Thermal Wightman Functions and Renormalized Stress Tensors in the Rindler Wedge

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Abstract

The Wightman functions in the Rindler portion of Minkowski space-time are presented for any value of the temperature and for massless spin fields up to $s = 1$ and the renormalized stress tensor relative to Minkowski vacuum is discussed. The failure of the method of images to reproduce the correct results for massless $s = 1$ fields is ponited out.

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1 Results

The Rindler regions can be defined with respect to any spacelike two-plane $\mathcal{P}$ in Minkowsi space-time, by taking the pair of wedges which are bounded by the two families of null geodesics which are orthogonal to both sides of the plane. We may choose rectangular coordinates (x, y, z, t), such that the plane is the set $x = t = 0$. The Rindler wedge we shall consider, denoted W_R, will then be defined by the inequality $x > |t|$. A global parametrization of W_R is obtained by setting $x = \xi \cosh \tau$, $t = \xi \sinh \tau$, for $\xi > 0$, so that $x^2 - t^2 = \xi^2$. Thus any line $\xi = \xi_0$, $y = y_0$, $z = z_0$ will be the trajectory of a uniformly accelerated particle, with proper acceleration $a = \xi_0^{-1}$ and proper time $s = a\tau$ along the trajectory. The space-time metric will take the form $ds^2 = -\xi^2 d\tau^2 + d\xi^2 + dx_t^2$, with $\xi > 0$ and $x_t = (y, z)$ standing for the transverse coordinates. The metric admits the timelike Killing field $K = \partial_\tau$ generating the isometry $\tau \to \tau + \tau_0$. The hypersurface $\xi = 0$ is an event horizon bifurcating in the transverse two-plane $\mathcal{P}$.

We shall find the thermal Wightman functions in the Rindler region W_R (the left region W_L is then covered by reflecting through the wedge, namely by sending $(t, x, x_t) \to (-t, -x, x_t)$). Hence it is understood that fields quantization in this region is defined by taking the Fock representation over a vacuum $|F>$ which is invariant under translations in τ (it is customary to call $|F>$ the Fulling vacuum[1, 2, 3, 4]. The vacuum Wightman functions for a general field $\phi_A(x)$ are then defined as the expectation values

$$W^+_{AB}(x, x') =< F|\phi_A(x)\phi_B(x')|F >, \qquad W^-_{AB}(x, x') =< F|\phi_B(x')\phi_A(x)|F > \qquad (1)$$

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The definition will be the same for other states as well, in particular for the Minkowski vacuum $|M>$. The situation will be rather different for a thermal equilibrium state, since then there is no obvious way to compute the expectation values. This is because the partition function for a quantum field is divergent in the infinite volume of the Rindler region. The thermal Wightman functions will then be defined as the periodic or anti-periodic solution of the field equations having the analyticy properties which are demanded by the KMS condition[5, 6]. An independent check will be then to recover the vacuum expectation values in the limit $\beta \to \infty$ of zero temperature. For future reference we define the quantity α by

$$\cosh\alpha = \frac{\xi^2 + \xi'^2 + |x_t - x_t^{'}|^2}{2\xi\xi'}$$

The Weyl and electromagnetic fields will be defined with respect to the natural orthonormal vierbein

$$e^a_{(0)} = \xi^{-1}\delta^a_0, \qquad e^a_{(i)} = \delta^a_i, \qquad i = 1, 2, 3$$

where $a, b, c, ..$ denotes coordinate indices and $i, j, k, ..$ anholonomic, or vierbein indices. The thermal Wightman functions at inverse temperature β for a massless field with elicity $s > 0$ will be denoted by $W^{(s)\pm}(\beta|x, x^{'})$ and simply by $W^{\pm}(\beta|x, x^{'})$ in the spin zero case. We give them first and then we discuss how they were obtained. They are given as follows:

1. the scalar $s = 0$ field

$$W^+(\beta|x, x^{'}) = \frac{1}{4\pi\beta\xi\xi'\sinh\alpha}\left[\frac{\sinh\frac{2\pi}{\beta}\alpha}{\cosh\frac{2\pi}{\beta}\alpha - \cosh\frac{2\pi}{\beta}(\tau - \tau' - i\varepsilon)}\right] \tag{2}$$

and $W^-(\beta|x, x^{'}) = [W^+(\beta|x, x^{'})]^*$. These are manifestly periodic in imaginary time with period β. To our knowledge, this result was first obtained by J.S.Dowker [7, 8].

   (a) the zero temperature limit is

$$W^+(x, x^{'}) =< F|\phi(x)\phi(x^{'})|F >= -\frac{1}{4\pi^2}\frac{\alpha}{\xi\xi'\sinh\alpha}\frac{1}{(\tau - \tau' - i\varepsilon)^2 - \alpha^2} \tag{3}$$

   and $W^-(x, x^{'}) =< F|\phi(x^{'})\phi(x)|F >= [W^+(x, x^{'})]^*$. Note that these functions are vacuum expectation values in the Fulling state.

   (b) the thermal function can be obtained by the sum over images

$$W^{\pm}(\beta|x, x^{'}) = -\frac{1}{4\pi^2}\frac{\alpha}{\xi\xi'\sinh\alpha}\sum_{n=-\infty}^{\infty}\frac{1}{(\tau - \tau' \mp i\varepsilon - in\beta)^2 - \alpha^2} \tag{4}$$

   (c) the value $\beta = 2\pi$ is distinguished by the property

$$\begin{aligned} W^+(2\pi|x, x^{'}) &= \frac{1}{4\pi^2}\frac{1}{\xi^2 + \xi'^2 + |x_t - x_t^{'}|^2 - 2\xi\xi'\cosh(\tau - \tau' - i\varepsilon)} \\ &= -\frac{1}{4\pi^2}\frac{1}{(t - t' - i\varepsilon)^2 - (x - x')^2 - |x_t - x_t^{'}|^2} \end{aligned} \tag{5}$$

   which is just the Wightman function which characterizes the Minkowski vacuum state. This means that this vacuum is a KMS state with respect to τ-translation [9, 2, 3, 10, 7, 11], with inverse temperature $\beta = 2\pi$.

(d) The thermal stress tensor relative to the Minkowski vacuum is [12, 13, 14]

$$T^{ab} = \frac{1}{1440\pi^2\xi^4}\left[\left(\frac{2\pi}{\beta}\right)^4 - 1\right][4v^a v^b + g^{ab}] \tag{6}$$

where $v^a = K^a/\sqrt{-K^2}$, $K = \partial_\tau$ being the Killing vector field of the Rindler region. The zero temperature stress tensor reduces to the one calculated in Ref.[15].

2. Weyl $s = 1/2$ fermions. There are two irreducible representations of the Dirac algebra. Denoting by σ the Pauli matrices, these are given by $\sigma^i = (e, \sigma)$ and $\tilde{\sigma}^i = (e, -\sigma)$, where e is the unit. In the tilde representation we find

$$W^{(1/2)+}(\beta|x, x^{'}) = i\tilde{\sigma}^a\tilde{\nabla}_a F^+_\beta(x, x^{'})$$

$$W^{(1/2)-}(\beta|x, x^{'}) = -i\tilde{\sigma}^a\tilde{\nabla}_a F^-_\beta(x, x^{'})$$

where

$$\begin{aligned} F^+_\beta(x, x^{'}) &= \frac{e}{4\pi\beta\xi\xi'\sinh\frac{\alpha}{2}}\left[\frac{\sinh\left(\frac{\pi}{\beta}\alpha\right)\cosh\left[\frac{\pi}{\beta}(\tau-\tau^{'})\right]}{\cosh\left(\frac{2\pi}{\beta}\alpha\right) - \cosh\left[\frac{2\pi}{\beta}(\tau-\tau'-i\varepsilon)\right]}\right] \\ &+ \frac{\sigma_1}{4\pi\beta\xi\xi'\cosh\frac{\alpha}{2}}\left[\frac{\cosh\left(\frac{\pi}{\beta}\alpha\right)\sinh\left[\frac{\pi}{\beta}(\tau-\tau^{'})\right]}{\cosh\left(\frac{2\pi}{\beta}\alpha\right) - \cosh\left[\frac{2\pi}{\beta}(\tau-\tau'-i\varepsilon)\right]}\right] \end{aligned} \tag{7}$$

These are manifestly anti-periodic in imaginary time with period β, in accord with the KMS condition.

(a) the zero temperature limit is

$$W^{(1/2)+}(x, x^{'}) = < F|\psi(x)\psi^\dagger(x^{'})|F > = i\tilde{\sigma}^a\tilde{\nabla}_a F^+(x, x^{'})$$

$$W^{(1/2)-}(x, x^{'}) = - < F|\psi^\dagger(x^{'})\psi(x)|F > = -i\tilde{\sigma}^a\tilde{\nabla}_a F^-(x, x^{'})$$

where

$$\begin{aligned} F^+(x, x^{'}) &= -\frac{e}{8\pi^2\xi\xi'\sinh\frac{\alpha}{2}}\left[\frac{\alpha}{(\tau-\tau'-i\varepsilon)^2 - \alpha^2}\right] - \\ &- \frac{\sigma_1}{8\pi^2\xi\xi'\cosh\frac{\alpha}{2}}\left[\frac{\tau-\tau^{'}}{(\tau-\tau'-i\varepsilon)^2 - \alpha^2}\right] \end{aligned} \tag{8}$$

and $F^- = (F^+)^*$. The sum over images with alternating signs will gives again the above thermal functions. Note that these functions are vacuum expectation values in the Fulling state $|F>$.

(b) The special value $\beta = 2\pi$ is again distinguished, since then

$$W^{(1/2)\pm}(2\pi|x, x^{'}) = \pm i\tilde{\sigma}^a\nabla_a W^\pm(x, x^{'})$$

where $W^\pm(x, x^{'})$ are the Wightman function for the massless scalar field. These are just the Wightman functions for neutrinos in the Minkoswki fermion vacuum, relative to a boosted tetrad.

(c) The stress tensor relative to the Minkowski vacuum has the perfect fluid form[16, 14]

$$T^{ab} = \frac{1}{11520\pi^2\xi^4}\left[7\left(\frac{2\pi}{\beta}\right)^4 + 10\left(\frac{2\pi}{\beta}\right)^2 - 17\right][4v^a v^b + g^{ab}] \tag{9}$$

The zero temperature tensor was calculated in Ref.[17] (see also Ref.[18]).

3. The electromagnetic $s = 1$ field. The tetrad components of the Wightman functions $W^{(1)\pm}_{ij'}(\beta|x, x')$ will be given in the Feynman gauge $\nabla_a A^a = 0$, where a prime over the indices means that the function is a bivector at x and x' respectively. They are given by the equations

$$W^{(1)+}_{00'}(\beta|x, x') = \frac{1}{4\pi\beta\xi\xi' \sinh\alpha} \frac{\cosh\left(\frac{2\pi}{\beta}(\tau - \tau')\right)\sinh\alpha + \sinh\left(\frac{2\pi}{\beta} - 1\right)\alpha}{\cosh\left(\frac{2\pi}{\beta}\alpha\right) - \cosh\left(\frac{2\pi}{\beta}(\tau - \tau' - i\varepsilon)\right)} \tag{10}$$

$$W^{(1)+}_{11'}(\beta|x, x') = -W^{(1)+}_{00'}(\beta|x, x') \tag{11}$$

$$W^{(1)+}_{10'}(\beta|x, x') = \frac{1}{4\pi\beta\xi\xi'} \frac{\sinh\left(\frac{2\pi}{\beta}(\tau - \tau')\right)}{\cosh\left(\frac{2\pi}{\beta}\alpha\right) - \cosh\left(\frac{2\pi}{\beta}(\tau - \tau' - i\varepsilon)\right)} \tag{12}$$

$$W^{(1)+}_{01'}(\beta|x, x') = -W^{(1)+}_{10'}(\beta|x, x') \tag{13}$$

$$W^{(1)+}_{22'}(\beta|x, x') = W^{(1)+}_{33'}(\beta|x, x') = -W^{+}(\beta|x, x') \tag{14}$$

where $W^{+}(\beta|x, x')$ is the scalar Wightman function, Eq. (2). The periodicity in imaginary time is again evident. In Ref.[14] the Green functions for any spin around a cosmic string have been given in the $(j, 0)$ representation of the Lorentz group. For $s = 1$ elicity fields these are Green functions for the fields $\mathbf{E} \pm i\mathbf{H}$, and subtle questions of gauge invariance were consequently avoided.

(a) the zero temperature limit is

$$W^{(1)+}_{00'}(x, x') = -W^{(1)+}_{11'}(x, x') = \frac{1}{4\pi^2\xi\xi'} \frac{\alpha\coth\alpha}{\alpha^2 - (\tau - \tau' - i\varepsilon)^2} \tag{15}$$

$$W^{(1)+}_{10'}(x, x') = -W^{(1)+}_{01'}(x, x') = \frac{1}{4\pi^2\xi\xi'} \frac{\tau - \tau'}{\alpha^2 - (\tau - \tau' - i\varepsilon)^2} \tag{16}$$

$$W^{(1)+}_{22'}(x, x') = W^{(1)+}_{33'}(x, x') = -W^{+}(x, x') \tag{17}$$

where $W^{+}(x, x')$ is the Wightman function for the scalar field, Eq. (2). They are expectation values in the Fulling vacuum state, which satisties $\nabla^a A^{(+)}_a|F >= 0$, namely

$$W^{(1)+}_{ij'}(x, x') =< F|A_i(x)A_j(x')|F >$$

The verification of this statement from canonical quantization is rather messy, due to an apparent divergence in the integral representation of the Rindler Wightman functions. This representation also appeared in Ref.[19], where the Fulling-Davies-Unruh thermal bath was shown to be exactly the bremsstrahlung radiation emitted by a uniformly accelerated charge.

(b) the value $\beta = 2\pi$ is distinguished since then

$$W^{(1)\pm}_{ij'}(2\pi|x, x') = -g_{ij} W^{\pm}(x, x')$$

where $W^{\pm}(x, x')$ is the scalar Wightman function. This is just the Wightman function in the Feynman gauge of Minkowski vacuum relative to a boosted tetrad.

(c) the Wightman functions obey the Ward identity. In the Feynman gauge this identity states that

$$\nabla^a W^{(1)\pm}_{ab'} + \nabla_{b'} W^{\pm}_{gh} = 0 \tag{18}$$

where $W^{\pm}_{gh}$ is the Wightman function for the ghost fields $\eta_1(x)$, $\bar{\eta}_2(x)$. This is actually equal to the scalar Wightman function because the ghosts equations of motion are $\Box\eta_{1,2}(x) = 0$. Though uncoupled to the electromagnetic field, their presence is essential in the finite temperature theory[20].

(d) The stress tensor relative to the Minkowski vacuum has the perfect fluid form[14]

$$T^{ab} = \frac{1}{720\pi^2\xi^4}\left[\left(\frac{2\pi}{\beta}\right)^4 + 10\left(\frac{2\pi}{\beta}\right)^2 - 11\right][4v^a v^b + g^{ab}] \tag{19}$$

2 Discussion

The above results were obtained from canonical quantization in the case of scalar and Weyl fields, but we used a different method for the electromagnetic field. In the former case, we obtained integral representations of the Wightman functions from the expansion of the fields into normal modes, then by summing over images we obtained the finite temperature results. In the latter case also we were able to solve the integral representation explicitly. Surprisingly then, the periodicity sum of the zero temperature Wightman functions so obtained failed to reduce to the Minkowski Wightmann functions when $\beta = 2\pi$ and thus failed to reproduce exactly the finite temperature result, since the thermal properties of the Minkowski vacuum relative to Rindler time translations can be established by independent arguments [9] and even in a model independent and rigorous way [21]. Just requiring this fact gives the correct result as given in the text. A closer scrutiny of the situation also reveals that the responsibility of the failure is due to the sector of the photon Fock space containing Rindler states with negative norm.

Nevertheless, the result obtained by periodicity sum obeys both the Ward identity and the wave equation and also it reproduces the correct result in the limit $\beta \to +\infty$. However, for $\beta < +\infty$, this behaves badly as $x_t \to \infty$, i.e., $W_{11'}(\beta)$ and $W_{00'}(\beta)$ do not vanish there as one might expect (in the case of $\beta = 2\pi$ at least). On the other hand, the Wightman functions for the field strength, are correctly given by periodicity summing over the zero temperature functions $< F|F_{ab}(x)F_{a'b'}(x')|F >$, and there are no subtleties associated with gauge invariance.

Then we noted Dowker tecnique for handling the scalar Green function on a conical space[22]. The Rindler metric tensor in euclidean time with β periodicity just represents a conical space of the form $\mathcal{C}_\beta \times I\!R^2$, $\mathcal{C}_\beta$ being a two dimensional cone with deficit angle $\gamma = 2\pi - \beta$. The Green function of a vector field can then be obtained in closed form on the cone after which by analytic continuation back to real time we have confirmed the given results. It is interesting to observe that, differently from the method of images, this euclidean approach forces automatically the Wightman functions to behave correctly at the infinity. A complete calculation for photons and gravitons was also presented in Ref.[23], for the case of a cosmic string background for which the conical singularity was rounded off. Upon translating their results to Rindler space, we founded complete agreement. The stress tensors were obtained by the cited authors using point splitting procedures. Here we give an argument which is based on the old observation[24] that the manifold $\mathcal{M} = I\!R \times H^3$, with the natural product metric, is conformal to the Rindler metric. The space H^3 here is the hyperbolic three space carrying a metric with constant negative curvature. The conditions under which this can be done were explained in Ref.[24]. They are fulfilled in the present case because our fields are described by conformally invariant wave equations and the two spaces have conformally related global Caucy surfaces (an extensive discussion of conformally invariant quantum field theory in hyperbolic universes were also given in Ref.[25, 12]).

The one-loop partition function (per unit volume) for a thermal state in $\mathcal{M}$ will be determined by the density of one-particle states in H^3, denoted $\nu^{(s)}(\omega)$ for a spin s field. Indeed

$$\log Z^{(s)}(\beta) = \xi^{-3} \int_0^\infty \log\left(1 \pm e^{\beta\omega}\right) \nu^{(s)}(\omega) d\omega - \beta U \tag{20}$$

where U is the vacuum energy density, the only quantity that needs a renormalization prescription in this contest. The conformal transformation back to Rindler only adds a β-linear term[26, 27], which may be absorbed into the definition of U. The density of states is thus the crucial quantity. In H^N and for the Laplace-Beltrami operator, it has long been known by mathematician where it is known as the Harish-Chandra or Plancherel measure (see Ref.[28] and Ref.s therein for a detailed account). In H^3 it is

$$\nu^{(0)}(\omega) = \frac{\omega^2}{2\pi^2} \tag{21}$$

$$\nu^{(1/2)} = \frac{(\omega^2 + 1/4)}{2\pi^2} \tag{22}$$

$$\nu^{(1)} = \frac{(\omega^2 + 1)}{\pi^2} \tag{23}$$

where the $s = 1$ case holds for transverse vector fields, i.e. in the Feynman gauge. The partition function is now easily computed from Eq. (20). We give the details for $s = 0$ only, the other cases being similar. We obtain

$$\log Z(\beta) = \frac{\pi^2}{90\xi^3}\beta^{-3} - \beta_T U(\Lambda), \qquad \beta_T = \xi\beta \tag{24}$$

where β_T is the Tolman inverse temperature and

$$U(\Lambda) = \frac{1}{4\pi^2\xi^4} \int_0^\Lambda \omega^3 d\omega$$

ie the regularized vacuum energy density. The linear term will not affects the entropy density while the energy density must vanishes at $\beta = 2\pi$, since this would correspond to the scalar vacuum in Minkowski space-time, whose energy density is defined to be zero in order to realize the Poincaré symmetry. The zero point of entropy will also vanishes at $\beta = \infty$ since the Fulling vacuum is a pure states. Once the zero point of entropy and energy density have been fixed, there is no further room left and all the thermodynamics densities are fixed. Thus we get the renormalized energy density and pressure

$$u(\beta) = 3p = \frac{1}{30\pi^2\xi^4}\left[\left(\frac{2\pi}{\beta}\right)^4 - 1\right] \tag{25}$$

the entropy density

$$s(\beta) = \frac{4\pi^2}{90\xi^3}\beta^{-3} \tag{26}$$

and the free energy density

$$f(\beta) = -\frac{\pi^2}{90\xi^4}[\beta^{-4} + 3(2\pi)^{-4}] \tag{27}$$

The cut-off dependence is now disappeared. Why should not we define the zero of entropy at $\beta = 2\pi$ which also is a pure state, namely the Minkowski vacuum? The reason is a well known consequence of quantum theory, first discovered by von Neumann[29], that a subsystem of a system in a pure state may has a non zero entropy if only the subsystem is being observed. Now while it is true that at $\beta = 2\pi$ we are computing quantities in the Minkowski vacuum, we must remember that we are observing only the right hand Rindler wedge since the field operators from which the above results were derived are restricted over there. Notice that the zero point free energy is equal to the zero point energy and that the Gibbs relation $Ts = u + p$ gets modified to $Ts = (u - u_0) + (p - p_0)$, in accord with thermodynamics. Eq. (6) for the stress tensor can now be derived by noting that the energy density and pressure must be the eigenvalues of the stress tensor in an orthonormal vierbein.

The Rindler metric also describes a static uniform gravitational field, locally identical to the field near the earth surface for example. There is no Minkowski vacuum then and so it appears an arbitrary procedure to fix the zero of entropy and energy at $\beta = \infty$ and $\beta = 2\pi g^{-1}$ respectively, where g is the local acceleration due to gravity. We think it is a very important question of principle to clarify the role of the distinguished temperature $\beta = 2\pi g^{-1}$ in such a case. Our discussion is thus intended to apply only to accelerating observers in Minkowski space-time.

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