and in fact, have two different decay modes. The first was discussed in [2] and corresponds to the pair creation of Kaluza-Klein monopoles. We will show that there is another decay mode which occurs at a much higher rate. It is a direct generalization of the "expanding bubble" found by Witten.

The semiclassical decay of a vacuum can be described by an instanton, i.e., a euclidean solution to the field equations which interpolates between the initial and final states. The leading approximation to the decay rate is simply e^{-I} where I is the instanton action. To show that $M^4 \times S^1$ is unstable, Witten constructed an appropriate instanton by analytically continuing the five dimensional Schwarzschild solution. We will use the five dimensional Kerr solution to construct an instanton which describes the decay of a Kaluza-Klein magnetic field. The subsequent evolution can be obtained by a further analytic continuation (as in the Schwarzschild case) and resembles an expanding bubble.

In [2], an instanton describing the pair creation of monopoles was constructed by analytically continuing the Kaluza-Klein Ernst solution [3]. Recall that in four dimensional Einstein-Maxwell theory, the Ernst solution [4] describes two oppositely charged black holes accelerating apart in a background magnetic field. Since the Kaluza-Klein monopole [5,6] is just an extremal magnetically charged black hole, the pair creation of monopoles can be described using an instanton constructed from the Kaluza-Klein analog of the Ernst solution.

Remarkably enough, it turns out that the instanton we construct from the Kerr metric is identical to the one previously constructed from the Ernst solution! At first sight, this appears impossible. Not only does the spacetime containing an expanding bubble seem very different from one containing two accelerating monopoles, but the actions for the two instantons are different: in the limit that the asymptotic magnetic field $B \to 0$, the rate for monopole creation vanishes, while the rate for bubble nucleation approaches the finite nonzero value associated with the standard Kaluza-Klein vacuum. We will resolve this apparent paradox in detail below. The essential point is that the magnetic field seen in four dimensions is not uniquely determined by the five dimensional solution. For axisymmetric configurations, one must choose an 'internal space' by specifying a Killing field with closed orbits; different choices yield different values of B. Physical considerations restrict this choice so that B is small compared to the compactification scale. For one range of parameters and one choice of internal space, the Kerr instanton yields a four dimensional solution with small B which resembles the one obtained by Witten. However, for another range of parameters, and a different choice of internal space, the Kerr instanton again yields a four dimensional solution with small B which now resembles a pair of accelerating monopoles.

A closer examination of these decay processes contain further surprises. First, as pointed out in [5] the "bubble of nothing" in five dimensions appears as a point-like singularity in four dimensions – it does not expand outward, instead space collapses in towards it. In five dimensions, it turns out that the bubble wall follows a geodesic, not a curve of uniform acceleration. Second, in the pair creation of monopoles, the spacetime between the monopoles dynamically decompactifies: the size of the fifth direction increases with time, so the four dimensional description eventually breaks down.

It has been suggested [1] that fundamental fermions could stabilize the standard Kaluza-Klein vacuum. It appears that the same mechanism eliminates bubble nucleation but allows the pair creation of monopoles.

The outline of this paper is as follows. In the next section we discuss the Kaluza-Klein solutions describing magnetic fields, and explain how a given five dimensional solution can give rise to different four dimensional descriptions. In section 3, we introduce the five dimensional Kerr instanton, examine its properties, and compute its action. Section 4 contains a review of the Kaluza-Klein Ernst instanton, and establishes its equivalence to the Kerr instanton of the previous section. The final section consists of a summary of our results and the arguments as to how spinors can rule out bubble formation but not pair creation.

2. Uniform Magnetic Field

In Einstein-Maxwell theory, the closest analogue to a uniform magnetic field is the Melvin spacetime [7], which describes a static cylindrically symmetric magnetic flux tube.

The generalisation of this solution to Kaluza-Klein theory was constructed by Gibbons and Maeda [8]. It was later realised that this solution can be obtained from M^5 by simply identifying points in a nonstandard way [3,2]. Explicitly, the spacetime is given by the flat metric in cylindrical coordinates

$$ds^2 = -dt^2 + dz^2 + d\rho^2 + \rho^2 d\varphi^2 + (dx^5)^2 \tag{2.1}$$

with the identifications

$$(t, z, \rho, \varphi, x^5) \equiv (t, z, \rho, \varphi + 2\pi n_1 RB + 2\pi n_2, x^5 + 2\pi n_1 R) \quad \forall n_1, n_2 \in Z\,. \tag{2.2}$$

The identification under shifts of $2\pi n_2$ for φ and $2\pi n_1 R$ for x^5 are, of course, standard. The new feature is that under a shift of x^5, one also shifts φ by $2\pi n_1 RB$. Since φ is already periodic with period 2π, changing B by a multiple of $1/R$ does not change the identifications. Inequivalent spacetimes are obtained only for $-1/2R < B \leq 1/2R$. More geometrically, one can obtain this spacetime by starting with (2.1) and identifying points along the closed orbits of the Killing vector $l = \partial_5 + B\partial_\varphi$.

To obtain the four dimensional description, one must reduce along a Killing field with closed orbits. An obvious candidate is l. Introducing the new coordinate $\tilde{\varphi} = \varphi - Bx^5$ which is constant along the orbits of l, the metric becomes

$$ds^2 = -dt^2 + dz^2 + d\rho^2 + \rho^2(d\tilde{\varphi} + Bdx^5)^2 + (dx^5)^2 \tag{2.3}$$

now with the points $(t, z, \rho, \tilde{\varphi}, x^5)$ and $(t, z, \rho, \tilde{\varphi} + 2\pi n_2, x^5 + 2\pi n_1 R)$ identified. In the new coordinates the Killing vector is simply $l = \partial_5$ and consequently it is straightforward to perform the dimensional reduction. We recast the metric in the following canonical form

$$ds^2 = e^{-4\phi/\sqrt{3}}(dx^5 + 2A_\mu dx^\mu)^2 + e^{2\phi/\sqrt{3}} g_{\mu\nu} dx^\mu dx^\nu \tag{2.4}$$

where x^μ are the four dimensional coordinates. Note that with this decomposition into four dimensional fields which do not depend on the fifth direction, the five dimensional Einstein-Hilbert action up to surface terms becomes

$$I = \frac{1}{16\pi G_5} \int d^5x \sqrt{-{}^5g}\; {}^5R = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} \left(R - 2(\nabla\phi)^2 - e^{-2\sqrt{3}\phi} F^2 \right) \tag{2.5}$$

where $G_5 = 2\pi R G_4$. We deduce that the unit of electric charge, in these units, is $e = 2/R$.

In terms of four dimensional fields, (2.3) is

$$\begin{aligned} ds_4^2 &= \Lambda^{1/2}\left[-dt^2+d\rho^2+dz^2\right]+\Lambda^{-1/2}\rho^2 d\tilde{\varphi}^2 \\ e^{-\frac{4}{\sqrt{3}}\phi} &= \Lambda, \qquad A_{\tilde{\varphi}} = \frac{B\rho^2}{2\Lambda} \\ \Lambda &= 1+B^2\rho^2 \end{aligned} \tag{2.6}$$

This solution describes a magnetic flux tube in the z direction and thus generalises the Melvin solution of Einstein-Maxwell theory. The parameter B gives the strength of the magnetic field on the axis via $B^2 = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}\big|_{\rho=0}$.

Although the choice of reducing to four dimensions along l seems natural, it is not unique. One could consider using the Killing vector $\hat{l} = l+(n/R)\partial_\varphi$ for any integer n, which also has closed orbits. The corresponding four dimensional solution is simply (2.6), with magnetic field parameter $B+n/R$. Recalling that the parameter B in the five dimensional metric is restricted to lie in the range $-1/2R < B \leq 1/2R$, it would appear that all values of the four dimensional magnetic field can be obtained.

However, we must consider the range of applicability of these spacetimes. From (2.4) and (2.6) we see that for every $B \neq 0$, the proper length of the circles in the fifth direction grows linearly with ρ for large ρ. This seems to cast doubt on their interpretation as Kaluza-Klein backgrounds. Fortunately, this is not a problem since physical magnetic fields are not infinite in spatial extent. We can view (2.6) as an approximation to a constant physical magnetic field which is valid only for $\rho << 1/|B|$, in which range three dimensional space is approximately flat and the internal circles have approximately constant length. (This is not new to Kaluza-Klein theory: even in Einstein-Maxwell theory, calculations of the decay of electromagnetic fields due to pair creation of black holes use the same assumption since the exact Melvin spacetime "curls up" far from the axis.) In addition, in order for the fifth direction to remain unobservable, we must consider length scales large compared to their size: $\rho >> R$. Comparing these two restrictions on ρ we see that there is a nontrivial range of applicability only for $|B| << 1/R$. (If R is of order the Planck scale, this includes large magnetic fields in conventional units.) In other words, if $|B| \sim 1/R$, the four dimensional metric is curved on scales of order the compactification scale, so a four dimensional interpretation is no longer appropriate. Since the different four dimensional reductions change B by multiples of $1/R$, we see that for fixed B at most one can be physically reasonable. Note that in contrast, due to the translational invariance, there is no limit on the length of a physical flux tube that can be well approximated by the Melvin solution.

3. The Five Dimensional Kerr Instanton

3.1. The Geometry

Myers and Perry have generalised the four dimensional Kerr solution to arbitrary dimensions $d \geq 4$ [9]. For $d = 5$, in addition to the mass, the solutions are labeled by two angular momentum parameters. Asymptotically we can think of these as describing a rotation in two orthogonal planes in R^4. For our purposes we are interested in the case when only one of the angular momentum parameters is non-zero. In this case the Lorentzian metric is given by

$$\begin{aligned} ds^2 =& - dt^2 + \sin^2\theta (r^2 + a^2) d\varphi^2 + \frac{\mu}{\rho^2}(dt + a\sin^2\theta d\varphi)^2 \\ &+ \frac{\rho^2}{r^2 + a^2 - \mu} dr^2 + \rho^2 d\theta^2 + r^2\cos^2\theta d\psi^2 \end{aligned} \tag{3.1}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, μ and a are the mass and angular momentum parameters and the range of the angular variables is $0 \leq \theta \leq \pi/2$, $0 \leq \varphi \leq 2\pi$, $0 \leq \psi \leq 2\pi$.

The instanton metric is obtained by setting $t = ix^5$ and $a = i\alpha$ with α real:

$$\begin{aligned} ds^2 =& (dx^5)^2 + \sin^2\theta (r^2 - \alpha^2) d\varphi^2 - \frac{\mu}{\rho^2}(dx^5 + \alpha\sin^2\theta d\varphi)^2 \\ &+ \frac{\rho^2}{r^2 - \alpha^2 - \mu} dr^2 + \rho^2 d\theta^2 + r^2\cos^2\theta d\psi^2 \end{aligned} \tag{3.2}$$

where now

$$\rho^2 = r^2 - \alpha^2\cos^2\theta, \tag{3.3}$$

This metric has a coordinate singularity at $r^2 = r_H^2 \equiv \mu + \alpha^2$ (the location of the black hole horizon in the Lorentzian metric). The potential conical singularity can be eliminated by a suitable periodic identification of the coordinates φ and x^5. To see this in detail, let us first introduce two quantities encountered in lorentzian black hole theory:

$$\Omega = \frac{\alpha}{\mu}, \qquad \kappa = \frac{\sqrt{\mu + \alpha^2}}{\mu} \tag{3.4}$$

where $\omega = i\Omega$ and κ are the lorentzian angular velocity and surface gravity, respectively, analytically continued to imaginary values of the parameter a. The norm of the Killing vector

$$l = \frac{\partial}{\partial x^5} + \Omega\frac{\partial}{\partial\varphi} \tag{3.5}$$

consequently vanishes at $r = r_H$. Introducing the new coordinate $\tilde{\varphi} = \varphi - \Omega x^5$, which is constant along the orbits of l, we note that near $r = r_H$, the metric (3.2) can be written

$$\begin{aligned} ds^2 \approx &(r - r_H) f(\theta)(dx^5)^2 + (r - r_H) g(\theta) d\tilde{\varphi} dx^5 + \frac{\mu^2 \sin^2\theta}{(\alpha^2 \sin^2\theta + \mu)} d\tilde{\varphi}^2 \\ &+ \frac{1}{4\kappa^2} \frac{f(\theta)}{(r - r_H)} dr^2 + \dots \end{aligned} \tag{3.6}$$

where

$$f(\theta) = \frac{2r_H(\alpha^2 \sin^2\theta + \mu)}{\mu^2} \qquad g(\theta) = \frac{4r_H \alpha \sin^2\theta(\alpha^2 \sin^2\theta + 2\mu)}{\mu(\alpha^2 \sin^2\theta + \mu)} \tag{3.7}$$

and the elipsis denote terms that are not important for the following argument. If at fixed $\tilde{\varphi}$ we assume that $0 \le x^5 \le 2\pi/\kappa$ we can introduce the coordinates

$$x = (r - r_H)^{1/2} \cos(x^5 \kappa) \qquad y = (r - r_H)^{1/2} \sin(x^5 \kappa) \tag{3.8}$$

The metric then takes the form

$$ds^2 \approx \frac{f(\theta)}{\kappa^2}(dx^2 + dy^2) + \frac{g(\theta)}{\kappa}(xdy - ydx) d\tilde{\varphi} + \frac{\mu^2 \sin^2\theta}{(\alpha^2 \sin^2\theta + \mu)} d\tilde{\varphi}^2 + \dots \tag{3.9}$$

which is clearly real and analytic at $r = r_H$ $(x = y = 0)$. Thus the conical singularity is eliminated by requiring that x^5 be periodic with period $2\pi R$ at fixed $\tilde{\varphi}$ where

$$R = \frac{1}{\kappa} = \frac{\mu}{\sqrt{\alpha^2 + \mu}} \tag{3.10}$$

In terms of of the $(x^5, \varphi, r, \theta, \psi)$ coordinates we deduce that the points $(x^5, \varphi, r, \theta, \psi)$ and $(x^5 + 2\pi n_1 R, \varphi + 2\pi n_1 \Omega R + 2\pi n_2, r, \theta, \psi)$ must be identified[1].

In the limit $r \to \infty$, the instanton metric (3.2) approaches

$$ds^2 = (dx^5)^2 + r^2 \sin^2\theta d\varphi^2 + dr^2 + r^2 d\theta^2 + r^2 \cos^2\theta d\psi^2 \tag{3.11}$$

Using the cylindrical coordinates $\rho = r\sin\theta, z = r\cos\theta$ with $z \ge 0$, we get

$$ds^2 = (dx^5)^2 + \rho^2 d\varphi^2 + d\rho^2 + dz^2 + z^2 d\psi^2 \tag{3.12}$$

[1] Demanding that the metric is smooth on the axis $g_{\varphi\varphi} = 0$ we deduce that φ has period 2π at fixed x^5.

This is clearly flat. Because of the identifications made on the angles x^5, φ, we conclude that asymptotically the instanton approaches a euclidean Kaluza-Klein magnetic field with magnetic field strength

$$B = \Omega = \frac{\alpha}{\mu} \tag{3.13}$$

Since $BR = \alpha/\sqrt{\alpha^2 + \mu}$, and α can take both positive and negative values, we see that B lies in the range $-1/R < B < 1/R$. In section 2, we saw that for a uniform magnetic field, inequivalent five-dimensional spacetimes were obtained only for $-1/2R < B \leq 1/2R$. This means that the Kerr instanton with $B < -1/2R$ ($B > 1/2R$) asymptotically approaches exactly the same magnetic field solution as the instanton with parameter $B + 1/R$ ($B - 1/R$). This will be an important point in the interpretation of the instanton as describing two modes of decay, to be discussed shortly.

3.2. The Euclidean Action

In this section we calculate the euclidean action of the above instantons. This will be used in the next section when we interpret them as mediating a decay of the Kaluza-Klein magnetic field. The full euclidean action with boundary terms included is only defined with respect to a reference background and is given by

$$I = -\frac{1}{16\pi G_5} \int \sqrt{g} R - \frac{1}{8\pi G_5} \oint \sqrt{h}(K - K_0) \tag{3.14}$$

where K is the trace of the extrinsic curvature of the boundary, and K_0 is the analogous quantity for the boundary embedded in the background geometry. For the Kerr instanton, the appropriate background is the (analytic continuation of the) Kaluza-Klein magnetic field solution (2.3). Since our instanton is Ricci flat, the only contribution to S comes from the surface term. The metric induced on the surface $r =$ constant in (3.2) is

$$ds^2 = \left(1 - \frac{\mu}{r^2}\right)(dx^5)^2 + \sin^2\theta\ (r^2 - \alpha^2) d\varphi^2 + \rho^2 d\theta^2 + r^2 \cos^2\theta d\psi^2 \tag{3.15}$$

where we have only kept terms of order $O(1/r^2)$. Computing the derivative of the volume element of this metric with respect to a unit radial vector yields

$$K\sqrt{h} = \sin(2\theta) \left[\frac{3}{2} r^2 - \mu - \frac{\alpha^2}{4}[3 - \cos(2\theta)]\right] \tag{3.16}$$

The background contribution is easily computed using the fact that the metric (3.2) approaches flat space in the limit $\mu = 0$ for all values of α [9]. (This is reasonable since the total angular momentum is proportional to μ.) It is clearly more convenient to use this representation of flat space to embed the boundary isometrically, than the standard one.[2] Since μ only enters the metric (3.15) in g_{55}, we can embed it in flat space by taking a surface of constant r in (3.2) with $\mu = 0$, and letting the flat space coordinate x^5 have period $2\pi R(1 - \mu/2r^2)$. We can then compute $K_0\sqrt{h}$ and take the difference with (3.16) to obtain

$$(K - K_0)\sqrt{h} = -\frac{\mu \sin(2\theta)}{4} \tag{3.17}$$

The euclidean action is therefore

$$I_{\text{Kerr}} = \frac{\pi^2 \mu R}{4G_5} \tag{3.18}$$

Using (3.10) and (3.13), one finds

$$I_{\text{Kerr}} = \frac{\pi R^2}{8G_4} \frac{1}{(1 - R^2 B^2)} \tag{3.19}$$

We can check the consistency of this result with the various thermodynamic formulae for five-dimensional black holes [9]. The mass, M, and angular momentum, J, are given by

$$M = \frac{3\pi}{8} \frac{\mu}{G_5}\ , \qquad J = \frac{2}{3} M a \tag{3.20}$$

where a is the Lorentzian rotation parameter. The Smarr relation is

$$M = \frac{3}{2}\left(TS + \omega J\right) \tag{3.21}$$

with the entropy given by $S = \frac{1}{4G_5} A_H$, A_H being the 3-area of the horizon. The thermodynamic potential W is

$$W = M - TS - \omega J = \frac{1}{3} M \tag{3.22}$$

and thus the euclidean action is

$$I_{\text{Kerr}} = \frac{W}{T} = \frac{1}{3}\frac{M}{T} = 2\pi R \frac{M}{3} \tag{3.23}$$

which agrees with our direct calculation. Note that we can obtain the action of five-dimensional Schwarzschild as the limit of I_{Kerr} for zero B: $I_{\text{Schw}} = \frac{\pi R^2}{8G_4}$. This differs by a factor of two from the result of Witten [1].

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[2] The identifications needed to convert this flat space into the magnetic field solution are identical to the ones with $\mu \neq 0$, and do not affect local calculations such as the extrinsic curvature.

3.3. Interpretation as a Decay

In order to show that the instanton (3.2) describes the semi-classical instability of a Kaluza-Klein magnetic field, it suffices to find a surface of zero extrinsic curvature (zero momentum). One can then use the fields on this surface as initial data to obtain a real lorentzian metric which describes the spacetime into which the static magnetic field decays.

Such a zero-momentum surface in (3.2) is easy to find, and is given by $\psi =$ constant. In fact, to obtain a surface which is complete, we need to take both $\psi = 0$ and π, since ψ is an angular coordinate with regular origin at $\theta = \pi/2$. The induced metric on this surface turns out to be just the four dimensional euclidean Kerr-Newman metric with zero mass. The two surfaces $\psi = 0, \pi$ each have $0 \leq \theta \leq \pi/2$ and cover half of the space. But they join at $\theta = \pi/2$, and the full zero-momentum slice is conveniently represented by letting θ take its usual range $0 \leq \theta \leq \pi$.

The lorentzian evolution of this initial data is obtained from (3.2) by rotating the coordinate ψ, $\psi \to it$:

$$\begin{aligned} ds^2 =&(dx^5)^2 + \sin^2\theta(r^2-\alpha^2)d\varphi^2 - \frac{\mu}{\rho^2}(dx^5+\alpha\sin^2\theta d\varphi)^2 \\ &+ \frac{\rho^2}{r^2-\alpha^2-\mu}dr^2 + \rho^2 d\theta^2 - r^2\cos^2\theta dt^2 \end{aligned} \tag{3.24}$$

To understand what this metric represents, let us first set $\alpha = 0$. The metric (3.24) then reduces to

$$ds^2 = \left(1-\frac{\mu}{r^2}\right)(dx^5)^2 + \left(1-\frac{\mu}{r^2}\right)^{-1}dr^2 + r^2[-\cos^2\theta dt^2 + d\theta^2 + \sin^2\theta d\varphi^2] \tag{3.25}$$

This is precisely the solution found by Witten in his study of the decay of the Kaluza-Klein vacuum. Witten presented the solution in a different set of coordinates

$$ds^2 = \left(1-\frac{\mu}{r^2}\right)(dx^5)^2 + \left(1-\frac{\mu}{r^2}\right)^{-1}dr^2 + r^2[-d\tilde{t}^2 + \cosh^2\tilde{t}(d\tilde{\theta}^2 + \sin^2\tilde{\theta}d\varphi^2)] \tag{3.26}$$

Both of these metrics can be obtained by starting with the five-dimensional euclidean Schwarzschild solution, and analytically continuing the round metric on the S^3. If one starts with $ds^2 = d\chi^2 + \sin^2\chi d\Omega_2$ and sets $\chi = (\pi/2) + i\tilde{t}$ one obtains the form (3.26). The metric in brackets is just three dimensional de Sitter space. If instead one starts with $ds^2 = d\theta^2 + \sin^2\theta d\varphi^2 + \cos^2\theta d\psi^2$ and sets $\psi = it$, one obtains the form (3.25). The metric in brackets is again three dimensional de Sitter space, but now in static coordinates. These do not cover the entire spacetime, but only the region inside the horizon at $\theta = \pi/2$. Note that the initial 4-surfaces $t = 0$ in the two metrics are identical. Because $\partial/\partial x^5$ vanishes at $r = r_H$ the initial spacelike surface is spherically symmetric and has topology $R^2 \times S^2$. The 2-surface $r = r_H$ is a totally geodesic (and hence minimal) submanifold of the initial 4-surface with area $4\pi\mu$. The subsequent evolution is easier to see in the metric (3.26) which covers the entire 5-dimensional spacetime. The surface $r = r_H$ expands outwards with area increasing like $\cosh^2 t$. This is Witten's expanding bubble. The isometry group is $U(1) \times SO(3,1)$.

Returning to the general solution (3.24), we see that at $t = 0$, the surfaces of constant r and x^5 again have a minimum area of $4\pi\mu$ which is obtained when $r = r_H$. Even though we no longer have spherical symmetry, this surface is geometrically singled out since it is a totally geodesic two-sphere. Although the metric appears static, it is directly analogous to (3.25). The time translation symmetry is a boost, since for large r the metric approaches the analytic continuation of (3.12)

$$ds^2 = (dx^5)^2 + \rho^2 d\varphi^2 + d\rho^2 + dz^2 - z^2 dt^2. \tag{3.27}$$

One can again introduce the coordinates $(\tilde{t}, \tilde{\theta})$ used in (3.26) which allow one to extend through the coordinate singularity at $\theta = \pi/2$ (which corresponds to a Killing horizon of the boost Killing vector field $\partial/\partial t$). Although these coordinates cover the entire spacetime, the $\partial/\partial\tilde{t}$ vector field is not hypersurface orthogonal when $\alpha \neq 0$. Nevertheless, we can still conclude that the "bubble" $r = r_H$ is a deformed version of the expanding three dimensional de Sitter metric.

As we mentioned earlier, there are two distinct Kerr instantons that asymptotically approach a given five-dimensional magnetic field solution, (2.1) with $|B_0| \leq 1/2R$. The obvious one has $\alpha/\mu = B_0$ while the less obvious one, "shifted Kerr", has $\alpha/\mu = B_0 \pm 1/R$ (where the upper sign is chosen when B_0 is negative and the lower sign when B_0 is positive). Thus there are two separate decay modes; the one that dominates will be the one with the lowest action. From (3.19) we see that if $|B| \equiv |\alpha/\mu| < 1/2R$ the first will dominate while if $|B| \equiv |\alpha/\mu| > 1/2R$ the second will. However, we argued in section 2, that these solutions are physically reasonable only if $|B| << 1/R$. Thus, the first instanton is physically the most important. Since even this instanton has a larger action than the one with $B = 0$, we see that the presence of a magnetic field tends to suppress the decay of the vacuum. We have plotted the action of the two instantons in figure 1.

It should be emphasised that the two instantons are the same five dimensional Kerr instanton but with different values of its parameters. On the other hand we will see in the next section that the more physical four dimensional interpretations differ substantially.

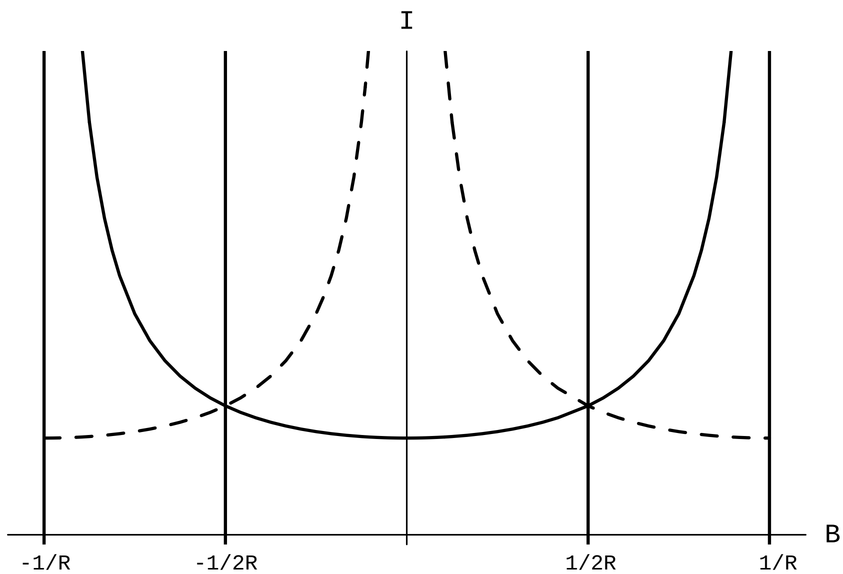

Fig. 1: Actions of two instantons mediating the decay of a Kaluza-Klein magnetic field versus magnetic field strength. The solid line is the Kerr instanton and the dotted line is the "shifted Kerr" instanton. The range of inequivalent magnetic fields is given by $-1/2R < B \leq 1/2R$ and the "physical" range is close to the I axis. Hence the "unshifted" decay dominates.

3.4. Four-Dimensional Description

We now wish to examine what the five-dimensional lorentzian solution (3.24) looks like in terms of four-dimensional fields: i.e. to relate it to physics. As we discussed in section 2, this requires a choice of Killing field l with closed orbits and the issues raised there regarding the physical justification of the Kaluza-Klein reduction will be relevant. If we use coordinates in which the Killing field is simply $l = \partial/\partial x^5$, then the four-dimensional fields can be read off after writing the five-dimensional metric in the form (2.4).

Let us start with the case $\alpha = 0$ (3.25). If we reduce along the symmetry $l = \partial/\partial x^5$, there is no four-dimensional Maxwell field, and the four-dimensional metric is

$$ds_4^2 = \left(1 - \frac{\mu}{r^2}\right)^{1/2} \left[\left(1 - \frac{\mu}{r^2}\right)^{-1} dr^2 + r^2(-\cos^2\theta dt^2 + d\theta^2 + \sin^2\theta d\varphi^2)\right] \tag{3.28}$$

In this metric, the "bubble" at $r = r_H$ has zero area and is a point-like singularity [3] which is timelike [5]. However this four dimensional spacetime differs from other static spacetimes with naked singularities such as the negative mass Schwarzschild solution. The reason is that the timelike symmetry is a boost, so the singularity is following the orbit of a boost and hits null infinity. More physically, one could view the singularity as being "at rest", with space "falling into it". Both future and past null infinity are incomplete.

If we reduce along the symmetry $\hat{l} = \partial_5 + (n/R)\partial_\varphi$ then the situation is different. To see this, we introduce the new coordinate $\tilde{\varphi} = \varphi - (n/R)x^5$ which is constant along $\hat{l}$. Since we have singled out one rotation direction, the four-dimensional spacetime will no longer have the full $SO(3,1)$ symmetry, but instead will have only a time translation and $U(1)$ symmetry. In the new coordinates

$$g_{55} = \left(1 - \frac{\mu}{r^2}\right) + \frac{n^2 r^2}{R^2}\sin^2\theta \tag{3.29}$$

$$g_{5\tilde{\varphi}} = \frac{nr^2}{R}\sin^2\theta$$

Notice that g_{55} no longer vanishes everywhere on the horizon but only at the poles $\theta = 0, \pi$. More geometrically, the Killing vector field $\hat{l}$ has a "nut" and an "antinut" at the north and south pole, respectively. In the case $n^2 = 1$ these nuts are self dual (anti-self dual) in the sense of [10] and correspond to monopoles (antimonopoles), as we will see in detail in section 4 in the context of the Kerr instanton. The four-dimensional gauge field, $A_{\tilde{\varphi}} = g_{5\tilde{\varphi}}/(2g_{55})$, is now nonzero and asymptotically describes a uniform magnetic field with

[3] Although we refer, here and subsequently, to this as a singularity in four dimensions, it should be noted that we are using this as short hand for the statement that it is a point in whose neighbourhood the four dimensional description breaks down and the true five dimensional nature of the spacetime, which is completely regular, necessarily manifests itself. It is not a singularity in the sense of a breakdown of the physics. The same comment applies to discussions of Kaluza-Klein monopoles.

strength $B = n/R$. This means that these four-dimensional reductions of Schwarzschild are in some sense unphysical: as discussed in Section 2, the Kaluza-Klein reduction only makes sense when the four-dimensional magnetic field strength, B, satisfies $B << 1/R$. In other words we, from our four-dimensional point of view, would never see magnetic fields of strength n/R and the question of how we would see them decay is moot. We will however continue to analyse these reductions as a simpler exercise before looking at the four-dimensional reductions of Kerr, some of which *will* be physically relevant.

Using (2.4) and (3.29), the four-dimensional metric is given by

$$ds^2 = g_{55}^{1/2}\left[-r^2\cos^2\theta dt^2 + \left(1-\frac{\mu}{r^2}\right)^{-1}dr^2 + r^2 d\theta^2 + \frac{(r^2-\mu)\sin^2\theta}{g_{55}}d\tilde{\varphi}^2\right] \tag{3.30}$$

The important point is that $g_{\tilde{\varphi}\tilde{\varphi}} = 0$ at the horizon while $g_{\theta\theta}$ remains nonzero. Thus the horizon is no longer a point, but rather a line. The endpoints of this line $\theta = 0, \pi$ are curvature singularities, but away from these points the metric near this line (on a t = constant surface) is simply proportional to

$$dz^2 + d\tilde{r}^2 + \frac{\tilde{r}^2}{n^2}d\tilde{\varphi}^2 \tag{3.31}$$

where we have set $z = r_H\theta$, $\tilde{r}^2 = r^2 - \mu$, and used the fact (3.10) that $R^2 = \mu$. Thus for $n = \pm 1$ the line is completely smooth, while for $|n| > 1$ there is a conical singularity. Since the deficit angle is positive, the conical singularity represents a string connecting the two singularities. Given the boost symmetry of the five-dimensional metric, it is clear that under time evolution, the two pointlike singularities will expand away from each other and hit null infinity, for all values of n.

We now consider the general case (3.24) with $\alpha \neq 0$. The natural choice of Killing field to reduce along is $l = \partial_5 + \Omega\partial_\varphi$ which vanishes at the horizon. Since $l = 0$ at the horizon, it follows that the four-dimensional metric will be singular there. Since the boost symmetry is preserved under dimensional reduction, we see that the four-dimensional metric resembles the first reduction of $\alpha = 0$ discussed above. There is a single naked singularity and null infinity is incomplete. The asymptotic value of the four-dimensional magnetic field is $\Omega = \alpha/\mu$ and so we can only interpret this as the four-dimensional view of the decay if $|\alpha/\mu| << 1/R$.

More generally, we can reduce along the Killing field $\hat{l} = l + (n/R)\partial_\varphi$ for any integer n. As before, this is accomplished by introducing the new coordinate $\tilde{\varphi} = \varphi - Bx^5$ where $B = (\alpha/\mu) + (n/R)$. We then find

$$g_{55} = 1 - \frac{\mu}{\rho^2}(1+\alpha B\sin^2\theta)^2 + B^2(r^2-\alpha^2)\sin^2\theta \tag{3.32}$$

$$g_{5\tilde{\varphi}} = B(r^2 - \alpha^2)\sin^2\theta - \frac{\mu\alpha\sin^2\theta}{\rho^2}(1 + \alpha B\sin^2\theta)$$

One can show that $g_{55} = 0$ on the horizon for $B = \alpha/\mu$ as expected, but is nonzero on the horizon for other values of B (except at $\theta = 0, \pi$). From this, and (2.4), we can compute the four-dimensional metric.

$$ds^2 = g_{55}^{1/2}\left[-r^2\cos^2\theta dt^2 + \frac{\rho^2}{r^2 - \alpha^2 - \mu}dr^2 + \rho^2 d\theta^2 + \frac{(r^2 - \alpha^2 - \mu)\sin^2\theta}{g_{55}}d\tilde{\varphi}^2\right] \quad (3.33)$$

We see that $g_{\tilde{\varphi}\tilde{\varphi}}$ again vanishes on the horizon so that the horizon is now a line which ends in two naked singularities. Those singularities, due to the boost symmetry, accelerate away to infinity. Furthermore, on the horizon

$$\rho^2 g_{55} = n^2 r_H^2 \sin^2\theta \quad (3.34)$$

so if we set $\tilde{r}^2 = r^2 - \alpha^2 - \mu$, the metric is again proportional to (3.31) which is regular for $|n| = 1$.

As before, we can only interpret these four-dimensional descriptions sensibly as Kaluza-Klein reductions when the four dimensional magnetic field is much less than $1/R$. However, unlike the Schwarzschild case, this condition can now be satisfied even with $n \neq 0$: We need $|(\alpha/\mu) + (n/R)| << 1/R$ which (since $|\alpha/\mu| < 1/R$ from (3.10)) will hold if either α/μ is positive, close to $1/R$ and $n = -1$, or α/μ is negative, close to $-1/R$ and $n = +1$.

To summarize, we have seen that there are two Kerr instantons which asymptotically approach a given five-dimensional Kaluza-Klein magnetic field. If we start with a four-dimensional magnetic field with strength $0 \leq B << 1/R$, then we can either use the Kerr instanton (3.2) with $\alpha/\mu = B$ and reduce along $l = \partial_5 + B\partial_\varphi$, or take the Kerr instanton with $\alpha/\mu = B - 1/R$ ("shifted Kerr") and reduce along the Killing field $\hat{l} = l + (1/R)\partial_\varphi$. (Similarly for small negative B, there are two decay modes.) In the first case, a single naked singularity appears in space, while in the second, there is a pair of naked singularities accelerating away from each other. For small B the second process is highly suppressed with respect to the first. The actions for these two instantons are given in figure 1 by the solid and dotted lines, respectively.

4. Kerr is Ernst in Kaluza-Klein

The four-dimensional picture of two objects accelerating away from each other in a magnetic field is reminiscent of another known solution in Kaluza-Klein theory: the Kaluza-Klein Ernst solution [3,2]. In this section we will prove that the five-dimensional Kerr instanton and the extremal Kaluza-Klein Ernst instanton are actually the same.

4.1. Review of Kaluza-Klein Ernst: Pair Creation of Monopoles

In Einstein-Maxwell theory, a Melvin magnetic solution can decay via the pair production of (extremal and non-extremal) charged black holes [11,12,2]. The instanton for this process is the euclidean section of a solution found by Ernst [4] which describes charged black holes accelerating in a magnetic field. Similarly, a Kaluza Klein magnetic field can also decay via the pair creation of Kaluza Klein monopole-anti-monopole pairs [2]. The instanton for this process is the euclideanisation of a solution that describes a Kaluza-Klein monopole and anti-monopole accelerating away from each other in a Melvin background[4] [3,2]

$$\begin{aligned} ds^2 =& \frac{\Lambda f(y)}{f(x)}\left(dx^5 + 2A_\Phi d\Phi\right)^2 \\ &+ \frac{1}{A^2(x-y)^2}\left[-f(x)^2\left(\frac{g(y)d\tau^2}{f(y)} + \frac{dy^2}{g(y)}\right) + f(y)\left(\frac{f(x)dx^2}{g(x)} + \frac{g(x)d\Phi^2}{\Lambda}\right)\right] \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} A_\Phi =& -\frac{1}{2b\Lambda}(1+2bq(x-\xi_3)) + \frac{1}{2b} \\ \Lambda =& (1+2bq(x-\xi_3))^2 + \frac{b^2 g(x)f(x)}{A^2(x-y)^2} \\ f(\xi) =& (1+r_-A\xi) \\ g(\xi) =& \left[1-\xi^2-r_+A\xi^3\right] \\ 4q^2 =& r_+r_- \,. \end{aligned} \tag{4.2}$$

[4] The form of the solution given here differs slightly from that in the references. Here we have chosen coordinates such that $\Lambda \to 1$ at infinity. This turns out to be much more convenient, and this new form of the solution can also be taken for all values of the dilaton coupling a (here, $a = \sqrt{3}$). It is obtained by the same generating transformation used in [3] but starting with a form of the C-metric in which the gauge potential vanishes on $x = \xi_3$. The formulae in [2] may be used here if some care is taken in making the substitutions $\Lambda(\xi_3) \to 1$ and $\Lambda(\xi_4) \to (1 + \frac{1}{2}(1+a^2)bq(\xi_4-\xi_3))^2$.

The roots of $g(\xi)$ are $\xi_2 < \xi_3 < \xi_4$. The surface $y = \xi_3$ is the acceleration horizon and the coordinates are restricted by $\xi_2 \leq y \leq \xi_3$ and $\xi_3 \leq x \leq \xi_4$. τ is a euclidean time coordinate whose period, $\Delta\tau$, is chosen so that the acceleration horizon is regular. The zero-momentum slice is $\tau = 0, \frac{1}{2}\Delta\tau$. x^5 is also periodic with period $2\pi R$. As infinity is approached, $x, y \to \xi_3$, the solution tends to the Melvin spacetime (2.6), euclideanised, with magnetic field parameter $\widehat{B} \equiv b\nu$, where we have defined $\nu \equiv \frac{1}{2}g'(\xi_3)f(\xi_3)^{-\frac{1}{2}} > 0$ for convenience.

The five parameters, R, b, $r_+ > 0$, $r_- > 0$ and $A > 0$ are restricted by three conditions. The first requirement is that the root of $f(\xi)$, ξ_1, be equal to the lowest root of $g(\xi)$, ξ_2. If this condition does not hold then the solution describes non-extremal, magnetically charged black holes, not monopoles. The second condition is

$$-g'(\xi_3)f(\xi_3)^{-\frac{1}{2}} = g'(\xi_4)f(\xi_4)^{-\frac{1}{2}}\Lambda(\xi_4)^{-\frac{1}{2}} . \tag{4.3}$$

which is needed to enforce regularity of the solution on the axis of symmetry. The range of the azimuthal angle, Φ, is $\Delta\Phi = \frac{2\pi}{\nu}$. The condition (4.3) ensures that choosing $\Delta\Phi$ as the range of Φ eliminates conical singularities at both $x = \xi_3$ and ξ_4. In four dimensions, in the weak field limit, this condition is physically transparent, it is simply Newton's law: $mA = qb$, where for weak fields we can identify m, A, q with the mass, acceleration and charge of the black holes and $\hat{B} \approx b$.

The final condition on the parameters is given by the geometrical analogue of the Dirac quantisation condition in the presence of magnetic charge, which is indeed its four-dimensional manifestation. We can reduce (4.1) to four dimensions along ∂_5, calculate the physical magnetic charge of the monopole, $\widehat{q}$ which must be an integer multiple of $R/4$ to eliminate conical singularities at the poles in five dimensions:

$$\widehat{q} \equiv q\frac{(\xi_4 - \xi_3)}{2\nu\left(1 + 2qb(\xi_4 - \xi_3)\right)} = k\frac{R}{4} \tag{4.4}$$

where k is an integer. Since the unit of electric charge is $e = 2/R$ we see that this gives us the Dirac quantisation condition. At the center of the monopole, $y = \xi_2$, the solution (4.1) approaches that of the static Kaluza-Klein monopole of charge $\widehat{q}$ [2]. Thus, when $k = \pm 1$ this corresponds to the Hopf fibration of S^3 and the spacetime is completely regular at the origin, whereas for $|k| > 1$, the higher Hopf fibrations, there is an orbifold singularity. We therefore restrict attention to the cases $k = \pm 1$. After imposing these conditions the independent parameters can be chosen to be $\hat{B}$ and R.

4.2. The Action

The action of (4.1) is [2]

$$I_{\rm mon} = \frac{2\pi\widehat{q}^2}{G_4}\Lambda(\xi_4)\frac{(\xi_3-\xi_2)}{(\xi_4-\xi_3)} \tag{4.5}$$

It is possible to express the action in terms of the physical magnetic field and monopole charge. We have

$$\Lambda(\xi_4) = \left(1+2bq(\xi_4-\xi_3)\right)^2 \tag{4.6}$$

and condition (4.3) gives us

$$\Lambda(\xi_4) = \frac{\xi_4-\xi_2}{\xi_3-\xi_2} \tag{4.7}$$

Then,

$$\frac{\Lambda(\xi_4)(\xi_3-\xi_2)}{(\xi_4-\xi_3)} = \frac{\xi_4-\xi_2}{\xi_4-\xi_3} = \left(1-\Lambda(\xi_4)^{-1}\right)^{-1} \tag{4.8}$$

From (4.4) and the definition of $\widehat{B}$ we have

$$\Lambda(\xi_4) = \left(1-4\widehat{B}\widehat{q}\right)^{-2} \tag{4.9}$$

and finally we obtain[5]

$$I_{\rm mon}(\widehat{B}) = \frac{2\pi\widehat{q}^2}{G_4}\frac{1}{1-\left(1-4\widehat{B}\widehat{q}\right)^2} = \frac{\pi R^2}{8G_4}\frac{1}{1-\left(1-|\widehat{B}|R\right)^2} \tag{4.10}$$

where in the last step we have used $\hat{q}\hat{B}\geq 0$ which follows from (4.3) and (4.4).

Comparing with (3.19), we see that the actions are equal up to a shift in the magnetic field parameter by an amount $1/R$: thus the action for the Ernst instanton is given precisely by the dotted line in figure 1.

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[5] We can do a similar calculation for the action of the instanton for pair creation of extremal black holes for all values of the dilaton coupling a [2], expressing it in terms of the physical magnetic field and charge:

$$I_{\rm ext} = \frac{2\pi\widehat{q}^2}{G_4}\frac{1}{1-\left(1-(1+a^2)\widehat{B}\widehat{q}\right)^2}$$

4.3. The Equivalence

The actions of (3.2) and (4.1) indicate that we should look for a coordinate transformation between (3.2) with $\alpha > 0$ and $B = \frac{\alpha}{\mu}$ and (4.1) with $\widehat{B} = b\nu = B - 1/R < 0$. This value of b requires, by (4.3) and (4.4) that $q < 0$ and

$$\widehat{q} \equiv q \frac{(\xi_4 - \xi_3)}{2\nu\,(1 + 2qb(\xi_4 - \xi_3))} = -\frac{R}{4} \tag{4.11}$$

and from (4.7)

$$1 + 2qb(\xi_4 - \xi_3) = \sqrt{\frac{(\xi_4 - \xi_2)}{(\xi_3 - \xi_2)}} \tag{4.12}$$

which, together with the definition ν, and (3.10) and (3.13) imply

$$\begin{aligned} \mu =& \frac{R^2}{1 - B^2R^2} = \frac{4q^2}{\nu^2}(\xi_3 - \xi_2)(\xi_4 - \xi_3) \\ \alpha =& \frac{BR^2}{1 - B^2R^2} = -\frac{2q}{\nu}(\xi_3 - \xi_2)\,. \end{aligned} \tag{4.13}$$

Next we note that g_{55} in both cases tends to 1 at infinity, suggesting we take the two x^5 coordinates to be equal. However, the cross term between x^5 and φ in (3.2) tends to zero at infinity whereas the cross term between x^5 and Φ in (4.1) gives rise to a Melvin magnetic field at infinity of strength $B - 1/R$. We conclude that in order to compare the two solutions we must change coordinates in (4.1) (we could choose to change coordinate in (3.2) but that turns out to be more complicated) so that the cross-term between the new azimuthal coordinate and x^5 vanishes at infinity. This is achieved by setting

$$\Phi = \varphi' - \frac{1}{\nu}(B - 1/R)x^5 \tag{4.14}$$

and we obtain

$$\begin{aligned} ds^2 =& \frac{f(y)}{f(x)}\left(dx^5 + 2A'_{\varphi'}d\varphi'\right)^2 \\ &+ \frac{1}{A^2(x-y)^2}\left[-f(x)^2\left(\frac{g(y)d\tau^2}{f(y)} + \frac{dy^2}{g(y)}\right) + f(y)\left(\frac{f(x)dx^2}{g(x)} + g(x)d\varphi'^2\right)\right] \end{aligned} \tag{4.15}$$

where $A'_{\varphi'} = q(x - \xi_3)$. This is locally the Kaluza-Klein C-metric [3]. However, it differs in that the identifications on the coordinates ϕ and x^5 are similar to that of Kerr as given after (3.10) and so it still asymptotically approaches a Kaluza-Klein magnetic field with strength B. The fact that the coordinate transformation (4.14) results in the C-metric is an immediate result of the observation that the Ernst solution is obtained from the C-metric by just the reverse of this transformation [3].

One can now verify that (3.2) can be transformed into (4.15) by the coordinate identifications:

$$\begin{aligned}\varphi =&\nu\varphi' \\ \psi =&\nu\tau \\ r^2 =&-\mu\frac{(y-\xi_4)(x-\xi_2)}{(\xi_4-\xi_3)(x-y)} \\ \cos^2\theta =&-\frac{(y-\xi_3)(x-\xi_2)}{(\xi_3-\xi_2)(x-y)}\end{aligned} \tag{4.16}$$

It follows that the lorentzian Ernst solution, (4.1) with $\tau = it'$, and the doubly continued lorentzian Kerr solution (3.24) are the same. Thus the five-dimensional solution which was previously interpreted as describing two Kaluza-Klein monopoles accelerating apart in a magnetic field, is in fact the same as the one describing an expanding bubble. Contrary to one's expectation, the five-dimensional space does not have two localized regions of curvature. These appear in four dimensions as a result of the reduction. In fact, the two singularities that appear in the "shifted" reduction of the Kerr solution are now revealed to be none other than a Kaluza-Klein monopole and anti-monopole with charges $\pm R/4$.

The surprising equivalence between these two solutions raises a number of issues which we now address. In [2] it was shown that the centers of the monopoles in the extreme Kaluza-Klein Ernst solution are not really accelerating, but in fact follow geodesics in five dimensions. How is this compatible with the fact that this spacetime is equivalent to an expanding bubble? Since the worldlines for the monopole centers are the North and South poles of the bubble, this is consistent only if the bubble itself is not accelerating! To confirm this, consider the bubble (3.26) obtained from the five dimensional Schwarzschild solution. The worldline corresponding to constant r, θ, φ has tangent vector $u = (1/r)(\partial/\partial t)$. The acceleration of this worldline is $A_\nu = (1/r)\nabla_\nu r$ whose norm vanishes as one approaches the bubble at $r^2 = \mu$. Thus, a five-dimensional observer near the bubble does not have to undergo large acceleration to stay away from the bubble. A more general argument that the bubble does not accelerate (which applies to Kerr as well) is simply that it is the fixed point set of a continuous isometry and therefore must be totally geodesic [10]. In four dimensions however, observers do need to accelerate more and more to stay away from the singularity.

One can clearly take the angular momentum parameter to zero in the Kerr solution and obtain the Schwarzschild metric. What is the analog of this limit for the Ernst solution? From equation (4.13) we have

$$\mu = \frac{\alpha^2}{(\xi_3 - \xi_2)}(\xi_4 - \xi_3) \tag{4.17}$$

Since $\xi_4 - \xi_3$ is always finite, we see that the limit $\alpha \to 0$ with μ fixed corresponds to the limit $\xi_2 \to \xi_3$. Since the range of y in the Ernst instanton is $\xi_2 \leq y \leq \xi_3$, this is clearly a singular limit. To obtain a regular limiting geometry, one has to also rescale the coordinates x and y. The appropriate rescalings can be derived from (4.16) since they just correspond to keeping r and θ finite and nonzero. The result is a description of the Schwarzschild metric in accelerating coordinates. In some sense the usual Ernst coordinates include a factor of the Kerr angular momentum which must be removed before taking the Schwarzschild limit.

In a similar vein, one might ask what is the analog of the extreme Kerr solution. One can see from (3.1) that in five dimensions, the lorentzian Kerr solution never has a degenerate horizon. If we make the angular momentum parameter too large, the horizon becomes singular. However, the analytically continued Kerr instanton (3.2) is regular for all values of μ and α and thus there is no analog of the extremal limit. Of course, the extreme Ernst solution is itself the limit of a more general non-extreme solution. It follows that there is a deformation of the five-dimensional Kerr metric under which it describes two non-extreme Kaluza-Klein black holes accelerating apart. To obtain it one can, for example, substitute in the non-extremal C-metric x and y as functions of r and θ given by (4.16).

We have seen that the Kaluza-Klein Ernst solution can be rewritten in a simpler way as the Kerr solution. It is thus natural to ask whether the original Ernst solution in Einstein-Maxwell theory can similarly be rewritten in a simpler form. More generally, consider the one parameter family of theories with metric, Maxwell field, and dilaton where the parameter a governs the coupling between the dilaton and Maxwell field. There is an analog of the Ernst solution for each value of the parameter a [3]. Kaluza-Klein theory corresponds to $a^2 = 3$. One can, for example, utilise the coordinate transformation (4.16) to obtain another form of these metrics for $a^2 \neq 3$, perhaps leading to new insights.

In [2], the topology of the Kaluza-Klein Ernst instanton was shown to be S^5 with an S^1 removed. How does this compare with the topology of the Kerr instanton? The

Kerr instanton has the topology of a five-dimensional euclidean black hole: $R^2 \times S^3$ (with metrically the R^2 in the shape of a cigar). But $S^5 - S^1$ is equivalent to R^4 with a line removed, which is indeed $R^2 \times S^3$. So the spacetimes are equivalent globally, and not just locally. The lorentzian analog of this statement explains how the positive energy theorem is violated. The extremal Ernst solution has zero total mass since there is a boost symmetry but it is clearly not the Melvin background. However, as first pointed out by Witten [1], the positive mass theorem holds in Kaluza-Klein theory only if the manifold is globally of the form $M \times S^1$, for some four manifold M.

The topology of the spatial slices, including the zero-momentum slice, of the Kaluza-Klein Ernst-Kerr solution is $S^4 - S^1 \cong R^2 \times S^2$. We can argue that this is the spatial topology of *any* monopole anti-monopole configuration, for example one which is asymptotically the Kaluza-Klein vacuum rather than Melvin, as follows. The topology of a monopole-anti-monopole configuration can be described generally as the union $A \cup B \cup C$ where A and B are both four balls D^4 corresponding to the monopole and antimonopole and C is the non-trivial $U(1)$ bundle over $R^3 \# D^3 \# D^3$ (R^3 with two three balls removed) which has zero winding over the sphere at infinity, and windings $+1$ and -1 over the other two S^2 boundaries [13]. This description fixes the topology uniquely and is independent of whether the metric tends to the vacuum or a magnetic field at infinity. Thus the topology must be $R^2 \times S^2$, since we know that is the topology of the pole-anti-pole configuration in Kaluza-Klein Ernst-Kerr.

A final interesting observation is that after two Kaluza-Klein monopoles are created and accelerate away to infinity, the spacetime dynamically decompactifies. This is most easily seen using the Ernst form of the metric (4.1). The coordinate y becomes timelike for $y > \xi_3$, and the late time behavior corresponds to fixing x ($\neq \xi_3$) and all coordinates other than y, and then taking the limit $y \rightarrow x$. It is easy to see that in this limit, g_{55} diverges. In other words the fifth dimension decompactifies and the four-dimensional reduction is no longer valid. We should note that this is the case at the level of the whole solution and since we are only considering the solution close to the axis, as an approximation to the decay of a real magnetic field, its physical significance is unclear.

5. Discussion

To summarise, we have seen that magnetic fields in Kaluza-Klein theory are described by five dimensional Minkowski space with twisted identifications. The four-dimensional reduction, however, is only valid for four-dimensional magnetic field parameter values $|B| << 1/R$ and for distances from the axis of symmetry that are $< 1/|B|$. We justify the latter by arguing that magnetic fields in the real world will have finite spatial extent.

We have demonstrated that the euclidean five-dimensional Kerr metric gives an instanton describing the decay of Kaluza-Klein magnetic fields, and argued that for a physical four-dimensional magnetic field (i.e. $|B| << 1/R$) there are two ways it can decay: by producing single singularities into which space "collapses," or by producing pairs of monopole-anti-monopole pairs which accelerate off to infinity. The former type of decay is much more likely, for small fields, than the latter. We have seen that this Kerr instanton is, in fact, identical to the Kaluza-Klein Ernst instanton, and discussed several consequences of this surprising fact. We have also shown that the fifth dimension tends to decompactify dynamically after the second, rarer decay by pair production.

Thus we arrive at the final four-dimensional picture. If the magnetic field is zero, then the vacuum decays by endlessly producing apparent naked singularities. From the five-dimensional point of view these correspond to Witten's "bubbles of nothing" which must eventually collide, and so in the four dimensional description the singularities will coalesce. If we start at time $t = 0$ in the Kaluza-Klein vacuum (though it is hard to imagine, given these instabilities, how we could have gotten into the vacuum in the first place) then at any finite time, there is still an infinite amount of flat space left and the process will continue forever. If there is any non-zero magnetic field present, then, as well as this decay, there is a small chance that a pair of monopoles will be pair created.

Since many currently popular unified theories include extra spatial dimensions, it is important to ask why the never-ending bubble nucleation discussed here is not a problem for these theories. One resolution was proposed in [1], and is applicable if the theory contains fundamental fermions. In five dimensional Kaluza-Klein theory, there are inequivalent ways to include elementary fermions; one must specify the periodicity of the fermions around the compact direction or in other words a spin structure. When spacetime has topology $R^4 \times S^1$ there are two choices for the spin structure (for simplicity, we will assume that the fermions are not coupled to any extra $U(1)$ symmetry). In the vacuum, the choices can be distinguished by asking how spinors transform as they are parallelly transported around the fifth direction. For one spin structure they come back to themselves, for the other they pick up a minus sign. As Witten pointed out, the five dimensional Schwarzschild instanton has topology $R^2 \times S^3$ and consequently a unique spin structure. Asymptotically, Schwarzschild tends to the Kaluza-Klein vacuum and one can ask which spin structure one obtains there. It turns out that a spinor picks up a minus sign under parallel propagation around the fifth direction. Thus, if one chooses the other spin structure for the vacuum (which is the conventional choice - required to have massless fermions and supersymmetry), it cannot decay via the Schwarzschild instanton.

What about magnetic fields? Since the Melvin solution again has topology $R^4 \times S^1$, there are two spin structures. Even though the spacetime is locally flat, the nontrivial indentifications imply that if a vector is parallelly propagated around the S^1, it will return rotated by an angle $2\pi RB$. It follows that for one spin structure, parallel propagation of a spinor around the fifth direction results in the spinor acquiring a phase $e^{\pi RB\gamma}$, where γ is a generator of the Lie algebra of the spin group Spin(5) and $\gamma^2 = -1$. For the other spin structure, parallel propagation gives a phase $-e^{\pi RB\gamma}$.

Since the topology of the Kerr instanton (3.2) is again $R^2 \times S^3$ it also has a unique spin structure. It tends to Melvin at infinity and one can show that spinors pick up the phase $-e^{\pi R\frac{\alpha}{\mu}\gamma}$ under parallel transport around the closed integral curves of l, (3.5), at infinity. Now, for a given four dimensional magnetic field of parameter B, there are two instantons that describe its decay, as we discussed: (3.2) with (i) $\alpha/\mu = B$ and reduced along l and (ii) $\alpha/\mu = B - 1/R$ and reduced along $\hat{l} = l + (1/R)\partial_\varphi$. Instanton (i) has spinors which pick up the phase $-e^{\pi RB\gamma}$ when parallelly transported around orbits of l. In the case of (ii) one might think that the extra rotation involved in the definition of the internal direction would introduce an extra minus sign into the phase. This is not the case. Since the spacetime is almost flat near infinity, this extra rotation has the same effect as parallelly propagating a spinor around a circle in flat spacetime. It does not introduce another minus sign. Thus spinors on (ii) pick up a phase $-e^{\pi R(B-1/R)\gamma} = +e^{\pi RB\gamma}$ on parallel transport around orbits of l' (and l). We see that the spin structures on the two different instantons correspond to the two inequivalent choices of spin structure on the Melvin background. Thus choosing the one in which spinors pick up the phase $e^{\pi RB\gamma}$ (which is the natural generalization, for small B, of the standard choice) rules out decay via instanton (i), the "bubble" type decay, but allows decay via (ii), the pair creation of monopoles.

It is natural to wonder what the implications of our results are for string theory. The Kaluza-Klein monopole solves the string equations of motion to leading order in α'. For large R the five dimensional curvature is small and we do not expect significant α' corrections [14]. Although it is not yet known how to determine the soliton spectrum in string theory, it is natural to assume that the Kaluza-Klein monopole solution corresponds to a state in the Hilbert space of toroidal compactifications (although the fact that this solution does not approach the standard Kaluza-Klein vacuum at infinity [15] is a subtlety that needs to be addressed). Supersymmetric toroidally compactified string theories are conjectured to be invariant under strong weak coupling duality (see for example [16],[17],[18]). The states dual to the Kaluza-Klein monopoles are electrically charged string winding states. It would be interesting to calculate the pair production rate for the elementary string states in the first quantised theory and compare this with the rate calculated using the space-time instanton techniques employed here. This is currently under investigation.

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