Octahedral and Dodecahedral Monopoles

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Abstract

It is shown that there exists a charge five monopole with octahedral symmetry and a charge seven monopole with icosahedral symmetry. A numerical implementation of the ADHMN construction is used to calculate the energy density of these monopoles and surfaces of constant energy density are displayed. The charge five and charge seven monopoles look like an octahedron and a dodecahedron respectively. A scattering geodesic for each of these monopoles is presented and discussed using rational maps. This is done with the aid of a new formula for the cluster decomposition of monopoles when the poles of the rational map are close together.

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1   Introduction

BPS monopoles are topological solitons in a three dimensional SU(2) Yang-Mills-Higgs gauge theory, in the limit of vanishing Higgs potential. They are solutions to the Bogomolny equation

$$D_A \Phi = \star F_A \tag{1.1}$$

where D_A is the covariant derivative, with A an $su(2)$-valued gauge potential 1-form, F_A its gauge field 2-form and $\star$ the Hodge dual on $\mathbb{R}^3$. The Higgs field, Φ, is an $su(2)$-valued scalar field and is required to satisfy

$$\|\Phi\| \xrightarrow{r \to \infty} 1 \tag{1.2}$$

where $r = |x|$ and $\|\Phi\|^2 = -\frac{1}{2}\mathrm{tr}\Phi^2$. The boundary condition (1.2) can be considered to be a residual finite energy condition, derived from the now vanished Higgs potential.

The Higgs field at infinity induces a map between spheres:

$$\Phi : S^2(\infty) \to S^2(1) \tag{1.3}$$

where $S^2(\infty)$ is the two-sphere at spatial infinity and $S^2(1)$ is the two-sphere of vacuum configurations given by $\{\Phi \in su(2) : \|\Phi\| = 1\}$. The degree of this map is a non-negative integer k which (in suitable units) is the total magnetic charge. We shall refer to a monopole with magnetic charge k as a k-monopole. The total energy of a k-monopole is equal to $8\pi k$ and the energy density may be expressed [19] in the convenient form

$$\mathcal{E} = \triangle \|\Phi\|^2 \tag{1.4}$$

where $\triangle$ denotes the laplacian on $\mathbb{R}^3$.

Monopoles correspond to certain algebraic curves, called spectral curves, in the mini-twistor space $\mathbb{T} \cong T\mathbb{CP}^1$ [19, 7, 8]. This space is isomorphic to the space of directed lines in $\mathbb{R}^3$. If ζ is the standard inhomogeneous coordinate on the base space, it corresponds to the direction of a line in $\mathbb{R}^3$. The fibre coordinate, η, is a complex coordinate in a plane orthogonal to this line. The spectral curve of a monopole is the set of lines along which the differential equation

$$(D_A - i\Phi)v = 0 \tag{1.5}$$

has bounded solutions in both directions. The spectral curve of a k-monopole takes the form

$$\eta^k + \eta^{k-1}a_1(\zeta) + \ldots + \eta^r a_{k-r}(\zeta) + \ldots + \eta a_{k-1}(\zeta) + a_k(\zeta) = 0 \tag{1.6}$$

where, for $1 \le r \le k$, $a_r(\zeta)$ is a polynomial in ζ of maximum degree $2r$. However, general curves of this form will only correspond to k-monopoles if they satisfy the reality condition

$$a_r(\zeta) = (-1)^r \zeta^{2r} \overline{a_r(-\frac{1}{\bar{\zeta}})} \tag{1.7}$$

and some difficult non-singularity conditions [7]. In [6] the concept of a strongly centred monopole is introduced. A strongly centred monopole is centred on the origin and its rational map has total phase one. If a monopole is strongly centred its spectral curve satisfies

$$a_1(\zeta) = 0. \tag{1.8}$$

Even though the Bogomolny equation is integrable, it is not easily solved. Explicit solutions are only known in the cases of 1-monopole [17], 2-monopoles [19, 20] and axisymmetric monopoles of higher charges [16]. Recently, progress has been made in understanding multi-monopoles. Hitchin, Manton and Murray [6] have demonstrated the existence of monopoles corresponding to the spectral curves

$$\eta^3 + i\frac{\Gamma(1/6)^3\Gamma(1/3)^3}{48\sqrt{3}\pi^{3/2}}\zeta(\zeta^4 - 1) = 0 \qquad (1.9)$$

$$\eta^4 + \frac{3\Gamma(1/4)^8}{64\pi^2}(\zeta^8 + 14\zeta^4 + 1) = 0. \qquad (1.10)$$

The first spectral curve (1.9) has tetrahedral symmetry, the second (1.10) has octahedral symmetry. In [9] we computed numerically and displayed surfaces of constant energy density for these monopoles. We noted that the charge four monopole looks like a cube, rather than an octahedron. We therefore refer to this 4-monopole as a cubic monopole.

Hitchin, Manton and Murray [6] also prove that although

$$\zeta_1^{11}\zeta_0 + 11\zeta_1^6\zeta_0^6 - \zeta_1\zeta_0^{11} \qquad (1.11)$$

is an icosahedrally invariant homogeneous polynomial of degree 12, the invariant algebraic curve

$$\eta^6 + a\zeta(\zeta^{10} + 11\zeta^5 - 1) = 0 \qquad (1.12)$$

does not correspond to a monopole for any value of a. However, based upon considerations of the symmetries of rational maps for infinite curvature hyperbolic monopoles, Atiyah has suggested, [1] [1] that there may be an icosahedrally invariant 7-monopole. In this paper, we prove that this suggestion is correct by demonstrating that the algebraic curve

$$\eta^7 + \frac{\Gamma(1/6)^6\Gamma(1/3)^6}{64\pi^3}\zeta(\zeta^{10} + 11\zeta^5 - 1)\eta = 0 \qquad (1.13)$$

is the spectral curve of a monopole. Using our numerical scheme introduced in [9], we then compute its energy density. On examining surfaces of constant energy density, we find that the charge seven monopole looks like a dodecahedron.

In each of the cases examined so far, the minimum charge monopole with the symmetry of a regular solid has charge $k = \frac{1}{2}(F + 2)$, where F is the smallest number of faces of a regular solid with that symmetry. This leads us to conjecture that the minimum charge monopole resembling a regular solid with F faces has charge $k = \frac{1}{2}(F+2)$. For the dodecahedron $F = 12$, which gives $k = 7$. In fact, this conjecture was one of the motivations for our consideration of charge seven when searching for an icosahedrally symmetric monopole. In this paper, we demonstrate that our conjecture is also correct for the octahedron by proving that the octahedrally symmetric algebraic curve

$$\eta^5 + \frac{3\Gamma(\frac{1}{4})^8}{16\pi^2}(\zeta^8 + 14\zeta^4 + 1)\eta = 0 \qquad (1.14)$$

is the spectral curve of a 5-monopole. We display its energy density and confirm that it looks like an octahedron. It remains to be verified that an icosahedrally symmetric monopole of charge eleven exists and resembles an icosahedron.

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[1] We thank Nick Manton for drawing this to our attention

It is interesting that numerical evidence suggests that similar results hold in the case of static minimum energy multi-skyrmion solutions. In [4] Braaten, Townsend and Carson use a discretization of the Skyrme model on a cubic lattice to calculate such solutions for baryon numbers $B = 3, 4, 5$ and 6. They find that surfaces of constant baryon number density resemble solids with $2B - 2$ faces. Furthermore, the fields describing solutions with $B = 3$ and $B = 4$ are seen to possess tetrahedral and octahedral symmetry. However, they conclude that the solution for $B = 5$ seems only to have D_{2d} symmetry. This contrasts with the existence of a charge five monopole with octahedral symmetry.

Approximations to the $B = 3$ and $B = 4$ skyrmions have been calculated by computing the holonomies of Yang-Mills instantons [13]. These instanton generated Skyrme fields also have tetrahedral and octahedral symmetry respectively. Given the numerical evidence for an apparent difference between charge five monopoles and skyrmions, it would be instructive to construct instanton-generated Skyrme fields with baryon number five. It may be that an octahedrally symmetric 5-skyrmion simply does not exist. However, the instanton construction could shed some light on other possibilities; for example, that such a skyrmion exists but it does not have minimum energy. A second possibility is that the numerical scheme used in [4] is responsible for no such skyrmion being found. For particular orientations, an octahedron will not fit inside a cubic lattice; in the sense of all the vertices of the octahedron sitting on lattice sites. The discretization could then result in the octahedron being squashed into a shape similar to that found in [4]. Of course, at the moment, all these possibilities are pure speculation. What is clear from our results is that the $B = 7$ skyrmion should now be investigated, as there is some interest in the possibility that this is icosahedrally symmetric.

In Section 2, we outline the ADHMN construction as applied to symmetric monopoles. In Sections 3 and 4, we present our results on dodecahedral and octahedral monopoles. Finally, in Section 5, we discuss rational maps and geodesic monopole scattering related to these symmetric monopoles. This is done with the aid of a new formula for the cluster decomposition of monopoles when the poles of the rational map are close together.

2 The Nahm Equations

The main difficulty in proving that an algebraic curve is the spectral curve of a monopole lies in demonstrating satisfaction of the non-singularity conditions. However, there is a reciprocal formulation of the Bogomolny equation in which non-singularity is manifest. This formulation is the Atiyah-Drinfeld-Hitchin-Manin-Nahm (ADHMN) construction [15, 8]. This is an equivalence between k-monopoles and Nahm data (T_1, T_2, T_3), which are three $k \times k$ matrices depending on a real parameter $s \in [0, 2]$ and satisfying:

(i) Nahm's equation

$$\frac{dT_i}{ds} = \frac{1}{2}\epsilon_{ijk}[T_j, T_k] \tag{2.1}$$

(ii) $T_i(s)$ is regular for $s \in (0, 2)$ and has simple poles at $s = 0$ and $s = 2$,

(iii) the matrix residues of (T_1, T_2, T_3) at each pole form the irreducible k-dimensional representation of SU(2),

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(iv) $T_i(s) = -T_i^\dagger(s)$,

(v) $T_i(s) = T_i^t(2-s)$.

It should be noted that in this paper we shall not search for a basis in which property (v) is explicit, but rely on a general argument that such a basis exists (see [6]).

Explicitly, the spectral curve may be read off from the Nahm data as the equation

$$\det(\eta + (T_1 + iT_2) - 2iT_3\zeta + (T_1 - iT_2)\zeta^2) = 0. \qquad (2.2)$$

It is obvious from (2.2) that the strong centering condition (1.8) is equivalent to

(vi) $\text{tr}T_i(s) = 0$.

To extract the monopole fields (Φ, A) from the Nahm data requires the computation of a basis for the kernel of a linear differential operator constructed out of the Nahm data, followed by some integrations. We have developed a numerical algorithm which can perform all these required tasks, the details are included in [9]. The algorithm takes as input the Nahm data and outputs the energy density of the corresponding monopole. It will be applied to the Nahm data which we construct in this paper.

As in [9] we use the discrete symmetry group G of the conjectured monopole to reduce the number of Nahm equations. Since the Nahm matrices are traceless, they transform under the rotation group as

$$
\begin{aligned}
\underline{3} \otimes sl(\underline{k}) \ &\cong\ \underline{3} \otimes (\underline{2k-1} \oplus \underline{2k-3} \oplus \ldots \oplus \underline{3}) \\
&\cong\ (\underline{2k+1}_u \oplus \underline{2k-1}_m \oplus \underline{2k-3}_l) \oplus \ldots \\
&\ldots\ \oplus(\underline{2r+1}_u \oplus \underline{2r-1}_m \oplus \underline{2r-3}_l) \oplus \ldots \oplus (\underline{5}_u \oplus \underline{3}_m \oplus \underline{1}_l) \qquad (2.3)
\end{aligned}
$$

where $\underline{r}$ denotes the unique irreducible r dimensional representation of $su(2)$ and the subscripts u, m and l (which stand for upper, middle and lower) are a convenient notation allowing us to distinguish between $2r+1$ dimensional representations occuring as

$$
\begin{aligned}
\underline{3} \otimes \underline{2r-1} \ &\cong\ \underline{2r+1}_u \oplus \underline{2r-1}_m \oplus \underline{2r-3}_l, \\
\underline{3} \otimes \underline{2r+1} \ &\cong\ \underline{2r+3}_u \oplus \underline{2r+1}_m \oplus \underline{2r-1}_l
\end{aligned}
$$

and

$$\underline{3} \otimes \underline{2r+3} \ \cong\ \underline{2r+5}_u \oplus \underline{2r+3}_m \oplus \underline{2r+1}_l.$$

We can then use invariant homogeneous polynomials over $\mathbb{CP}^1$ to construct G-invariant Nahm triplets. The vector space of degree $2r$ homogeneous polynomials $a_{2r}\zeta_1^{2r} + a_{2r-1}\zeta_1^{2r-1}\zeta_0 + \ldots + a_0\zeta_0^{2r}$ is the carrier space for $\underline{2r+1}$ under the identification

$$X = \zeta_1 \frac{\partial}{\partial\zeta_0}; \qquad Y = \zeta_0 \frac{\partial}{\partial\zeta_1}; \qquad H = -\zeta_0 \frac{\partial}{\partial\zeta_0} + \zeta_1 \frac{\partial}{\partial\zeta_1}. \qquad (2.4)$$

where X, Y and H are the basis of $su(2)$ satisfying

$$[X,Y] = H, \qquad [H,X] = 2X, \qquad [H,Y] = -2Y. \qquad (2.5)$$

As explained in [6, 9] if $p(\zeta_0, \zeta_1)$ is a G-invariant homogeneous polynomial we can construct a G-invariant $\underline{2r+1}_u$ charge k Nahm triplet by the following scheme.

(i) The inclusion

$$\underline{2r+1} \hookrightarrow \underline{3} \otimes \underline{2r-1} \cong \underline{2r+1}_u \oplus \underline{2r-1}_m \oplus \underline{2r-3}_l \tag{2.6}$$

is given on polynomials by

$$p(\zeta_0, \zeta_1) \mapsto \xi_1^2 \otimes p_{11}(\zeta_0, \zeta_1) + 2\xi_0\xi_1 \otimes p_{10}(\zeta_0, \zeta_1) + \xi_0^2 \otimes p_{00}(\zeta_0, \zeta_1) \tag{2.7}$$

where we have used the notation

$$p_{ab}(\zeta_0, \zeta_1) = \frac{\partial^2 p}{\partial \zeta_a \partial \zeta_b}(\zeta_0, \zeta_1). \tag{2.8}$$

(ii) The polynomial expression $\xi_1^2 \otimes p_{11}(\zeta_0, \zeta_1) + 2\xi_0\xi_1 \otimes p_{10}(\zeta_0, \zeta_1) + \xi_0^2 \otimes p_{00}(\zeta_0, \zeta_1)$ is rewritten in the form

$$\xi_1^2 \otimes q_{11}(\zeta_0 \frac{\partial}{\partial \zeta_1})\zeta_1^{2r} + (\xi_o \frac{\partial}{\partial \xi_1})\xi_1^2 \otimes q_{10}(\zeta_0 \frac{\partial}{\partial \zeta_1})\zeta_1^{2r} + \frac{1}{2}(\xi_o \frac{\partial}{\partial \xi_1})^2 \xi_1^2 \otimes q_{00}(\zeta_0 \frac{\partial}{\partial \zeta_1})\zeta_1^{2r}. \tag{2.9}$$

(iii) This then defines a triplet of $k \times k$ matrices. Given a $k \times k$ representation of X, Y and H above, the invariant Nahm triplet is given by:

$$(S_1', S_2', S_3') = (q_{11}(\mathrm{ad}Y)X^r, q_{10}(\mathrm{ad}Y)X^r, q_{00}(\mathrm{ad}Y)X^r), \tag{2.10}$$

where $\mathrm{ad}Y$ denotes the adjoint action of Y and is given on a general matrix M by $\mathrm{ad}Y M = [M, Y]$.

(iv) The Nahm isospace basis is transformed. This transformation is given by

$$(S_1, S_2, S_3) = (\frac{1}{2}S_1' + S_3', -\frac{i}{2}S_1' + iS_3', -iS_2'). \tag{2.11}$$

Relative to this basis the $SO(3)$-invariant Nahm triplet corresponding to the $\underline{1}_l$ representation in (2.3) is given by (ρ_1, ρ_2, ρ_3) where

$$\rho_1 = X - Y; \qquad \rho_2 = i(X + Y); \qquad \rho_3 = iH. \tag{2.12}$$

It is also necessary to construct invariant Nahm triplets lying in the $\underline{2r+1}_m$ representations. To do this, we first construct the corresponding $\underline{2r+1}_u$ triplet. We then write this triplet in the canonical form

$$\begin{aligned}[c_0 + c_1(\mathrm{ad}Y \otimes 1 + 1 \otimes \mathrm{ad}Y) \quad &+ \quad \ldots + c_i(\mathrm{ad}Y \otimes 1 + 1 \otimes \mathrm{ad}Y)^i \\ &+ \quad \ldots + c_{2r}(\mathrm{ad}Y \otimes 1 + 1 \otimes \mathrm{ad}Y)^{2r}] X \otimes X^r\end{aligned} \tag{2.13}$$

and map this isomorphically into $\underline{2r+1}_m$ by mapping the highest weight vector $X \otimes X^r$ to the highest weight vector

$$X \otimes \mathrm{ad}Y X^{r+1} - \frac{1}{r+1}\mathrm{ad}Y X \otimes X^{r+1}. \tag{2.14}$$

3 Dodecahedral Seven Monopole

The minimum degree icosahedrally invariant homogeneous polynomial is [12]

$$\zeta_1^{11}\zeta_0 + 11\zeta_1^6\zeta_0^6 - \zeta_1\zeta_0^{11}. \tag{3.1}$$

Polarizing this gives

$$\xi_1^2 \otimes (110\zeta_1^9\zeta_0 + 330\zeta_1^4\zeta_0^6) + 2\xi_1\xi_0 \otimes (11\zeta_1^{10} + 396\zeta_1^5\zeta_0^5 - 11\zeta_0^{11}) + \xi_0^2 \otimes (330\zeta_1^6\zeta_0^4 - 110\zeta_1\zeta_0^9). \tag{3.2}$$

This is proportional to

$$\xi_1^2 \otimes (\zeta_0\frac{\partial}{\partial\zeta_1} \quad + \quad \frac{1}{5040}(\zeta_0\frac{\partial}{\partial\zeta_1})^6)\zeta_1^{10} + 2\xi_1\xi_0 \otimes (1 + \frac{1}{840}(\zeta_0\frac{\partial}{\partial\zeta_1})^5 - \frac{1}{10!}(\zeta_0\frac{\partial}{\partial\zeta_1})^{10})\zeta_1^{10}$$

$$+ \quad \xi_0^2 \otimes (\frac{1}{168}(\zeta_0\frac{\partial}{\partial\zeta_1})^4 - \frac{1}{9!}(\zeta_0\frac{\partial}{\partial\zeta_1})^9)\zeta_1^{10} \tag{3.3}$$

which gives matrices

$$X \otimes (\mathrm{ad}Y \quad + \quad \frac{1}{5040}(\mathrm{ad}Y)^6)X^5 + \mathrm{ad}YX \otimes (1 + \frac{1}{840}(\mathrm{ad}Y)^5 - \frac{1}{10!}(\mathrm{ad}Y)^{10})X^5$$

$$+ \quad \frac{1}{2}(\mathrm{ad}Y)^2 X \otimes (\frac{1}{168}(\mathrm{ad}Y)^4 - \frac{1}{9!}(\mathrm{ad}Y)^9)X^5. \tag{3.4}$$

We choose the basis given by

$$H = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -6 \end{bmatrix}, \tag{3.5}$$

$$Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{10} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 \end{bmatrix}, \; X = \begin{bmatrix} 0 & \sqrt{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{10} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using MAPLE the invariant Nahm triplet is calculated, relative to the basis (2.11), to give the $\underline{13}_u$ invariant

$$Z_1 = \begin{bmatrix} 0 & 5\sqrt{6} & 0 & 0 & 7\sqrt{6}\sqrt{10} & 0 & 0 \\ -5\sqrt{6} & 0 & -9\sqrt{10} & 0 & 0 & 0 & 0 \\ 0 & 9\sqrt{10} & 0 & 5\sqrt{12} & 0 & 0 & -7\sqrt{6}\sqrt{10} \\ 0 & 0 & -5\sqrt{12} & 0 & 5\sqrt{12} & 0 & 0 \\ -7\sqrt{6}\sqrt{10} & 0 & 0 & -5\sqrt{12} & 0 & -9\sqrt{10} & 0 \\ 0 & 0 & 0 & 0 & 9\sqrt{10} & 0 & 5\sqrt{6} \\ 0 & 0 & 7\sqrt{6}\sqrt{10} & 0 & 0 & -5\sqrt{6} & 0 \end{bmatrix}$$

$$Z_2 = i \begin{bmatrix} 0 & 5\sqrt{6} & 0 & 0 & -7\sqrt{6}\sqrt{10} & 0 & 0 \\ 5\sqrt{6} & 0 & -9\sqrt{10} & 0 & 0 & 0 & 0 \\ 0 & -9\sqrt{10} & 0 & 5\sqrt{12} & 0 & 0 & 7\sqrt{6}\sqrt{10} \\ 0 & 0 & 5\sqrt{12} & 0 & 5\sqrt{12} & 0 & 0 \\ -7\sqrt{6}\sqrt{10} & 0 & 0 & 5\sqrt{12} & 0 & -9\sqrt{10} & 0 \\ 0 & 0 & 0 & 0 & -9\sqrt{10} & 0 & 5\sqrt{6} \\ 0 & 0 & 7\sqrt{6}\sqrt{10} & 0 & 0 & 5\sqrt{6} & 0 \end{bmatrix}$$

$$Z_3 = i \begin{bmatrix} -12 & 0 & 0 & 0 & -14\sqrt{6} & 0 & 0 \\ 0 & 48 & 0 & 0 & 0 & 0 & -14\sqrt{6} \\ 0 & 0 & -60 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 60 & 0 & 0 \\ -14\sqrt{6} & 0 & 0 & 0 & 0 & -48 & 0 \\ 0 & -14\sqrt{6} & 0 & 0 & 0 & 0 & 12 \end{bmatrix}$$

To calculate the $\underline{13}_m$ invariant we put (3.4) in the form (2.14). It is proportional to

$$[11!(\mathrm{ad}Y \otimes 1 + 1 \otimes \mathrm{ad}Y) + 7920(\mathrm{ad}Y \otimes 1 + 1 \otimes \mathrm{ad}Y)^6 - (\mathrm{ad}Y \otimes 1 + 1 \otimes \mathrm{ad}Y)^{11}]X \otimes X^5. \quad (3.6)$$

Then using the isomorphism mentioned earlier we obtain matrices

$$Y_1 = \begin{bmatrix} 0 & \sqrt{6} & 0 & 0 & -\sqrt{6}\sqrt{10} & 0 & 12 \\ \sqrt{6} & 0 & -3\sqrt{10} & 0 & 0 & 12 & 0 \\ 0 & -3\sqrt{10} & 0 & 5\sqrt{12} & 0 & 0 & -\sqrt{6}\sqrt{10} \\ 0 & 0 & 5\sqrt{12} & 0 & -5\sqrt{12} & 0 & 0 \\ -\sqrt{6}\sqrt{10} & 0 & 0 & -5\sqrt{12} & 0 & 3\sqrt{10} & 0 \\ 0 & 12 & 0 & 0 & 3\sqrt{10} & 0 & -\sqrt{6} \\ 12 & 0 & -\sqrt{6}\sqrt{10} & 0 & 0 & -\sqrt{6} & 0 \end{bmatrix}$$

$$Y_2 = i \begin{bmatrix} 0 & \sqrt{6} & 0 & 0 & \sqrt{6}\sqrt{10} & 0 & 12 \\ -\sqrt{6} & 0 & -3\sqrt{10} & 0 & 0 & -12 & 0 \\ 0 & 3\sqrt{10} & 0 & 5\sqrt{12} & 0 & 0 & \sqrt{6}\sqrt{10} \\ 0 & 0 & -5\sqrt{12} & 0 & -5\sqrt{12} & 0 & 0 \\ -\sqrt{6}\sqrt{10} & 0 & 0 & 5\sqrt{12} & 0 & 3\sqrt{10} & 0 \\ 0 & 12 & 0 & 0 & -3\sqrt{10} & 0 & -\sqrt{6} \\ -12 & 0 & -\sqrt{6}\sqrt{10} & 0 & 0 & \sqrt{6} & 0 \end{bmatrix}$$

$$Y_3 = i \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -10\sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10\sqrt{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10\sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -10\sqrt{6} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In order to derive the reduced Nahm equations we examine, the commutation relations. The required relations involving ρ matrices and Z matrices are

$$\begin{aligned} [\rho_1, \rho_2] &= 2\rho_3 \\ [Z_1, Z_2] &= -750\rho_3 + 90Z_3 \\ [Z_1, \rho_2] + [\rho_1, Z_2] &= -10Z_3. \end{aligned} \quad (3.7)$$

Because of the closed form of these relations, it is possible to derive a consistent set of Nahm equations from the icosahedrally invariant Nahm data

$$T_i(s) = x(s)\rho_i + z(s)Z_i, \qquad i \in \{1, 2, 3\}. \tag{3.8}$$

That is, we can consistently ignore the invariant Nahm triplet (Y_1, Y_2, Y_3). In fact, if we add $y(s)Y_i$ to (3.8), we cannot simultaneously satisfy $T_i(s) = -T_i^\dagger(s)$ and the reality condition (1.7) for non-trivial $y(s)$. Combining (3.7) and (3.8) gives the reduced Nahm equations

$$\frac{dx}{ds} = 2x^2 - 750z^2 \tag{3.9}$$

$$\frac{dz}{ds} = -10xz + 90z^2$$

with corresponding spectral curve

$$\eta[\eta^6 + a\zeta(\zeta^{10} + 11\zeta^5 + 1)] = 0 \tag{3.10}$$

where

$$a = 552960(14xz - 175z^2)(x + 5z)^4 \tag{3.11}$$

is a constant.

To solve equations (3.9), let $u = x + 5z$ and $v = x - 30z$ so that

$$\frac{du}{ds} = 2uv$$

$$\frac{dv}{ds} = 6u^2 - 4v^2$$

$$a = 110592(u^6 - v^2u^4) \equiv 110592\kappa^6. \tag{3.12}$$

Using the constant to eliminate v, the equation for u becomes

$$\frac{du}{ds} = -2u^2\sqrt{1 - \kappa^6 u^{-6}}. \tag{3.13}$$

If we let $u = -\kappa\sqrt{\wp(t)}$, where $t = 2\kappa s$, then $\wp(t)$ is the Weierstrass function satisfying

$$\wp'^2 = 4(\wp^3 - 1) \tag{3.14}$$

where, in the above and what follows, primed functions are differentiated with respect to their arguments. Thus the Nahm equations are solved by

$$x(s) = \frac{2\kappa}{7}\left[-3\sqrt{\wp(2\kappa s)} + \frac{\wp'(2\kappa s)}{4\wp(2\kappa s)}\right] \tag{3.15}$$

$$z(s) = -\frac{\kappa}{35}\left[\sqrt{\wp(2\kappa s)} + \frac{\wp'(2\kappa s)}{2\wp(2\kappa s)}\right]. \tag{3.16}$$

These functions are analytic in $s \in (0, 2)$ and have simple poles at $s = 0, 2$ provided $\kappa = \omega$, where 2ω is the real period of $\wp(t)$. Since ω is explicitly known for this Weierstrass function, we have

$$\kappa = \frac{\Gamma(1/6)\Gamma(1/3)}{8\sqrt{3\pi}} \tag{3.17}$$