Biharmonic Conformal Field Theories

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Abstract

The main result of this paper is the construction of a conformally covariant operator in two dimensions acting on scalar fields and containing fourth order derivatives. In this way it is possible to derive a class of Lagrangians invariant under conformal transformations. They define conformal field theories satisfying equations of the biharmonic type. Two kinds of these biharmonic field theories are distinguished, characterized by the possibility or not of the scalar fields to transform non-trivially under Weyl transformations. Both cases are relevant for string theory and two dimensional gravity. The biharmonic conformal field theories provide in fact higher order corrections to the equations of motion of the metric and give a possibility of adding new terms to the Polyakov action.

1 Introduction

The main result of this paper is the construction of an operator in two dimensions with fourth order derivatives. In this way it is possible to derive a class of Lagrangians invariant under the Virasoro group [3]. They define conformal field theories (CFT's) satisfying equations of the biharmonic type. The biharmonic CFT's discussed here are interesting under several points of view, from string theory [4] to two dimensional gravity [5]. The point of view chosen in this paper starts from the two dimensional $(2-d)$ Maxwell field theory. In the Lorentz gauge, after eliminating the longitudinal degrees of freedom, the Maxwell theory becomes a biharmonic scalar field theory, satisfying biharmonic equations of motion $\triangle^2\varphi = 0$. Here $\triangle = \partial_\mu\partial^\mu$ denotes the $2-d$ Laplacian and φ is a scalar field. Of course, the biharmonic equation is not conformally invariant. This is another way to state that conformal transformations are not a symmetry of $2-d$ Quantum Electrodynamics. Exploiting an old theorem [1], it is however possible to make the biharmonic equation invariant under the $SL(2,\mathbf{C})$ subgroup of the Virasoro group, allowing for point transformations in the scalar fields φ. In this way one obtains a curious example of $SL(2,\mathbf{C})$ invariant gauge field theory, without the need of introducing vector fields [2].

Much more difficult is the task of extending the set of symmetries of the biharmonic equation to the whole Virasoro group. One may think for example to construct a conformal invariant operator of the fourth order K adding to the biharmonic operator $\triangle^2$ a suitable linear combination of lower order operators, built out of the metric $g_{\mu\nu}$, the covariant derivatives ∇_μ and the Ricci tensor $R_{\mu\nu\rho\sigma}$ [6]:

$$K = \triangle^2 + aR^{\mu\nu}\nabla_\mu\nabla_\nu + bR\,\triangle + cg^{\mu\nu}(\nabla_\mu R)\nabla_\nu + dR^2 +$$

$$eR^{\mu\nu}R_{\mu\nu} + fR^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} + g\,\triangle R$$

Unfortunately this approach fails in two dimensions, since the coefficients $a - g$, chosen in such a way that K becomes invariant under Weyl rescalings of the metric $g_{\mu\nu}$, diverge.

To solve this problem, we define here a generalized Laplacian $D[A_{\mu\nu}, a_\mu]$ which transforms as follows:

$$D\varphi' = e^\phi D\varphi \tag{1}$$

under a rescaling of the fields of the kind $\varphi' = e^\phi \varphi$. D is constructed using the same strategy of the covariant derivatives in gauge theories but here, besides a vector gauge field a_μ, also a spin two gauge field $A_{\mu\nu}$ is needed in order to fulfill 1. For a conformal transformation $\zeta = \zeta(z)$, where ζ and z are complex coordinates, the real function ϕ appearing in eq. 1 is defined by $\phi = -\log \left| \frac{dz}{d\zeta} \right|$.

Starting from the generalized Laplacian D, we are able to construct a set of interacting conformal field theories, called here of type I, with equations of motion of the biharmonic type

$$D^2 \varphi = \frac{\delta V(\varphi)}{\delta \varphi} \tag{2}$$

Here $V(\varphi)$ represents an interaction potential. The above equations of motion are invariant under the Virasoro group. However, Weyl rescalings of the metric are a symmetry of the theory only in the free case, i.e. when $V(\varphi) = 0$. The action of type I field theories is renormalizable if the gauge fields $A_{\mu\nu}$ and a_μ are external. The free energy–momentum tensor can be easily computed and it is traceless. The problem of introducing kinetic terms also for the gauge fields without spoiling renormalizability, locality or conformal invariance seems instead without solution.

In a slight variation of the biharmonic CFT's presented here, called type II for convenience, Weyl rescalings are admitted for the scalar fields, i.e. $\varphi' = e^\phi \varphi$ if the metric undergoes a Weyl transformation of the kind $g'_{\mu\nu} = e^{2\phi} g_{\mu\nu}$. Clearly, the generalized Laplacian D is covariant with respect to this transformation but, as we will see, it is no longer possible to define Weyl covariant equations of motion also at the free level. Despite of this fact, if a conformal metric $g_{\mu\nu} = h(x)\delta_{\mu\nu}$ is chosen, the type II biharmonic CFT's continue to remain invariant under a conformal change of variables $\zeta = \zeta(z)$. The energy–momentum tensor is however not traceless. The second type of biharmonic CFT's plays an important role in $2 - d$ gravity, representing possible corrections to Liouville field theory with higher order derivatives.

The second point of view from which the biharmonic CFT's can be considered is that of string theory and conformal field theory. Let us in fact consider the string action in the Polyakov formulation [7]:

$$S_{str} = \int d^2 x \left[\partial_\mu X \partial^\mu X + RX \right] \tag{3}$$

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How much can this action be generalized, maintaining the properties of conformal invariance and locality? This is a physically important question, since the coupling of string theory with other CFT's yields remarkable consequences, as it has been shown for instance in the case of the Thirring model [8] discussed in ref. [9], where a new method of spontaneous symmetry breaking was found in this way. Unfortunately, the set of CFT's admitting a Lagrangian formulation and which may be coupled to string theory is very limited. The biharmonic CFT's introduced in this paper enlarge this set to a new class of conformal field theories. Some interesting possibilities of letting both type I and type II CFT's interact with string theory will be given below.

A third and final point of view which will be only briefly discussed here is that of $2 - d$ gravity. In the second type of biharmonic CFT's, where the scalar fields φ are allowed to rescale according to a Weyl transformation of the metric explained above, it is possible to perform the following identification: $\varphi = |g|^{-\frac{1}{4}}$, where g is the determinant of the metric. Thus the type II biharmonic CFT's provide local equations of motion in $|g|^{-\frac{1}{4}}$ and consequently in the metric. As mentioned before, the type II equations of motion are no longer Weyl invariant. Therefore, conversely to what happens in string theory, also the conformal factor of the metric can be determined adding biharmonic CFT's to the Polyakov action 3. With respect to the Liouville field theory, the biharmonic corrections to $2 - d$ gravity have the advantage of providing simple equations, e.g. in the free case, in which the potential $V(\varphi)$ is set to zero.

In the following, the biharmonic CFT's will be treated in details, both in the Weyl invariant and Weyl noninvariant versions. Possible ways to couple them to string theory will be discussed, together with their interpretation as generalizations of the action for $2 - d$ gravity containing higher order derivatives.

2   Biharmonic Conformal Field Theories

Let us start by considering the action of $2 - d$ Quantum Electrodynamics (QED). In complex coordinates $z = x_1 + ix_2$, $\overline{z} = x_1 - ix_2$, we have:

$$S_{Maxwell} = \int d^2z F_{z\overline{z}}^2 \tag{4}$$

with $F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z$. Complex indices will be denoted with the first letters of the Greek alphabet, for instance $\alpha = z, \bar{z}$. Exploiting the Hodge decomposition for the gauge fields A_α:

$$A_z = \partial_z \chi + \partial_z \varphi \qquad\qquad A_{\bar{z}} = \partial_{\bar{z}} \chi - \partial_{\bar{z}} \varphi \qquad (5)$$

one obtains:

$$S_{Maxwell} = \int d^2 z (\triangle \varphi)^2 \qquad (6)$$

Thus the QED action is not conformally invariant in $2-d$. To see this more in details, let us consider the biharmonic equations of motion coming from 6:

$$\triangle(\triangle \varphi) = 0 \qquad (7)$$

Performing a conformal change of variables $\zeta = \zeta(z)$, the Laplacian transforms as follows: $4\partial_z \partial_{\bar{z}} = \frac{4}{J^2} \partial_\zeta \partial_{\bar{\zeta}}$, where $J = \left| \frac{dz}{d\zeta} \right|$. Therefore, the biharmonic equation 7 becomes in the new coordinates:

$$\frac{1}{J^2} \partial_\zeta \partial_{\bar{\zeta}} \left(\frac{1}{J^2} \partial_\zeta \partial_{\bar{\zeta}} \varphi \right) = 0$$

and it is clearly not conformally invariant unless J is a constant.

One way to extend the set of transformations leaving the biharmonic equation invariant is to allow for point transformations in the field φ of the kind [1]:

$$\varphi(z, \bar{z}) = J \varphi' \left(\zeta, \bar{\zeta} \right) \qquad (8)$$

If φ obeys the above requirement, the action 6 becomes invariant under the Möbius group of transformations $\zeta = \frac{az+b}{cz+d}$, with a, b, c, d being arbitrary complex numbers satisfying the condition $ad - bc \neq 0$. As pointed out in ref. [2], this is a remarkable theory, having an SL(2,**C**) group of local symmetries and no gauge fields. On a sphere, the gauge group coincides with the SL(2,**C**) group of automorphisms admitted on a genus zero Riemann surface. Other symmetries of the biharmonic equations have been obtained in [10]. In order to obtain invariance under the whole set of conformal transformations, however, the introduction of gauge fields is necessary.

To this purpose, we define a new Laplacian $D_{z\bar{z}}$, requiring that

$$D_{z\bar{z}} \varphi'(z, \bar{z}) = e^{\phi(z, \bar{z})} D_{z\bar{z}} \varphi(z, \bar{z}) \qquad (9)$$

if φ is rescaled by an arbitrary real factor:

$$\varphi'(z,\overline{z}) = e^{\phi(z,\overline{z})}\varphi(z,\overline{z}) \tag{10}$$

To fulfill requirement 9, let us try the following ansatz:

$$D_{z\overline{z}}\varphi = [\partial_z\partial_{\overline{z}} + aA_{z\overline{z}} + b(a_z\partial_{\overline{z}} + a_{\overline{z}}\partial_z) + c(a_z a_{\overline{z}})]\varphi \tag{11}$$

where $A_{z\overline{z}}$ and a_α are gauge fields undergoing suitable transformations in order to reabsorb all the unwanted terms in $\phi(z,\overline{z})$ which arise in the rhs of eq. 9 after exploiting the derivatives $\partial_z\partial_{\overline{z}}$. In the case of a conformal change of variables $\zeta = \zeta(z)$, the scalar field φ transforms as in eq. 8. Therefore, putting $\phi(z,\overline{z}) = -\log J$, with $J = \left|\frac{dz}{d\zeta}\right|$ in eqs. 9 and 10, it is easy to realize that the new Laplacian $D_{z\overline{z}}$ is covariant also under conformal transformations:

$$D_{\zeta\overline{\zeta}}\left[A_{\zeta\overline{\zeta}}, a_\zeta, a_{\overline{\zeta}}\right]\varphi'\left(\zeta,\overline{\zeta}\right) = JD_{z\overline{z}}\left[A_{z\overline{z}}, a_{\overline{z}}\right]\varphi(z,\overline{z}) \tag{12}$$

Eqs. 9 and 12 are verified performing in eq. 11 the substitutions:

$$a = -1 \qquad b = -1 \qquad c = 1 \tag{13}$$

Moreover, the fields $A_{z\overline{z}}$ and a_α should obey the following transformations:

$$A_{\zeta\overline{\zeta}}d\zeta d\overline{\zeta} = (A_{z\overline{z}} + \partial_z\partial_{\overline{z}}\phi(z,\overline{z}))\,dzd\overline{z} \tag{14}$$

$$a_\zeta d\zeta + a_{\overline{\zeta}}d\overline{\zeta} = (a_z + \partial_z\phi(z,\overline{z}))\,dz + (a_{\overline{z}} + \partial_{\overline{z}}\phi(z,\overline{z}))\,d\overline{z} \tag{15}$$

Starting from the modified Laplacian 11, we can easily build the free action of type I biharmonic CFT's:

$$S_{conf} = \int d^2z\,(D_{z\overline{z}}\varphi)^2 \tag{16}$$

Apparently, the above Lagrangian density

$$L_{zz\overline{zz}} = (D_{z\overline{z}}\varphi)^2$$

has the wrong tensorial properties. Proper Lagrangians in complex $2-d$ coordinates should be in fact $(1,1)$ tensors $T_{z\overline{z}}$ with one complex and a complex conjugate indices. For instance, the Lagrangian of massless scalar fields

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$X(z, \overline{z})$ is given by $L_{z\overline{z}} = \partial_z X \partial_{\overline{z}} X$. However, due to eq. 8, the scalar fields φ behave as $\varphi \sim \frac{1}{\sqrt{g_{z\overline{z}}}}$, i.e. they may be regarded as $\left(\frac{1}{2}, \frac{1}{2}\right)$ differentials. This fact assures the covariance of S_{conf} also under general diffeomorphisms, as we will see below. Let us notice that if we insist in the interpretation of φ as a spinor, no ambiguity with the spin structures arises, because φ transforms as an element of $\left(\frac{1}{2}, 0\right) \otimes \left(0, \frac{1}{2}\right)$. As a consequence, the multivaluedness of the $\left(\frac{1}{2}, 0\right)$ component induced by the presence of an hypothetical spin structure cancels against the phase of the $\left(0, \frac{1}{2}\right)$ component. Thus S_{conf} describes a well defined conformal field theory with higher order derivatives. As in string theory, the dependence on the metric disappears once conformal coordinates are chosen. In $2 - d$ this is a great advantage, since every Riemannian manifold is conformally flat. Let us now discuss possible kinetic terms for the gauge fields $A_{z\overline{z}}$ and a_α. A simple way to introduce a dynamics for the gauge fields is suggested by the transformation rules 14–15 and consists in rewriting $A_{z\overline{z}}$ and a_α in terms of two other scalar fields A and a:

$$A_{z\overline{z}} = \partial_z \partial_{\overline{z}} A$$

$$a_z = \partial_z a \qquad a_{\overline{z}} = \partial_{\overline{z}} a$$

For consistency with eqs. 14–15, A and a should transform as follows under a conformal change of variables:

$$A' = A + \phi(z, \overline{z}) \qquad a' = a + \phi(z, \overline{z})$$

It is now easy to define a conformally invariant kinetic term for A and a:

$$S_{kin} = \int d^2 z \partial_z (A - a) \partial_{\overline{z}} (A - a)$$

In this way, however, the action $S = S_{conf} + S_{kin}$ becomes no longer renormalizable. Other attempts to build local kinetic terms for the gauge fields either lead to non-renormalizable actions or break the conformal symmetry. Hence, we will assume in the following that the fields $A_{z\overline{z}}$ and a_α are external fields. This is not a serious limitation. Effective Lagrangians without the presence of those gauge fields can be always constructed solving the classical equations of motion in $A_{z\overline{z}}$ and a_α. Alternatively, one may integrate out the external fields in the path integral using suitable weights in order to assure convergence.

At this point, we compute the free energy momentum tensor of the conformal field theories of type I defined by the action S_{conf}. Using a general, not conformally flat metric, we obtain in real coordinates the following expression for the functional S_{conf}:

$$S_{conf} = \int d^2x \, g \left[\triangle\varphi - g^{\mu\nu} \left(A_{\mu\nu} + a_\mu\partial_\nu - a_\mu a_\nu \right) \varphi \right]^2 \tag{17}$$

where $g = |\det g^{\mu\nu}|$ and $\triangle\varphi = \frac{1}{\sqrt{g}}\partial_\mu \left(\sqrt{g}g^{\mu\nu}\partial_\nu\varphi \right)$. In eq. 17 the determinant g of the metric and not its square root appears. The reason is the presence of the field φ with the transformation rule 8, suitably generalized to the case of general diffeomorphisms $y^\mu = y^\mu(x)$ putting $J = \det\left|\frac{dx^\mu}{dy^\nu}\right|$. Clearly the action 17 is invariant under general diffeomorphisms and also under a Weyl rescaling of the metric $g'_{\mu\nu} = e^{\phi(x)}g_{\mu\nu}$. Setting $l_{\mu\nu}\varphi = -\left(A_{\mu\nu} + a_\mu\partial_\nu - a_\mu a_\nu \right)\varphi$, the energy momentum tensor $(T_{conf})_{\kappa\lambda} \equiv \frac{\delta S_{conf}}{\delta g_{\mu\nu}}$ can be written as follows:

$$(T_{conf})_{\kappa\lambda} = 2\sqrt{g} \left(\triangle\varphi + g^{\mu\nu}l_{\mu\nu}\varphi \right) \cdot$$

$$\left[\partial_\mu \left(\frac{1}{2}\sqrt{g}g^{\kappa\lambda}g^{\mu\nu}\partial_\nu\varphi - \sqrt{g}g^{\mu\kappa}g^{\nu\lambda}\partial_\nu\varphi \right) + \left(\frac{1}{2}\sqrt{g}g^{\kappa\lambda}g^{\mu\nu} - \sqrt{g}g^{\mu\kappa}g^{\nu\lambda} \right) l_{\mu\nu}\varphi \right] \tag{18}$$

From eq. 18 it turns out that $(T_{conf})_{\kappa\lambda}$ is traceless, as it should be in a conformal field theory.

From the transformation law 8, one can also give to the field φ an interpretation as the inverse of the fourth root of the metric. Let us discuss this case, performing the identification

$$\varphi = g^{-\frac{1}{4}} \tag{19}$$

Since g is a semipositive definite quantity, (we allow also for degenerate metrics), there is no ambiguity in taking its fourth root. Moreover, as explained before, all the problems with the spin structures are absent, despite of the fact that now φ has spinorial indices. Substituting eq. 19 in eq.16, S_{conf} becomes

$$S_{conf} = \int d^2z \left[D_{z\bar{z}} \; g^{-\frac{1}{4}} \right]^2 \tag{20}$$

and it is local in $g^{-\frac{1}{4}}$. Thus S_{conf} may be considered in this case as a possible correction of the $2-d$ gravity action, characterized by the presence of higher

order derivatives. Remarkably, after the identification 19, S_{conf} remains conformally invariant, but clearly the invariance under Weyl rescalings of the metric is lost. Therefore, contrary to the Polyakov formulation of string theory, the equations of motion coming from S_{conf} are also able to determine the conformal factor of the metric. In principle, we can define an action for $\varphi = (g)^{-\frac{1}{4}}$ that satisfies both Weyl and conformal symmetries:

$$S'_{conf} = \int d^2x \ g^{\frac{1}{4}} \left[(\triangle - g^{\mu\nu} l_{\mu\nu}) \ g^{\frac{1}{4}} \right] \tag{21}$$

In a conformally flat metric:

$$S'_{conf} = \int d^2z \ g^{-\frac{1}{4}} \left(D_{z\bar{z}} \ g^{\frac{1}{4}} \right)$$

As we see, S'_{conf} is unfortunately non–local in the field $\varphi = (g)^{-\frac{1}{4}}$ and therefore we will neglect it. In general, for a field φ transforming as $(g)^{-\frac{1}{4}}$, we will have that the free action of type II biharmonic CFT's coincides with S_{conf}.

Self-interactions of the metric field can be added for both type I and type II theories introducing the following functional:

$$S_{int} = \int d^2z V_n(\varphi) \tag{22}$$

Substituting

$$G_{z\bar{z}} = A_{z\bar{z}} - \frac{1}{2} \left(\partial_z a_{\bar{z}} + \partial_{\bar{z}} a_z \right)$$

the most general possible potential $V_n(\varphi)$ is given by:

$$V_n(\varphi) = a_{-1}\varphi^{-2} + a_0 G_{z\bar{z}} + a_1 G_{z\bar{z}}^2 \varphi^2 + a_2 G_{z\bar{z}}^3 \varphi^4 + \ldots + a_n G_{z\bar{z}}^n \varphi^{2n} \tag{23}$$

with $a_1 \ldots a_n$ denoting a set of real coupling constants. Eqs. 22–23 describe a well defined conformal action, invariant under the transformations 8, 12 and 14–15. However, in both cases of type I and II biharmonic CFT's, the action S_{int} is not invariant for Weyl rescalings. The above potential is interesting also because it yields several possibilities of coupling the biharmonic CFT's with the basic fields of string theory, i.e. the scalar fields X, the metric $g_{\mu\nu}$ and, in the super case , the spin currents of the kind $J_{\theta\bar{\theta}} = \overline{\psi}_\theta \psi_{\bar{\theta}}$.

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3   Conclusions

Concluding, let us briefly summarize our results. Starting from the new Laplacian defined by eqs. 9, 11, 13 and 14–15, we are able to construct conformal field theories with biharmonic equations of motion. Type I biharmonic CFT's are characterized by the free action 16. After a conformal transformation, the scalar fields φ undergo the point transformation defined in eq. 10. The type I free biharmonic CFT is both Weyl and conformal invariant. Its energy–momentum tensor is traceless. The Weyl invariance is however lost if a self–interacting Lagrangian is added (eqs. 22–23).

In type II biharmonic CFT's φ is allowed to rescale under Weyl transformations. The action is again provided by eq. 16, but in this case the Weyl invariance is lost. It is however possible to define an action with second order derivatives which is both invariant under Weyl and conformal transformations (eq. 21). Unfortunately, this action is non-local in the scalar fields. After the identification 19, the type II biharmonic CFT's become particularly relevant in $2-d$ gravity and string theory. Also for type II theories one can add to the free action an interacting term S_{int} of the kind given in eqs. 22–23.

Finally, many interesting issues could not be treated here because they are outside of the aims of this letter. For instance the relation, if it exists, between the biharmonic CFT's and the usual ones [11]. Also the problem of their quantization and several aspects connected to possible consequences in string theory could not be fully developed due to the lack of space. We hope however to have drawn with this letter the attention to the interesting class of biharmonic conformal field theories. They provide in fact local and simple equations of motion for the metric in two dimensions (see e.g. eq. 20) and can be easily coupled with the fields entering superstring theory. In this way it seems possible to extend the Polyakov action introducing new terms and new free parameters. Hopefully this will provide more degrees of freedom, that will at least be sufficient to settle some of the remaining problems of string theories.

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