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# Quantum groups and deformed special relativity

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## Abstract

The structure and properties of possible  $q$ -Minkowski spaces is discussed, and the corresponding non-commutative differential calculi are developed in detail and compared with already existing proposals. This is done by stressing its covariance properties as described by appropriate reflection equations. Some isomorphisms among the space-time and derivative algebras are demonstrated, and their representations are described briefly. Finally, some physical consequences and open problems are discussed.

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# 1 Introduction

The question of the quantization of space has been discussed by physicists from the very early days of quantum theory. Due to the recent emergence of a far-reaching generalization of Lie groups and Lie algebras [1, 2, 3, 4], known under the name of quantum groups, it is tempting to introduce suitable quantum Lorentz  $L_q$  and Poincaré  $P_q$  groups to arrive to the quantum Minkowski space-time  $\mathcal{M}_q$  by extending the classical construction  $\mathcal{M} \sim P/L$  to the quantum case. This program, however, is not completely straightforward. A rigorous (unique) definition of quantum groups was given [1, 2, 3] only for the simple Lie groups and algebras; for inhomogeneous groups many problems appear. Using the well known classical homomorphism  $SL(2, C)/Z_2 \approx L$  it was proposed [5] to define a quantum Lorentz group  $L_q$  by using the simplest quantum group,  $SL_q(2)$ , which is a  $q$ -deformed analog of the classical ( $q = 1$ ) commutative algebra of functions on the Lie group  $SL(2, C)$ . Using  $SL_q(2)$ , a quantum Minkowski space  $\mathcal{M}_q$  was then introduced by means of a quadratic combination of  $q$ -spinors transforming homogeneously under the quantum group  $SL_q(2)$  [6, 7, 8, 9].

Let us start by writing down some simple algebraic relations to introduce some quantum group aspects relevant for our discussion. The essential feature in the field of *quantum groups* (we shall omit for a while their dual objects or *quantum algebras*, which may look more familiar for physicists) is in some sense similar to the relation between classical and quantum mechanics, where the commutative algebra of functions on phase space (the algebra of observables) becomes non-commutative after quantization. In the case of Lie groups, the commutative algebra of functions on the group manifold is replaced by a non-commutative algebra after quantization (or  $q$ -deformation); in particular, the matrix elements generating the group algebra become non-commutative. Let us recall the case of  $SL_q(2)$  which will be extensively used below. The quantum group  $GL_q(2)$  is defined as the associative *algebra* (quantum groups are not really group manifolds) generated by four elements  $a, b, c, d$  satisfying the homogeneous quadratic relations ( $\lambda \equiv q - q^{-1}$ ,  $q \neq 0$ )

$$\begin{aligned} ab &= qba, & bd &= qdb, & bc &= cb, \\ ac &= qca, & cd &= qdc, & [a, d] &= \lambda bc. \end{aligned} \quad (1.1)$$

The relations (1.1), however, include four generators, while  $SL(2, C)$  depends on three parameters. To obtain  $SL_q(2)$  one notices that the element

$$ad - qbc = da - q^{-1}bc := \det_q T, \quad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (1.2)$$

is a central (commuting) element of the algebra which defines the  $q$ -determinant of the matrix  $T$ ; then, the addition of the constraint  $\det_q T = 1$  to eqs. (1.1) consistently reduces the number of generators to three. If the entries of  $T$  satisfy (1.1), those of  $T^n$  satisfy analogous relations with  $q$  replaced by  $q^n$

(this product should not be confused with the comultiplication, which preserves (1.1) [10, 11]). In the limit  $q=1$ , (1.1) just expresses that the algebra generated by the elements of  $SL(2, C)$  is commutative, and (1.2) is the usual determinant.

It is not apparent why eqs. (1.1) plus  $\det_q T=1$  define  $SL_q(2)$ , nor how to generalize them to the  $SL_q(n)$  case. This becomes clearer in the framework of the  $R$ -matrix formalism [3] developed in the framework of the quantum inverse scattering method [12]. Indeed, eqs. (1.1) may be rewritten as ‘RTT’ relations,

$$R_{12}T_1T_2 = T_2T_1R_{12} \quad (1.3)$$

where  $T_1 = T \otimes I$ ,  $T_2 = I \otimes T$  (see Appendix A for notation) and the  $4 \times 4$   $c$ -number matrix  $R_{12}$  is given in (A.7). In this way, they may be generalized to  $SL_q(n)$  by means of the appropriate  $n^2 \times n^2$   $R$ -matrix [3]. One could insert any matrix into (1.3) as  $R$ . However, the natural ones (as (A.7)) satisfy the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \quad (1.4)$$

which ensures the consistency of (1.3). It may be shown that no further conditions may be derived from (1.3) and the associativity requirement. As a result the set of relations (1.3) for  $T_{ij}$  define an associative algebra.

Since quantum groups are very close to the algebra of functions on a Lie group, we may expect them to have other characteristics pertaining to the group multiplication rule, inverse (antipode) and unit elements, etc. In fact, they may be characterized as Hopf algebras [1, 2, 3] (see [13, 14, 15, 16] for a review). Here we shall just underline some properties which will be relevant for the  $q$ -Minkowski space below. It is possible to introduce a quantum or  $q$ -plane  $C_q^2$  as an associative *algebra* generated by two elements  $x, y$  subjected to the commutation property

$$xy = qyx \quad (1.5)$$

(notice that a  $q$ -plane is not a manifold). If we now define a two-component  $q$ -vector  $X = (x, y)$ , the commutation properties of the components of  $X' = TX$  satisfy again (1.5) since it is assumed that the entries of  $T$  and of  $X$  commute among each other. This compatibility, not evident a priori, permits to look at quantum groups in general as symmetries of *quantum spaces* [17] (see also [11]). In the general of  $SL_q(n)$  case, the analogue relations for an  $n$ -component  $X$  may be extracted from the  $R$ -matrix.

We may now extend this to the more elaborated situation in which the generators of a quantum space may be put in matrix form  $K$  and the action  $\phi$  is given by  $\phi : K \mapsto K' = TKT^{-1}$ , where the elements of  $T$  commute, as in the  $q$ -plane case, with the entries of  $K$ ,  $[K_{ij}, T_{mn}] = 0$  ( $i, j, m, n=1, 2$ ). A possibility to organize the commutation properties which define the algebra

generated by the elements  $K_{ij}$  is to write them [19, 20, 21] in the form of a reflection equation<sup>1</sup> (RE) (see also [25] in the framework of braided algebras),

$$R_{12}K_1R_{21}K_2 = K_2R_{12}K_1R_{21} . \quad (1.6)$$

Due to eq. (1.3) (and its consequence  $R_{21}T_2T_1 = T_1T_2R_{21}$ ) it is not difficult to see that  $K' = TKT^{-1}$  satisfies the same equation (1.6). In fact, the RE formalism is a convenient framework to discuss the invariance of commutation relations under a certain action  $\phi$ . However, it should be mentioned at this stage that the non-commutative character of the algebra generated by  $K_{ij}$  puts forward some questions as to its physical interpretation. If we consider the elements of  $K$  as non-commutative generators of the algebra, one has to find their irreducible representations  $\mathcal{H}_K$ ; there could be more than one. Also, the non-commutative transformation coefficients (the generators of the quantum group algebra) have their own irreducible representation  $\mathcal{H}_T$ . Hence, after the quantum group transformation, the new entries of  $K' = TKT^{-1}$  act, generally speaking, as operators on a larger space  $\mathcal{H}_T \otimes \mathcal{H}_K$ : the quantum group *coacts* rather than acts. This is a rather unusual situation for the symmetries of physical systems, where the action of a symmetry on the space-time coordinates, say, does not entail an enlargement of the corresponding algebra or of the space of physical states. We will not develop this important and interesting question here, which we shall bypass by stressing the isomorphism among the algebras generated by  $K_{ij}$  and  $K'_{ij}$ ; instead, we shall restrict ourselves to the technical problems of constructing quantum Minkowski space-time algebras generated by elements transforming covariantly under the quantum Lorentz transformations.

Many relations and formulas presented in this paper were found previously [5, 6, 8, 9, 7, 26, 27, 28], but here they will be obtained in a more systematic way due to the flexibility of the  $R$ -matrix and the reflection equation formalism. In particular, our approach is particularly suited to incorporate other  $q$ -Minkowski space proposals, as well as for discussing the non-commutative differential calculus on other quantum spaces as, for instance, with  $SO_q(3)$  symmetry (see in this respect [29, 30]).

The plan of the paper is as follows. First, the procedure for defining a quantum Minkowski space-time algebra  $\mathcal{M}_q$  is given in Sec.2 where some special elements, such as the  $q$ -Minkowski length, are introduced. Other possibilities for  $\mathcal{M}_q$  (some of them are new) may be found using our formalism. In Sec.3, arguments of covariance and consistency establish the commutation relations among  $q$ -‘coordinates’ (generators of  $\mathcal{M}_q$ ),  $q$ -derivatives (id. of  $\mathcal{D}_q$ ) and  $q$ -one-forms (id. of  $\Lambda_q$ ). It is also shown there that in the non-commutative case there cannot be a linear hermitean structure for both coordinates and

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<sup>1</sup>The name is due to the fact that an equation with such a form appeared originally in the factorizable scattering theory on the half line with a reflection from a boundary [22] (see [20]). Originally the reflection equation was written in a ‘spectral parameter’-dependent form [22, 23] (see also [24]); here we shall consider only constant solutions.

derivatives, recovering previous results [9, 31]. In Sec.4, a convenient set of generators is picked up by taking into account their  $q$ -tensor properties with respect to the  $q$ -Lorentz transformations. Sec.5 is devoted to developing the non-commutative differential calculus for  $\mathcal{M}_q$ . A number of elaborated questions concerning the mutual interrelations between quantum Poincaré group  $P_q$  and algebra  $\mathcal{P}_q$ , the representation theory and the physical interpretation are briefly considered in Sec.6, where in particular the mass and momenta spectrum will be calculated. From the discussion it looks that it may not be easy to reconcile the notion of a quantum Minkowski space with the standard treatment of the special relativity or relativistic quantum theory. These points, as well as a classical counterpart of  $\mathcal{M}_q$  are discussed shortly at the end.

Dealing with non-commutative and Hopf algebras to define  $q$ -Minkowski spaces requires using a number of results and explicit expressions from quantum group theory [1, 2, 3]. To facilitate the reading of the paper, some useful facts and formulae are collected in Appendix A; Appendix B contains the proof of certain important algebra isomorphisms and algebra properties used in the main text.

There are, certainly, other approaches to define deformations of kinematical groups and algebras and their realizations (see, e.g., [5, 32, 33, 34, 35, 36] and references therein). In particular, a widely used approach utilizes the contraction procedure [37] to obtain the  $\kappa$ -Poincaré algebra [38, 39]. The approach to  $q$ -relativity based on the  $\kappa$ -Poincaré algebra and the  $q$ -Minkowski spaces  $\mathcal{M}_q^{(i)}$  to be discussed here are, however, unrelated.

## 2 Deformed Minkowski space-time algebras

The group of transformations preserving the Minkowski metric  $diag(1, -1, -1, -1)$  is the Lorentz group  $L$  or  $SO(1,3)$ . Its universal covering group  $SL(2, C)$  possesses two fundamental representations,  $D^{\frac{1}{2}, 0}$  and  $D^{0, \frac{1}{2}}$ , realized by  $2 \times 2$  matrices  $A$  and  $\tilde{A} = (A^{-1})^\dagger$  which act on undotted and dotted spinor respectively,

$$\xi'_\alpha = A_\alpha^\beta \xi_\beta \quad , \quad \bar{\xi}'^{\dot{\alpha}} = \tilde{A}^{\dot{\alpha}}_{\dot{\beta}} \bar{\xi}^{\dot{\beta}} \quad . \quad (2.1)$$

The vector representation on space-time may be obtained through

$$K' = AK\tilde{A}^{-1} = AKA^\dagger \quad . \quad (2.2)$$

where  $K$  is hermitean. Writing  $K = \sigma^\mu x_\mu$  ( $\sigma^\mu = (\sigma^0, \sigma^i)$ ;  $\sigma^0 = I$  and  $\sigma^i$  are the Pauli matrices),  $det K = (x^0)^2 - \vec{x}^2$ . Clearly,  $K' = \sigma^\mu x'_\mu$  with  $x'^\mu = \Lambda^\mu_\nu x^\nu$ , and the correspondence  $\pm A \mapsto \Lambda \in L$  realizes the covering homomorphism  $SL(2, C)/Z_2 = SO(1,3)$ . The antisymmetric matrix  $\epsilon = i\sigma^2$  satisfies the property  $\epsilon A \epsilon^{-1} = (A^{-1})^t$ ; hence  $\epsilon A^* \epsilon^{-1} = (A^{-1})^\dagger$  and

$$K'^\epsilon = \tilde{A} K^\epsilon A^{-1} \quad , \quad K^\epsilon \equiv \epsilon K^* \epsilon^{-1} \quad . \quad (2.3)$$

Clearly, since  $\epsilon(\sigma^\mu)^*\epsilon^{-1} = (\sigma^0, -\sigma^i) \equiv \rho^\mu$ , another four vector of hermitean matrices (not related by a similarity transformation to  $\sigma^\mu$ ) may be introduced. Including the appropriate indices, eqs. (2.2) and (2.3) read

$$K'_{\alpha\delta} = A_\alpha^\beta K_{\beta\gamma}(\tilde{A}^{-1})_{\delta}^{\gamma} \quad , \quad K'^{\epsilon\dot{\alpha}\delta} = \tilde{A}_{\beta}^{\dot{\alpha}} K^{\epsilon\dot{\beta}\gamma}(A^{-1})_{\gamma}^{\delta} \quad , \quad (2.4)$$

and  $\sigma^\mu$  and  $\rho^\mu$  read  $(\sigma^\mu)_{\alpha\dot{\beta}}$ ,  $(\rho^\mu)^{\dot{\alpha}\beta}$ ; they satisfy

$$\frac{1}{2}\text{tr}(\rho_\mu\sigma_\nu) = g_{\mu\nu} \quad , \quad (2.5)$$

and the Lorentz matrix given by  $\pm A$  is

$$\Lambda_\nu^\mu = \frac{1}{2}\text{tr}(\rho^\mu A\sigma_\nu A^\dagger) \quad . \quad (2.6)$$

If  $K$  is now restricted to be traceless, this condition will be preserved only if  $U = (U^{-1})^\dagger$  i.e., by the  $SU(2)$  subgroup. Then, the homomorphism  $SU(2)/Z_2 = SO(3)$  is realized by  $R_j^i = \frac{1}{2}\text{tr}(\sigma^i U \sigma_j U^\dagger)$ . As mentioned in the introduction, we wish to explore in this paper the consequences of  $q$ -deforming the above relations.

The crucial idea [6, 40, 7, 8, 9] to deform the Lorentz group was to replace the  $SL(2, C)$  matrices  $A$  in (2.1) by the generator matrix  $M$  of the quantum group  $SL_q(2)$ . Due to the fact that the hermitean conjugation ( $M_{ij}^\dagger = M_{ji}^*$ ) includes the  $*$ -operation, an extra copy  $\tilde{M}$  of  $SL_q(2)$  was introduced, with entries not commuting with those of  $M$ . The  $R$ -matrix form of the commutation relations among the quantum group generators  $(a, b, c, d)$  of  $M$  and  $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$  of  $\tilde{M}$  are then expressed by

$$\begin{aligned} R_{12}M_1M_2 &= M_2M_1R_{12} \quad , \\ R_{12}\tilde{M}_1\tilde{M}_2 &= \tilde{M}_2\tilde{M}_1R_{12} \quad , \\ R_{12}M_1\tilde{M}_2 &= \tilde{M}_2M_1R_{12} \quad . \end{aligned} \quad (2.7)$$

As in the classical ( $q = 1$ ) case, the reality condition is expressed [6, 7] by requiring  $\tilde{M}^{-1} = M^\dagger$ , which implies

$$\begin{pmatrix} \tilde{d} & -\tilde{b}/q \\ -q\tilde{c} & \tilde{a} \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \quad . \quad (2.8)$$

This condition is consistent with all relations in (2.7) provided that the deformation parameter  $q$  is real,  $q \in R$ . The set of generators  $(a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$  satisfying  $\det_q M = 1 = \det_q \tilde{M}$ , the commutation relations (2.7) and the conditions (2.8) define the quantum Lorentz group  $L_q$ . There are, however, other possibilities if we allow for different  $R$  matrices in eqs. (2.7); these will be discussed at the end of the section.

To introduce the  $q$ -Minkowski algebra  $\mathcal{M}_q$  it is natural to extend (2.2) to the quantum case by stating that  $K$  is a comodule for the coaction  $\phi$  defined by

$$\phi : K \longmapsto K' = MK\tilde{M}^{-1}, \quad K'_{is} = M_{ij}\tilde{M}_{ls}^{-1}K_{jl}, \quad (2.9)$$

where it is assumed that the matrix elements of  $K$  commute with those of  $M$  and  $\tilde{M}$  but not among themselves<sup>2</sup>. Much in the same way the commuting properties of  $q$ -two-vectors (or, better said here,  $q$ -spinors) are preserved by the coaction of  $M$  and  $\tilde{M}$ , we now demand that the commuting properties of the entries of  $K$  are preserved by (2.9). More specifically, in order to identify the elements of  $K$

$$K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad (2.10)$$

with the generators of the  $q$ -Minkowski algebra  $\mathcal{M}_q$  we require, as in the classical case,

- a) a reality property preserved by (2.9),
- b) a (real)  $q$ -Minkowski length, defined through the  $q$ -determinant  $\det_q K$  of  $K$ , invariant under the  $q$ -Lorentz transformation (2.9),
- c) a set of commutation relations for the elements of  $K$  (a ‘presentation’ of the algebra) preserved by (2.9).

The reality condition  $K = K^\dagger$  is consistent with (2.9) since  $\tilde{M}^{-1} = M^\dagger$  as in the classical case. The  $q$ -determinant of  $K$  and the  $q$ -Minkowski metric will be given below but, using (2.7), it is not difficult to check (it will be discussed in detail later) that the commutation properties expressed by the RE (1.6) or, equivalently (using the permutation operator  $\mathcal{P}$  and  $\hat{R} \equiv \mathcal{P}R$ , eqs. (A.6), (A.7)) by

$$\hat{R}K_1\hat{R}K_1 = K_1\hat{R}K_1\hat{R} \quad , \quad (2.11)$$

are preserved by (2.9) and are consistent with the condition  $K=K^\dagger$ . In components, eq. (1.6) reads

$$R_{ij,kl}K_{kf}R_{lf,gh}K_{gm} = K_{jd}R_{id,cs}K_{ct}R_{st,mh} \quad . \quad (2.12)$$

For  $K$  given by (2.10), eq. (1.6) is equivalent to the six basic relations [6, 7, 25]

$$\begin{aligned} \alpha\beta &= q^{-2}\beta\alpha \quad , & [\delta, \beta] &= q^{-1}\lambda\alpha\beta \quad , \\ \alpha\gamma &= q^2\gamma\alpha \quad , & [\beta, \gamma] &= q^{-1}\lambda(\delta - \alpha)\alpha \quad , \\ [\alpha, \delta] &= 0 \quad , & [\gamma, \delta] &= q^{-1}\lambda\gamma\alpha \quad , \end{aligned} \quad (2.13)$$

which characterize the algebra  $\mathcal{M}_q$ ; we shall adopt the point of view that this ‘quantum space’ (algebra) is the primary object on which the non-commutative differential calculus will be constructed. We may then give the following

*Definition*

The quantum Minkowski space-time algebra is the associative algebra  $\mathcal{M}_q$  generated by the four elements of  $K$ , subjected to the reality conditions  $\alpha =$

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<sup>2</sup>A dotted and undotted index notation may also be introduced in the deformed case. However, since the dotted and undotted indices are always located, as in (2.4), to make multiplication of matrices always possible, we shall only use latin indices from now on.

$\alpha^*$ ,  $\delta = \delta^*$ ,  $\beta^* = \gamma$ ,  $\gamma^* = \beta$ , and satisfying the commutation relations (1.6) (or (2.13)).

The central (commuting) elements of  $\mathcal{M}_q$  may be obtained by using the  $q$ -trace  $tr_q$  [3, 41] which, for a  $2 \times 2$  matrix  $B$  with elements commuting with those of  $M$  (as it is the case of  $K$ ) is defined by

$$tr_q B = tr(DB) = q^{-1}b_{11} + qb_{22} \quad , \quad D = \text{diag}(q^{-1}, q) . \quad (2.14)$$

The  $q$ -trace is invariant under the coaction  $B \mapsto MBM^{-1}$ ,

$$tr_q(MBM^{-1}) = tr_q(B) . \quad (2.15)$$

This follows easily by using the preservation of the  $q$ -symplectic metric  $\epsilon^q$  [10] (which replaces  $i\sigma^2$  for  $q = 1$ ) by the  $SL_q(2)$  matrices,

$$M^t \epsilon^q M = \epsilon^q \quad , \quad \epsilon^q = \begin{bmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{bmatrix} = -(\epsilon^q)^{-1} , \quad (2.16)$$

since  $D = \epsilon^q \epsilon^{qt}$ ; we shall drop the superindex  $q$  henceforth. The matrix  $D$  satisfies

$$M^t D (M^{-1})^t = D \quad , \quad \tilde{M}^t D (\tilde{M}^{-1})^t = D . \quad (2.17)$$

In the general  $R$ -matrix case  $D$  may be expressed (cf. [3]) as  $D = q^2 tr_{(2)}\{\mathcal{P}[(R_{12}^t)^{-1}]^{t_1}\}$ , where the ordinary trace is taken in the second space and the transpositions refer to the first space; the factor  $q^2$  has been chosen to reproduce (2.14) for the  $GL_q(2)$   $R$ -matrix.

The centrality of the  $q$ -trace follows from (1.6), which after left (right) multiplication by  $R_{12}^{-1}$  ( $R_{21}^{-1}$ ) and a similarity transformation with the permutation operator  $\mathcal{P}$  (eqs. (A.6), (A.7)) may be rewritten as

$$K_2 R_{12} K_1 R_{12}^{-1} = R_{21}^{-1} K_1 R_{21} K_2 . \quad (2.18)$$

Indeed, since  $R_{12}$  and  $R_{21}^{-1}$  provide representations of the  $SL_q(2)$  quantum group when considered as  $2 \times 2$  matrices in the first space of  $C^2 \times C^2$  (although not faithful:  $b = 0$  for  $R_{12}$  and  $c = 0$  for  $R_{21}^{-1}$ ), taking the  $q$ -trace in the first space it follows that

$$K c_1 = c_1 K \quad (2.19)$$

with

$$c_1 \equiv tr_q K = tr_{q(1)}(R_{12} K_1 R_{12}^{-1}) = tr_{q(1)}(R_{21}^{-1} K_1 R_{21}) = q^{-1}\alpha + q\delta . \quad (2.20)$$

By iterating the procedure which lead to (2.18) it is found that

$$K_2 R_{12} K_1^n R_{12}^{-1} = R_{21}^{-1} K_1^n R_{21} K_2 \quad (2.21)$$

and hence, after taking  $tr_{q(1)}$  in this relation,

$$K c_n = c_n K \quad , \quad c_n \equiv tr_q K^n . \quad (2.22)$$



The first two central elements  $c_1$  and  $c_2$  are algebraically independent, but the  $c_n$  for  $n > 2$  are polynomial functions of them due to the characteristic equation for  $K$  [19, 21],

$$qK^2 - c_1K + \frac{q}{[2]}(q^{-1}c_1^2 - c_2)I = 0. \quad (2.23)$$

The  $q$ -determinant  $\det_q K$  of  $K$  is obtained by means of the  $q$ -antisymmetrizer  $P_-$  [(A.11), (A.12)], which is a rank one  $4 \times 4$  projector. It is defined [19] through

$$(\det_q K)P_- = -qP_-K_1\hat{R}K_1P_- = (\alpha\delta - q^2\gamma\beta)P_-, \quad (2.24)$$

although one of the projectors  $P_-$  may be dropped since it is easy to check that both  $P_-K_1\hat{R}K_1$  and  $K_1\hat{R}K_1P_-$  are proportional to  $P_-$ . It is central since

$$\det_q K = \frac{q^2}{[2]}(q^{-1}c_1^2 - c_2) \quad , \quad (2.25)$$

and, provided it is not zero,

$$K^{-1} = (\det_q K)^{-1} \begin{bmatrix} q^2\delta - q\lambda\alpha & -q^2\beta \\ -q^2\gamma & \alpha \end{bmatrix} \quad . \quad (2.26)$$

Since it may be expressed in terms of  $q$ -traces,  $\det_q K$  is obviously preserved under a similarity transformation  $K \mapsto MKM^{-1}$  where  $M$  and  $M^{-1}$  belong to the same quantum group. But, despite of the fact that the central elements  $c_n$  are not invariant with respect the  $q$ -Lorentz transformation (2.9) because it involves  $M$  and  $\tilde{M}^{-1}$ ,  $\det_q K$  is nevertheless preserved under this coaction. Using (2.24) we find

$$\begin{aligned} \det_q \phi(K) &= \det_q(MK\tilde{M}^{-1}) = -q^{-1}P_-M_1K_1M_2\hat{R}\tilde{M}_2^{-1}K_1\tilde{M}_1^{-1}P_- \\ &= -q^{-1}(\det_q M)P_-K_1\hat{R}K_1(\det_q \tilde{M})^{-1} = (\det_q M)\det_q K(\det_q \tilde{M})^{-1} \quad , \end{aligned}$$

where we have used  $\tilde{M}_1^{-1}\hat{R}M_1 = M_2\hat{R}\tilde{M}_2^{-1}$  (from the last equality in (2.7)) as well as the definition (A.13) of  $\det_q M$ . Since  $\det_q M = 1 = \det_q \tilde{M}$ ,  $\det_q \phi(K) = \det_q K$ , and we may identify this real and central invariant element with the square  $l_q$  of the  $q$ -Minkowski invariant length [6, 7, 8, 9]

$$l_q \equiv \det_q K = \alpha\delta - q^2\gamma\beta \quad , \quad l_q \in \mathcal{M}_q \quad . \quad (2.27)$$

The  $q$ -trace  $tr_q K = c_1$  is central but not invariant; it will be later identified with the time coordinate. It is invariant under the  $SU_q(2)$  ‘subgroup’ as in the classical case since then  $M=U$ ,  $\tilde{M}^{-1}=M^\dagger=U^\dagger$ , the matrices  $U$  satisfy the ‘unitarity’ condition  $U^\dagger=U^{-1}$  and the  $SU_q(2)$  coaction is defined by  $K \mapsto UKU^{-1}$  (it will be seen in Sec. 5, however, that there is no consistent reduction to  $SU_q(2)$  in the whole  $\mathcal{M}_q \times \mathcal{D}_q$  algebra). We shall not need the explicit form of the  $4 \times 4$  and  $3 \times 3$   $q$ -Lorentz and  $q$ -rotation matrices; the interested reader may find them in [6, 42].

Let us now find the expression for the  $q$ -Minkowski metric. First, we notice that if we define  $\hat{R}^\epsilon$  by

$$\hat{R}_{ij,kl}^\epsilon = \epsilon_{js}^t \hat{R}_{is,kt} (\epsilon^t)_{tl}^{-1} \quad (\hat{R}^\epsilon = (1 \otimes \epsilon^t) \hat{R} (1 \otimes (\epsilon^{-1})^t)), \quad (2.28)$$

it follows from the last eq. in (2.7), written in the form  $\hat{R}M_1\tilde{M}_2 = \tilde{M}_1M_2\hat{R}$  or  $\hat{R}M \otimes \tilde{M} = \tilde{M} \otimes M\hat{R}$ , that

$$\hat{R}^\epsilon M_1 (\tilde{M}_2^{-1})^t = \tilde{M}_1 (M_2^{-1})^t \hat{R}^\epsilon \quad (\hat{R}^\epsilon M \otimes (\tilde{M}^{-1})^t = \tilde{M} \otimes (M^{-1})^t \hat{R}^\epsilon) \quad (2.29)$$

since  $\epsilon^t M (\epsilon^{-1})^t = (M^{-1})^t$ , etc. (cf. (2.16)); notice that in the  $q \neq 1$  case,  $(M^{-1})^t \neq (M^t)^{-1}$ , although  $(M^{-1})^\dagger = (M^\dagger)^{-1}$ . Now, since the  $q$ -Lorentz transformation (2.9) may be written as

$$K'_{is} = \Lambda_{is,jl} K_{jl}, \quad \Lambda_{is,jl} = (M \otimes (\tilde{M}^{-1})^t)_{is,jl}, \quad (2.30)$$

(in this form, the reality of  $\Lambda$  means that  $\Lambda_{is,jl}^* = \Lambda_{si,lj}$  or  $\Lambda^* = \mathcal{P}\Lambda\mathcal{P}$ ). As a result, if  $K$  transforms by (2.9) say, contravariantly, then

$$K_{ij}^\epsilon = \hat{R}_{ij,kl}^\epsilon K_{kl} \quad (K^\epsilon = \hat{R}^\epsilon K) \quad (2.31)$$

transforms covariantly,

$$K^{\epsilon t} = \tilde{M} K^\epsilon M^{-1} \quad (2.32)$$

From (2.28) and (A.7) we find

$$\hat{R}^\epsilon = \begin{bmatrix} \lambda & 0 & 0 & -q \\ 0 & q & 0 & 0 \\ 0 & 0 & q & 0 \\ -q^{-1} & 0 & 0 & 0 \end{bmatrix}, \quad (\hat{R}^\epsilon)^{-1} = \begin{bmatrix} 0 & 0 & 0 & -q \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ -q^{-1} & 0 & 0 & -\lambda \end{bmatrix}, \quad (2.33)$$

so that (2.31) gives

$$K^\epsilon = \begin{bmatrix} \lambda\alpha - q\delta & q\beta \\ q\gamma & -q^{-1}\alpha \end{bmatrix} \quad (2.34)$$

and  $KK^\epsilon = -q^{-1}(\det_q K)I$  is  $q$ -Lorentz invariant (this relation is trivially checked when  $\det_q K \neq 0$  since in this case  $K^\epsilon = -q^{-1}(\det_q K)K^{-1}$ , eq. (2.26)). The scalar product is thus given by the  $q$ -trace of  $K^\epsilon K$

$$l_q = \det_q K = \frac{-q}{[2]} \text{tr}_q(KK^\epsilon) = \frac{-q}{[2]} \text{tr}_q(K^\epsilon K) \quad , \quad [l_q, K] = 0, \quad (2.35)$$

which also follows from (2.24) since  $(P_-)_{ij,kl} = [2]^{-1} \epsilon_{ij} \epsilon_{kl}$  (eq. (A.12)) and eq. (2.28) give  $(K_1 \hat{R} K_1 P_-)_{ij,kl} = (K_1 K_1^\epsilon P_-)_{ij,kl}$  so that  $\det_q(K)I = -q(KK^\epsilon)$ . We may define a  $q$ -Minkowski tensor  $g_{ij,kl}$  by means of the expression

$$l_q = q^2 g_{ij,kl} K_{ij} K_{kl} = \frac{-q}{[2]} D_{si} \hat{R}_{js,kl}^\epsilon K_{ij} K_{kl} \quad (2.36)$$

Explicitly,

$$g_{ij,kl} = \frac{-q^{-1}}{[2]} D_{si} \hat{R}_{js,kl}^\epsilon = \frac{-q^{-1}}{[2]} \epsilon_{im} \hat{R}_{jm,kt} \epsilon_{lt}^{-1} \quad , \quad (g = \frac{-q^{-1}}{[2]} D_1 \mathcal{P} \hat{R}_{12}^\epsilon) \quad , \quad (2.37)$$

and, on account of (2.29) and (2.17),  $g$  satisfies

$$\Lambda^t g \Lambda = g \quad , \quad \Lambda_{rs,ij} g_{rs,mn} \Lambda_{mn,kl} = g_{ij,kl} \quad . \quad (2.38)$$

Let us now analyze in generality the possible commutation properties of the entries of a matrix  $K$  generating a  $q$ -Minkowski algebra. We may describe them [43] by means of a general RE (see [20] and references therein and [44])

$$R^{(1)} K_1 R^{(2)} K_2 = K_2 R^{(3)} K_1 R^{(4)} \quad . \quad (2.39)$$

The four  $4 \times 4$   $R^{(i)}$  matrices have now to be determined by demanding the invariance of (2.39) under (2.9),

$$\begin{aligned} R^{(1)} M_1 M_2 &= M_2 M_1 R^{(1)} \quad , \quad R^{(2)} M_2 \tilde{M}_1 = \tilde{M}_1 M_2 R^{(2)} \quad , \\ R^{(3)} M_1 \tilde{M}_2 &= \tilde{M}_2 M_1 R^{(3)} \quad , \quad R^{(4)} \tilde{M}_2 \tilde{M}_1 = \tilde{M}_1 \tilde{M}_2 R^{(4)} \quad , \end{aligned} \quad (2.40)$$

Using the permutation operator  $\mathcal{P}$ , the first equation may be rewritten as

$$\mathcal{P}(R^{(1)})^{-1} \mathcal{P} M_1 M_2 = M_2 M_1 \mathcal{P}(R^{(1)})^{-1} \mathcal{P} \quad (2.41)$$

(and similarly for  $R^{(4)}$  and tilded  $\tilde{M}'$ s). Comparing these equations with (2.7) we find the solutions

$$R^{(1)} = R_{12} \text{ or } R_{21}^{-1} \quad , \quad R^{(2)} = R_{21} \quad , \quad R^{(3)} = R_{12} \quad , \quad R^{(4)} = R_{21} \text{ or } R_{12}^{-1} \quad . \quad (2.42)$$

The solution  $R^{(1)}=R_{12}$ ,  $R^{(4)}=R_{21}$  [43] was already used in (1.6) [(2.11)]. The possibility  $R^{(1)}=R_{12}$ ,  $R^{(4)}=R_{12}^{-1}$  implies replacing (2.11) by

$$\hat{R} K_1 \hat{R} K_1 = q^2 K_1 \hat{R} K_1 \hat{R}^{-1} \quad . \quad (2.43)$$

However, it is shown in Appendix B1 that (2.43) also leads to (2.13) with the restriction  $\det_q K = 0$  and thus it may be discarded. The other two solutions  $R^{(1)}=R_{21}^{-1}$ ,  $R^{(4)}=R_{12}^{-1}$  and  $R^{(1)}=R_{21}^{-1}$ ,  $R^{(4)}=R_{21}$ , are, respectively, the same as (2.11) and (2.43); thus, the assumptions on the  $q$ -Lorentz group reflected by eqs. (2.7) lead uniquely to (1.6) or (2.13) as the relations defining the  $q$ -Minkowski algebra  $\mathcal{M}_q$ .

The above is not the only possibility, since we may introduce another  $q$ -Lorentz group  $L_q^{(2)}$  generated by the same non-commuting entries of  $M$  and  $\tilde{M}$  as reflected by the first two equations in (2.7), with the same  $*$ -operation, but for which the third relation in (2.7) is replaced by  $M_1 \tilde{M}_2 = \tilde{M}_2 M_1$ , so that the elements of  $M$  and  $\tilde{M}$  commute in between. This corresponds to taking  $R^{(2)} = R^{(3)} = I$ , and leads to the possibilities

$$R_{12} K_1^{(2)} K_2^{(2)} = K_2^{(2)} K_1^{(2)} R_{21} \quad ; \quad (2.44)$$

$$R_{12}K_1^{(2)}K_2^{(2)} = q^2K_2^{(2)}K_1^{(2)}R_{12}^{-1} , \quad (2.45)$$

(the superindex has been added to distinguish  $K^{(2)}$  from  $K \equiv K^{(1)}$ ). It is simple to see that eqs. (2.44), (2.45) are consistent with the condition  $K^{(2)} = K^{(2)\dagger}$ . Eqs. (2.44) (and (2.45), which corresponds to  $\det_q K^{(2)}=0$ ) lead to the following independent commutation relations for the entries of  $K^{(2)}$  generating the  $\mathcal{M}_q^{(2)}$  algebra

$$\begin{aligned} \alpha^{(2)}\beta^{(2)} &= q^{-1}\beta^{(2)}\alpha^{(2)} , & \alpha^{(2)}\gamma^{(2)} &= q\gamma^{(2)}\alpha^{(2)} , & [\alpha^{(2)}, \delta^{(2)}] &= 0 , \\ [\beta^{(2)}, \gamma^{(2)}] &= \lambda\alpha^{(2)}\delta^{(2)} , & \beta^{(2)}\delta^{(2)} &= q\delta^{(2)}\beta^{(2)} , & \gamma^{(2)}\delta^{(2)} &= q^{-1}\delta^{(2)}\gamma^{(2)} , \end{aligned} \quad (2.46)$$

(cf. (2.13)). In these cases, there is an invariant and central element in the  $\mathcal{M}_q^{(2)}$  algebra which determines the Minkowski length and metric. The determinant (which is zero for (2.45)) is given through

$$\det_q(K^{(2)})P_- = -qK_1^{(2)}\mathcal{P}K_1^{(2)}P_- , \quad \det_q(K^{(2)}) = \alpha^{(2)}\delta^{(2)} - q\gamma^{(2)}\beta^{(2)} , \quad (2.47)$$

(cf. (2.27)). This definition guarantees that  $\det_q(K'^{(2)})=\det_q(K^{(2)})$  for the coaction  $\phi^{(2)}$  of  $L_q^{(2)}$ ; notice that in (2.24)  $\hat{R}$  is replaced by  $\mathcal{P}$  since now  $M_1\hat{M}_2=\hat{M}_2M_1$ . For  $L_q^{(2)}$ , however, there is no linear central element. This follows from the fact that eq. (2.44) can be linearly transformed into a ‘ $RTT$ ’ relation [(1.3), (1.1)] by the change  $T = K^{(2)}\sigma^1$ , where  $\sigma^1$  is the Pauli matrix, and that there is no linear central element for the  $GL_q(2)$  algebra.

A third possibility is obtained by setting equal to unity the  $R$  matrices in the two first equations in (2.7) but not in the third. In this case, the matrix elements of  $M$  (and  $\tilde{M}$ ) are commuting (they define a  $SL(2, C)$  group each), and the non-commutativity of the entries of  $M$  and  $\tilde{M}$  is described by a certain matrix  $V$  through (cf. (2.7))

$$VM_1\tilde{M}_2 = \tilde{M}_2M_1V . \quad (2.48)$$

There is no  $SU_q(2)$  subgroup in this case: the identification  $M = (M^{-1})^\dagger = \tilde{M}$  would imply  $VM_1M_2 = M_2M_1V$  contradicting the assumed commutativity of the entries of  $M$ . The commuting properties of the corresponding  $K^{(3)} \equiv Z$ , with entries and transformation properties given by

$$Z = \begin{bmatrix} z^1 & z^4 \\ z^2 & z^3 \end{bmatrix} = Z^\dagger , \quad Z' = MZ\tilde{M}^{-1} , \quad (2.49)$$

( $z^1=z^{1*}$ ,  $z^2=z^{4*}$ ,  $z^4=z^{2*}$ ,  $z^3=z^{3*}$ ) are determined by setting  $R^{(2)}=V=R^{(3)}$ ,  $R^{(1)}=I=R^{(4)}$  in eq. (2.39),

$$Z_1VZ_2 = Z_2VZ_1 . \quad (2.50)$$

For instance, for  $V = \text{diag}(p^2, 1, 1, p^2)$  (we have used  $p$  rather than  $q$  to stress the trivial deformation character of this algebra, see below), we obtain

$$\begin{aligned} z^1z^2 &= p^2z^2z^1 , & z^2z^3 &= p^2z^3z^2 , & z^3z^4 &= p^2z^4z^3 , \\ z^4z^1 &= p^2z^1z^4 , & z^4z^2 &= z^2z^4 , & z^1z^3 &= z^3z^1 . \end{aligned} \quad (2.51)$$

This  $p$ -Minkowski algebra  $\mathcal{M}_p^{(3)}$  may be obtained [45] from the analysis of a deformation of the conformal group  $SU(2, 2)$  as a real form of a multiparametric deformation of  $SL(4, C)$ , which justifies the previous form for  $V$ . However, this algebra and the corresponding deformed Lorentz group have been shown to be [46] a simple transformation (twisting [47, 48]) of the usual ones.

### 3 Deformed derivatives and $q$ -De Rham complex

The development of a non-commutative differential calculus (see, e.g., [49, 18, 50, 41, 51, 52]) requires including derivatives and differentials. We shall now do this first for the  $q$ -Minkowski space  $\mathcal{M}_q$  [(2.11)] by extending the RE to accommodate in them derivatives and differentials. Consider first an object  $Y$  transforming covariantly i.e.,

$$Y \longmapsto Y' = \tilde{M}Y M^{-1} \quad , \quad Y = \begin{bmatrix} u & v \\ w & z \end{bmatrix} . \quad (3.1)$$

The invariance of the commutation properties of the matrix elements of  $Y$  (now described by (2.39) with  $Y$  replacing  $K$ ) gives, on account of (2.7), the solutions

$$R^{(1)} = R_{12} \text{ or } R_{21}^{-1} , \quad R^{(2)} = R_{12}^{-1} , \quad R^{(3)} = R_{21}^{-1} , \quad R^{(4)} = R_{21} \text{ or } R_{12}^{-1} . \quad (3.2)$$

These four possibilities again reduce to two,

$$R_{12}Y_1R_{12}^{-1}Y_2 = Y_2R_{21}^{-1}Y_1R_{21} \quad \text{or} \quad \hat{R}Y_1\hat{R}^{-1}Y_1 = Y_1\hat{R}^{-1}Y_1\hat{R} ; \quad (3.3)$$

$$q^2\hat{R}^{-1}Y_1\hat{R}^{-1}Y_1 = Y_1\hat{R}^{-1}Y_1\hat{R} , \quad (3.4)$$

of which we shall retain only (3.3) since (3.4) leads to the same algebra plus the condition  $\det_q Y = 0$  (see (3.6) below and Appendix A). Eq. (3.3) gives

$$\begin{aligned} [u, v] &= q\lambda v z , & v z &= q^2 z v , & [u, z] &= 0 , \\ [w, u] &= q\lambda z w , & [v, w] &= q\lambda(u - z)z , & w z &= q^{-2} z w . \end{aligned} \quad (3.5)$$

The (central and  $q$ -Lorentz invariant)  $q$ -determinant is defined through

$$(\det_q Y)P_- = (-q^{-1})P_-Y_1\hat{R}^{-1}Y_1P_- = (uz - q^{-2}vw)P_- \quad (3.6)$$

so that, when it is non-zero,

$$Y^{-1} = (\det_q Y)^{-1} \begin{bmatrix} z & -q^{-2}v \\ -q^{-2}w & q^{-2}u + q^{-1}\lambda z \end{bmatrix} . \quad (3.7)$$

Since  $Y$  is covariant, we may define a contravariant  $Y^\epsilon$  by (cf. (2.31))

$$Y^\epsilon = (\hat{R}^\epsilon)^{-1} Y = \begin{bmatrix} -qz & q^{-1}v \\ q^{-1}w & -q^{-1}u - \lambda z \end{bmatrix} \quad (3.8)$$

(when  $\det_q Y \neq 0$ ,  $Y^\epsilon = -q(\det_q Y)Y^{-1}$ ); then, (cf. (2.35))

$$\det_q Y = -\frac{q^{-1}}{[2]} \text{tr}_q(YY^\epsilon) = -\frac{q^{-1}}{[2]} \text{tr}_q(Y^\epsilon Y) \equiv_q, \quad [{}_q, Y] = 0, \quad (3.9)$$

where  ${}_q$  becomes the ( $L_q$ -invariant)  $q$ -D'Alembertian once the components of  $Y$  are associated with the  $q$ -derivatives. Indeed, since we have already associated  $K$  (contravariant) with  $\mathcal{M}_q$  and we wish to have the equivalent of the classical Lorentz invariant  $x^\mu \partial_\mu$  we shall associate  $Y$  (covariant) with the algebra  $\mathcal{D}_q$  of the  $q$ -Minkowski derivatives; it is clear, however, that one could proceed reciprocally. The commutation properties of the elements of  $K$ ,  $Y^\epsilon$  and  $Y^{-1}$  (when  $\det_q Y \neq 0$ ) are governed by an equation of the type (1.6-2.11); similarly, those of  $Y$ ,  $K^\epsilon$  and  $K^{-1}$  (when  $\det_q K \neq 0$ ) obey a relation like (3.3) (for instance, inverting (2.11) one obtains  $\hat{R}K_1^{-1}\hat{R}^{-1}K_1^{-1} = K_1^{-1}\hat{R}^{-1}K_1^{-1}\hat{R}$ , cf.(3.3)). Thus, the entries of  $K$ ,  $Y^\epsilon$  and  $Y^{-1}$  (resp.  $Y$ ,  $K^\epsilon$ ,  $K^{-1}$ ) satisfy the same commutation relations, and the algebras they generate are isomorphic. Symbolically,  $K \sim Y^\epsilon \sim Y^{-1}$  and  $Y \sim K^\epsilon \sim K^{-1}$ , a fact that may be explicitly checked using eqs. (2.13), (3.5), (3.8), (3.7) and (2.35), (2.26). Moreover, it is proved in Appendix B2 that the following isomorphisms among these algebras hold

$$\mathcal{M}_q \approx \mathcal{M}_{q^{-1}} \approx \mathcal{D}_q \approx \mathcal{D}_{q^{-1}}, \quad (3.10)$$

where the subindex  $q^{-1}$  indicates that the corresponding algebras are defined by (2.13) and (3.5) where  $q$  is replaced by  $q^{-1}$ .

The next step in constructing the non-commutative  $q$ -Minkowski differential calculus is to establish the commutation properties among coordinates and derivatives. We need extending the classical relation  $\partial_\mu x^\nu = x^\nu \partial_\mu + \delta_\mu^\nu$ ,  $\partial^\dagger = -\partial$ , to the non-commutative case in a  $q$ -Lorentz covariant way. This requires considering an inhomogeneous RE of the form (more complicated expressions are possible [43])

$$Y_2 R^{(1)} K_1 R^{(2)} = R^{(3)} K_1 R^{(4)} Y_2 + \eta J, \quad (3.11)$$

where  $\eta$  is a constant and  $\eta J \rightarrow I_4$  in the  $q \rightarrow 1$  limit, invariant under the transformation

$$J \mapsto \tilde{M}_2 M_1 J \tilde{M}_1^{-1} M_2^{-1} = J, \quad \tilde{M}_2 M_1 J = J M_2 \tilde{M}_1. \quad (3.12)$$

This equation exhibits the need of having  $K$  and  $Y$  transforming contravariantly/covariantly; indeed, the assumption that they transform in the same manner (both as  $K$ , say) leads to  $M_1 M_2 J = J \tilde{M}_1 \tilde{M}_2$  which cannot be fulfilled already in the  $q = 1$  case since it would imply the equivalence of inequivalent representations.

An analysis similar to those of Sec. 2 shows that the invariance of the non-linear terms in (3.11) under (2.9), (3.1) is achieved with

$$R^{(1)} = R_{12} \text{ or } R_{21}^{-1}, \quad R^{(2)} = R_{21}, \quad R^{(3)} = R_{12}, \quad R^{(4)} = R_{12}^{-1} \text{ or } R_{21}. \quad (3.13)$$

As for  $J$ , setting  $J \equiv J'\mathcal{P}$  in eq. (3.12) gives  $\tilde{M}_2 M_1 J' = J' M_1 \tilde{M}_2$ , hence  $J = R_{12} \mathcal{P}$  (the same result follows if we set  $J = \mathcal{P} J'$ ). This means that there are, in principle, four basic possibilities consistent with covariance expressing the commutation properties of coordinates (elements of  $K$ ) and derivatives (entries of  $Y$ ). Using again  $\hat{R} = \mathcal{P} R$ , these read

$$Y_1 \hat{R} K_1 \hat{R} = \hat{R} K_1 \hat{R}^{-1} Y_1 + \eta_1 \hat{R} ; \quad (3.14)$$

$$Y_1 \hat{R}^{-1} K_1 \hat{R} = \hat{R} K_1 \hat{R} Y_1 + \eta_2 \hat{R} ; \quad (3.15)$$

$$Y_1 \hat{R}_1 K_1 \hat{R} = \hat{R} K_1 \hat{R} Y_1 + \eta_3 \hat{R} ; \quad (3.16)$$

$$Y_1 \hat{R}_1^{-1} K_1 \hat{R} = \hat{R} K_1 \hat{R}^{-1} Y_1 + \eta_4 \hat{R} . \quad (3.17)$$

Due to the fact that these expressions now involve  $K$  and  $Y$ , they are all inequivalent. In fact, we do not need assuming that the four  $Y$ 's appearing in each of the equations (3.14-3.17) are the same; all that it is demanded is that all they transform as  $Y \mapsto \tilde{M} Y M^{-1}$ .

Let us now look at the hermiticity properties of  $K$  and  $Y$ . It is clear that, since  $\hat{R} = \hat{R}^\dagger$  ( $q$  is real), eqs. (1.6) and (3.3) are consistent with the hermiticity of  $K$  and the antihermiticity of  $Y$ . However, this is no longer the case if the inhomogeneous equations are included. Keeping the physically reasonable assumption that  $K$  is hermitean, eq. (3.14) gives

$$Y_1^\dagger \hat{R}^{-1} K_1 \hat{R} = \hat{R} K_1 \hat{R} Y_1^\dagger - \eta_1 \hat{R} \quad (3.18)$$

i.e.,  $Y^\dagger$  satisfies the commutation relations given by the second inhomogeneous equation (3.15) for  $\eta_2 = -\eta_1$  (of course,  $Y'^\dagger = \tilde{M} Y^\dagger M^{-1}$  again since  $\tilde{M} = (M^{-1})^\dagger$ ). Thus, we need accommodating  $Y^\dagger$  by means of *another* reflection equation, eq. (3.15) for  $Y^\dagger$ . Having then selected (3.14) for  $Y$  and (3.15) for  $Y^\dagger$ , we may now consider the other possibilities (3.16), (3.17). It turns out that these possibilities are inconsistent with the previous relations (1.6) and (2.40), what may be seen with a little effort by acting on (3.16), (3.17) with an additional  $K$  or  $Y$ .

In order to have the inhomogeneous term in the simplest form (the analogue of the  $\delta_{\nu}^{\mu}$  of the  $q = 1$  case) it is convenient to take  $\eta_1 = q^2 = \eta$  and to redefine  $Y^\dagger$  as  $\tilde{Y} = -q^{-4} Y^\dagger$ . In this way, the full set of equations describing the commutation relations of the generators of the algebras of coordinates ( $K$ ), derivatives ( $Y$ ) and their hermitean conjugates ( $Y^\dagger \propto \tilde{Y}$ ) are given by

$$\begin{aligned} \hat{R} K_1 \hat{R} K_1 &= K_1 \hat{R} K_1 \hat{R} , \\ \hat{R} Y_1 \hat{R}^{-1} Y_1 &= Y_1 \hat{R}^{-1} Y_1 \hat{R} \quad , \quad \hat{R} \tilde{Y}_1 \hat{R}^{-1} \tilde{Y}_1 = \tilde{Y}_1 \hat{R}^{-1} \tilde{Y}_1 \hat{R} ; \\ Y_1 \hat{R} K_1 \hat{R} &= \hat{R} K_1 \hat{R}^{-1} Y_1 + \eta \hat{R} ; \quad \eta = q^2 ; \\ \tilde{Y}_1 \hat{R}^{-1} K_1 \hat{R} &= \hat{R} K_1 \hat{R} \tilde{Y}_1 + \tilde{\eta} \hat{R} , \quad \tilde{\eta} = q^{-2} . \end{aligned} \quad (3.19)$$

Notice that, although we have identified  $Y$  with the derivatives and  $\tilde{Y}$  with their hermiteans, the reciprocal assignment is also possible.

To determine now the commutation relations for the  $q$ -De Rham complex we now introduce the exterior derivative  $d$  following [9]; we shall assume that  $d^2=0$  and that it satisfies the Leibniz rule (other possibilities for  $q$ -differential calculus are occasionally considered [53, 54]). To the four generators of the  $\mathcal{M}_q$  (coordinates) and of  $\mathcal{D}_q$  (derivatives) Minkowski algebras we now add the four entries of  $dK$  ( $q$ -one-forms), which generate the algebra of the  $q$ -forms  $\Lambda_q$  (the degree of a form is defined as in the classical case). Clearly,  $d$  commutes with the  $q$ -Lorentz coaction (2.9), so that

$$dK' = MdK\tilde{M}^{-1} \quad . \quad (3.20)$$

Applying  $d$  to the first equation in (3.19) we obtain

$$\hat{R}dK_1\hat{R}K_1 + \hat{R}K_1\hat{R}dK_1 = dK_1\hat{R}K_1\hat{R} + K_1\hat{R}dK_1\hat{R} \quad . \quad (3.21)$$

We now use that  $\hat{R} = \hat{R}^{-1} + \lambda I$  to replace one  $\hat{R}$  in each side in such a way that the terms in  $dK_1\hat{R}K_1$  may be cancelled. In this way we obtain that a solution to (3.21) is given by

$$\hat{R}K_1\hat{R}dK_1 = dK_1\hat{R}K_1\hat{R}^{-1} \quad , \quad (3.22)$$

from which follows that

$$\hat{R}dK_1\hat{R}dK_1 = -dK_1\hat{R}dK_1\hat{R}^{-1} \quad . \quad (3.23)$$

Again, we may check that these relations are not invariant under hermitean conjugation, since they lead, respectively, to

$$\hat{R}dK_1^\dagger\hat{R}K_1 = K_1\hat{R}dK_1^\dagger\hat{R}^{-1} \quad , \quad \hat{R}dK_1^\dagger\hat{R}dK_1^\dagger = -dK_1^\dagger\hat{R}dK_1^\dagger\hat{R}^{-1} \quad . \quad (3.24)$$

Notice again that the first equation in (3.24) is also a legitimate solution of (3.21) for a generic  $dK$  ( $dK$  and  $dK^\dagger$  transform in the same manner); in fact, it is obtained by replacing two  $\hat{R}$ 's in (3.21) in such a way that now the terms  $K_1\hat{R}dK_1^\dagger$  are cancelled. We expect the ' $q$ -determinant' of  $dK$  to vanish; using (3.23) we check that

$$tr_q(dK \ dK^\epsilon) = 0 \quad , \quad (3.25)$$

where  $dK^\epsilon = \hat{R}^\epsilon dK$  (cf. (2.31)) and, in fact,  $P_-dK_1\hat{R}dK_1 = 0$ .

Finally, to complete the full set of commutation relations, we need those of  $dK$  and  $Y$  (and their hermiteans). They are given in general by

$$Y_2R^{(1)}dK_1R^{(2)} = R^{(3)}dK_1R^{(4)}Y_2 \quad , \quad (3.26)$$

which has the same transformation properties as (3.11) with  $J = 0$  and hence the same solutions (3.13). Thus, we may take

$$Y_1\hat{R}^{-1}dK_1\hat{R} = \hat{R}dK_1\hat{R}Y_1 \quad . \quad (3.27)$$



Its hermitean conjugated relation has the form

$$\tilde{Y}_1 \hat{R} dK_1^\dagger \hat{R} = \hat{R} dK_1^\dagger \hat{R}^{-1} \tilde{Y}_1 \quad , \quad (3.28)$$

and it corresponds to another possible solution for (3.26) now written for  $\tilde{Y}$ . Notice that the RE (3.26) is, as always, characterized by the transformation properties of its entries, and that  $Y$  and  $\tilde{Y}$  and  $dK$  and  $dK^\dagger$ , respectively, transform in the same manner due to the reality condition  $\tilde{M} = (M^{-1})^\dagger$ . The other two solutions (with  $Y$  ( $dK$ ) replaced by  $\tilde{Y}$  ( $dK^\dagger$ )) correspond to the commutation properties of  $\tilde{Y}$ ,  $dK$  and  $Y$ ,  $dK^\dagger$

$$\tilde{Y}_1 \hat{R} dK_1 \hat{R} = \hat{R} dK_1 \hat{R} \tilde{Y}_1 \quad \iff \quad \hat{R} dK_1^\dagger \hat{R} Y_1 = Y_1 \hat{R} dK_1^\dagger \hat{R} ; \quad (3.29)$$

$$Y_1 \hat{R}^{-1} dK_1^\dagger \hat{R} = \hat{R} dK_1^\dagger \hat{R}^{-1} Y_1 \quad \iff \quad \hat{R} dK_1 \hat{R}^{-1} \tilde{Y}_1 = \tilde{Y}_1 \hat{R}^{-1} dK_1 \hat{R} . \quad (3.30)$$

Eqs. (3.19), (3.22-3.24) and (3.27-3.30) [43] define the full differential calculus on  $\mathcal{M}_q$  [9]. We may now give a compact expression for  $d$ . As its invariance suggests, it has the form

$$d = tr_q(dKY) . \quad (3.31)$$

We shall use (3.31) to obtain further expressions for the non-commutative differential calculus on  $\mathcal{M}_q$  in Sec. 5 but, before doing so, it will be convenient to discuss in the next section how to select a more natural basis in  $\mathcal{M}_q$  and  $\mathcal{D}_q$ .

To exhibit the generality of the previous reasonings, we conclude this section by applying briefly our framework to e.g., the  $\mathcal{M}_p^{(3)}$  algebra discussed at the end of Sec.2. For the derivatives (which transform by  $\phi : D \mapsto D' = (M^\dagger)^{-1} D \tilde{M}$ ) we find that

$$D_1 V^{-1} D_2 = D_2 V^{-1} D_1 \quad , \quad D = \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_4 & \partial_3 \end{pmatrix} , \quad (3.32)$$

since  $\mathcal{P}V\mathcal{P}=V$ ; thus,  $\mathcal{M}_{p^{-1}}^{(3)} \approx \mathcal{D}_p^{(3)}$  since  $V(p)^{-1}=V(p^{-1})$ . Here  $Z$  and  $D$  may be taken hermitean and antihermitean simultaneously, and there is only one possibility for the mixed  $Z, D$  relation of the form

$$D_1 Z_2 V = V Z_2 D_1 + \mathcal{P}V \quad (3.33)$$

since  $M_2^{-1} M_1^\dagger J M_1 (M_2^{-1})^\dagger = J$  with  $J \propto \mathcal{P}V$  using (2.48). Clearly,  $d = tr(dZD)$ . For the basic relations of the  $q$ -De Rham complex we get

$$dZ_1 V Z_2 = Z_2 V dZ_1 \quad , \quad dZ_1 V dZ_2 = -dZ_2 V dZ_1 \quad ; \quad (3.34)$$

all other relations are found easily.

## 4 $q$ -Tensors and covariant ‘coordinates’

Since neither of the  $q$ -Minkowski spaces  $\mathcal{M}_q^{(i)}$  are manifolds (they are non-commutative algebras), we cannot define ‘coordinates’ for  $\mathcal{M}_q^{(i)}$  in the usual sense. However, there are some sets of generators which are more convenient than the generic ones provided by the entries of  $K$ , eq. (2.10). To find more suitable set of generators for  $\mathcal{M}_q^{(1)} = \mathcal{M}_q$  we now look for the quantum equivalent to the classical splitting  $K = \sigma_\mu x^\mu$ . We shall introduce the  $q$ -sigma matrices by imposing the condition that they are  $q$ -tensors [55, 56]. Consider the simplest case of  $SU_q(2)$ . The statement that the  $\sigma^i$  ( $i = 1, 2, 3$ ) constitute a  $q$ -tensor (in fact, a  $q$ -vector under the  $q$ -deformed rotation group) means that the adjoint action  $\rho(X)$  may be expressed in the form

$$(\rho(X)\vec{\sigma})_{il} = \langle X, U\vec{\sigma}U^{-1} \rangle_{il} = \langle X, U_{ij}U_{kl}^{-1} \rangle \vec{\sigma}_{jk} , \quad (4.1)$$

where  $U \in SU_q(2)$  and  $X$  is an element of its dual quantum algebra. Using the product/coproduct duality [(A.14)] and that  $\Delta(X) = \sum_r X_1^r \otimes X_2^r$ , this may be written as

$$\begin{aligned} (\rho(X)\vec{\sigma})_{ij} &= \sum_r \langle X_1^r, U_{ij} \rangle \vec{\sigma}_{jk} \langle X_2^r, U_{kl}^{-1} \rangle = \\ &= \sum_r \langle X_1^r, U_{ij} \rangle \vec{\sigma}_{jk} \langle S(X_2^r), U_{kl} \rangle , \end{aligned} \quad (4.2)$$

since  $S(U)_{kl} = U_{kl}^{-1}$  and  $\langle X, S(U) \rangle = \langle S(X), U \rangle$ ,  $S$  denoting the antipode. Written in this way (4.2) corresponds to the general definition of a tensor operator  $T$  [56]

$$\rho(X)T = \sum_r (X_1^r)_w T S(X_2^r)_w , \quad (4.3)$$

where the subindex  $w$  here indicates the representation of  $X_1^r$  and  $S(X_2^r)$ . It is simple to see that this expression reduces to the more familiar one given in terms of commutators with the generators of the  $q$ -algebra. Using the expressions for the coproduct of the  $su_q(2)$  quantum algebra generators

$$\begin{aligned} \Delta(J_3) &= J_3 \otimes 1 + 1 \otimes J_3 \quad , \quad \Delta(q^{J_3}) = q^{J_3} \otimes q^{J_3} , \\ \Delta(J_\pm) &= J_\pm \otimes q^{-J_3} + q^{J_3} \otimes J_\pm , \\ S(J_3) &= -J_3 , \quad S(q^{\pm J_3}) = q^{\mp J_3} , \quad S(J_\pm) = -q^{\mp 1} J_\pm , \end{aligned} \quad (4.4)$$

eqs. (4.2) or (4.3) give for a  $SU_q(2)$  tensor  $T_m^j$  the explicit conditions [57]

$$\begin{aligned} [J_3, T_m^j] &= mT_m^j \\ (J_\pm T_m^j - q^m T_m^j J_\pm) q^{J_3} &= \sqrt{[j \pm m + 1][j - m]} T_{m\pm 1}^j \end{aligned} \quad (4.5)$$

which reduce to the usual commutators for  $q = 1$ . Note, however, that the generators of the algebra do not constitute a  $q$ -vector in the quantum case. Using

now the representations of  $J_{\pm}, J_3$  given by  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  (see Appendix A2) it is easy to check that the ‘ $q$ -Pauli’ matrices [42]  $\sigma_{\pm}, \sigma_3$  in the set

$$\begin{aligned} \sigma_0 &= \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} ; & \sigma_+ &= [2]^{1/2} \begin{bmatrix} 0 & -q^{-1/2} \\ 0 & 0 \end{bmatrix} , \\ \sigma_3 &= \begin{bmatrix} -q & 0 \\ 0 & q^{-1} \end{bmatrix} , & \sigma_- &= [2]^{1/2} \begin{bmatrix} 0 & 0 \\ -q^{1/2} & 0 \end{bmatrix} , \end{aligned} \quad (4.6)$$

constitute an  $SU_q(2)$   $q$ -vector; the additional  $q$ -sigma matrix,  $\sigma_0=qI$ , is an  $SU_q(2)$  scalar. Using (4.6), the  $K$  matrix adopts the form

$$K = \sigma_{\mu} x^{\mu} = \begin{bmatrix} q(x^0 - x^3) & -[2]^{1/2} q^{-1/2} x^+ \\ -[2]^{1/2} q^{1/2} x^- & qx^0 + q^{-1} x^3 \end{bmatrix} \equiv \begin{bmatrix} qD & B \\ A & C/q \end{bmatrix} , \quad (4.7)$$

where  $(A, B, C, D)$  is the basis used in [6, 7, 9]; thus, the time  $x^0=q^{-1}[2]^{-1}tr_q K$  is central.

The  $q$ -Minkowski tensor (2.37) gives in the  $(\alpha, \beta, \gamma, \delta)$  basis

$$g = \frac{-q^{-1}}{[2]} \begin{bmatrix} \lambda q^{-1} & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & q^2 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} ; \quad (4.8)$$

in the  $x^{\mu}$  ( $\mu=0, \pm, 3$ ) and  $x^I=(A, B, C, D)$  basis the metric reads, respectively,

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -q^{-1} & 0 \\ 0 & -q & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = g^{\mu\nu} , \quad g_{IJ} = \frac{-q^{-1}}{[2]} \begin{bmatrix} 0 & q^2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & q\lambda \end{bmatrix} . \quad (4.9)$$

Similar expressions are given in [6, 40, 8, 9]; the overall factor in (4.8) has been fixed so that in the ‘physical’  $x^{\mu}$  basis  $g_{\mu\nu}$  has determinant one.

Using (4.7) and (2.13), we find that the six basic commutation relations for the  $x$ 's are given by [6, 8]

$$\begin{aligned} [x^0, x^{\pm}] &= [x^0, x^3] = 0 , & x^-(x^0 - x^3) &= q^{-2}(x^0 - x^3)x^- , \\ [x^+, x^-] &= \lambda x^3(x^0 - x^3) , & x^+(x^0 - x^3) &= q^2(x^0 - x^3)x^+ , \end{aligned} \quad (4.10)$$

and in the  $(A, B, C, D)$  basis by [6, 7, 9]

$$\begin{aligned} [A, B] &= -q^{-1}\lambda CD + q\lambda D^2 , & [B, C] &= -q^{-1}\lambda BD , \\ [A, C] &= q\lambda AD , & BD &= q^2 DB , \\ AD &= q^{-2}DA , & [D, C] &= 0 . \end{aligned} \quad (4.11)$$

Since the metric  $g_{\mu\nu}$  is not symmetric, a convention is needed to raise and lower indices. We shall adopt the convention that  $g$  acts on coordinates from

the left and on the  $q$ -sigmas from the right,  $K = \sigma_\mu x^\mu = \sigma^\nu g_{\nu\mu} x^\mu = \sigma^\nu x_\nu$ . Thus,

$$\begin{aligned}\sigma^0 &= \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} = \sigma_0, & \sigma^+ &= [2]^{1/2} \begin{bmatrix} 0 & 0 \\ q^{3/2} & 0 \end{bmatrix} = -q\sigma_-, \\ \sigma^3 &= \begin{bmatrix} q & 0 \\ 0 & -q^{-1} \end{bmatrix} = -\sigma_3, & \sigma^- &= [2]^{1/2} \begin{bmatrix} 0 & q^{-3/2} \\ 0 & 0 \end{bmatrix} = -q^{-1}\sigma_+.\end{aligned}\tag{4.12}$$

Then,  $K = \sigma^\nu x_\nu$  is given by

$$K = \begin{bmatrix} qx_0 + qx_3 & q^{-3/2} [2]^{1/2} x_- \\ [2]^{1/2} q^{3/2} x_+ & qx_0 - q^{-1} x_3 \end{bmatrix},\tag{4.13}$$

and  $l_q = q^2 g_{\mu\nu} x^\mu x^\nu = q^2 x_\mu x_\nu g^{\nu\mu}$  [(2.35) and (2.36)]

$$\begin{aligned}l_q &= q^2[(x^0)^2 - q^{-1}x^+x^- - qx^-x^+ - (x^3)^2] \\ &= q^2[(x_0)^2 - qx_+x_- - q^{-1}x_-x_+ - (x_3)^2];\end{aligned}\tag{4.14}$$

$$l_q = CD - q^2AB \quad .$$

A four vector basis  $\rho^\mu$  for the covariant matrices  $Y$  is immediately obtained from the contravariant  $\sigma^\mu$  by means of  $\hat{R}^\epsilon$ . Since all these matrices are defined up to a proportionality constant, we introduce a factor  $-q^{-1}$  by convenience to define

$$\rho_\mu = -q^{-1}\hat{R}^\epsilon \sigma_\mu \quad , \quad \rho^\mu = -q^{-1}\hat{R}^\epsilon \sigma^\mu \quad ,\tag{4.15}$$

where  $(\hat{R}^\epsilon \sigma)_{ij} = \hat{R}^\epsilon_{ij,kl} \sigma_{kl}$ . Using (2.33) we obtain  $\rho^0 = \rho_0 = q^{-1}I$  and

$$\rho_\mu = (\rho_0, -\sigma_i) \quad , \quad \rho^\mu = (\rho^0, -\sigma^i)\tag{4.16}$$

and check that the equivalent to (2.5) is satisfied,

$$\frac{1}{[2]} tr_q(\rho_\mu \sigma_\nu) = g_{\mu\nu} \quad , \quad \frac{1}{[2]} tr_q(\rho^\mu \sigma^\nu) = g^{\nu\mu} \quad ,\tag{4.17}$$

where  $g_{\mu\nu}$  is given by the first expression in (4.9). In terms of the  $\rho$ 's, we find  $Y = \rho^\mu \partial_\mu = \rho_\mu \partial^\mu$

$$\begin{aligned}Y &= \begin{bmatrix} q^{-1}\partial_0 - q\partial_3 & -q^{-3/2}[2]^{1/2}\partial_- \\ -q^{3/2}[2]^{1/2}\partial_+ & q^{-1}(\partial_0 + \partial_3) \end{bmatrix} = \begin{bmatrix} q^{-1}\partial^0 + q\partial^3 & q^{-1/2}[2]^{1/2}\partial^+ \\ q^{1/2}[2]^{1/2}\partial^- & q^{-1}(\partial^0 - \partial^3) \end{bmatrix} \\ &\equiv \begin{bmatrix} \partial_D & q^{-1}\partial_A \\ q\partial_B & \partial_C \end{bmatrix}\end{aligned}\tag{4.18}$$

( $\partial_\mu \equiv \partial/\partial x^\mu$ ,  $\partial/\partial x_0 = \partial^0$ ,  $\partial/\partial x_+ = q^2\partial^+$ ,  $\partial/\partial x_- = q^{-2}\partial^-$ ,  $\partial/\partial x_3 = \partial^3$ ) and then (3.3) or (3.5) gives the commutation relations [8]

$$\begin{aligned}[\partial_0, \partial_\pm] &= [\partial_0, \partial_3] = 0 \quad , & \partial_+(\partial_0 + \partial_3) &= q^{-2}(\partial_0 + \partial_3)\partial_+ \quad , \\ [\partial_+, \partial_-] &= \lambda\partial_3(\partial_3 + \partial_0) \quad , & \partial_-(\partial_0 + \partial_3) &= q^2(\partial_0 + \partial_3)\partial_- \quad ,\end{aligned}\tag{4.19}$$

which in the  $\partial_I$  basis read [9]

$$\begin{aligned} [\partial_A, \partial_B] &= q\lambda(\partial_D\partial_C - \partial_C\partial_D) \quad , & [\partial_C, \partial_D] &= 0 \quad , \\ [\partial_A, \partial_D] &= -q^3\lambda\partial_C\partial_A \quad , & \partial_A\partial_C &= q^2\partial_C\partial_A \quad , \\ [\partial_B, \partial_D] &= q\lambda\partial_C\partial_B \quad , & \partial_C\partial_B &= q^2\partial_B\partial_C \quad . \end{aligned} \quad (4.20)$$

In these basis, the definition (3.31) leads to

$$d = dx^I\partial_I = [2]dx^\mu\partial_\mu = [2]dx_\mu\partial/\partial x_\mu \quad (x^I = A, B, C, D; \mu = 0, \pm, 3) \quad , \quad (4.21)$$

(the factor [2] may be absorbed in the definition of the  $\sigma$ 's and  $\rho$ 's) and the D'Alembertian operator (3.9) reads

$${}_q = \partial_C\partial_D - q^{-2}\partial_A\partial_B = q^{-2}(\partial_0^2 - q\partial_+\partial_- - q^{-1}\partial_-\partial_+ - \partial_3^2) \quad . \quad (4.22)$$

The relations (4.19) are, we note in passing, the same that we would have obtained for the coordinates  $x_\mu$ , and those of (4.10) are the same as those for the derivatives  $\partial^\mu$ . This fact, not evident a priori, becomes obvious once we notice that  $\hat{R}^\epsilon$  transforms  $\sigma$ 's into  $\rho$ 's, and that accordingly the entries of  $\rho^\mu x_\mu$ , whose non-commutativity properties are already fixed by those of the coordinates  $x_\mu$ , have the same commutation properties as  $\rho^\mu\partial_\mu$  in  $Y$ . It may be checked that the  $K\text{-}\tilde{Y}$  and  $K\text{-}Y$  equations in (3.19) reproduce, respectively, the quadratic coordinates-derivatives relations in the  $x^\mu, \tilde{\partial}_\nu$  basis [8] (once the  $\partial$ 's in [8] are identified with the  $\tilde{\partial}$ 's in  $\tilde{Y}$  and some misprints in formulae (3.9) of this reference are corrected) and in the  $x^I, \partial_J$  basis [9].

The above covariant operators may now be used to construct a covariant  $q$ -Dirac operator. Without making any explicit reference to a basis, it is given by

$$\mathcal{D}_q = \begin{bmatrix} 0 & -qY^\epsilon \\ Y & 0 \end{bmatrix} \quad (4.23)$$

and transforms covariantly under the reducible  $q$ -Dirac spinorial representation

$$\mathcal{S}(\Lambda) = \begin{bmatrix} M & 0 \\ 0 & \tilde{M} \end{bmatrix} \quad (4.24)$$

i.e.,  $\mathcal{S}(\Lambda) \mathcal{D}_q \mathcal{S}^{-1}(\Lambda) = \mathcal{D}_q'$ . The Dirac gamma matrices may be extracted from (4.23) easily by recurring to a specific basis. In the  $\partial_\mu$  basis,  $\mathcal{D}_q = \gamma^\mu\partial_\mu$  leads to

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \rho^\mu & 0 \end{bmatrix} \quad , \quad (4.25)$$

where the  $q$ -matrices  $\sigma^\mu, \rho^\mu$  are given by (4.12) and (4.16); in the  $(A, B, C, D)$  basis they are obtained immediately from (4.18) and the analogous expression for  $Y^\epsilon$ , with the result that

$$\mathcal{D}_q = \begin{bmatrix} 0 & 0 & q^2\partial_C & -q^{-1}\partial_A \\ 0 & 0 & -q\partial_B & \partial_D + q\lambda\partial_C \\ \partial_D & q^{-1}\partial_A & 0 & 0 \\ q\partial_B & \partial_C & 0 & 0 \end{bmatrix} \quad . \quad (4.26)$$

Factoring out the derivatives in (4.26), the obtained  $\gamma^I$  coincide with those given in [28, 58]. The operator (4.23) may be used for a  $q$ -Dirac equation; by extension (for instance, via the Bargmann-Wigner procedure) we may obtain a covariant operator suitable for higher spin  $q$ -relativistic invariant equations.

In fact, an immediate and physically necessary application is the analysis of the deformed relativistic equations. In the preliminary free case, they may be looked at (specially in momentum space) as the set of constraints defining the Hilbert space which is the support of an (almost) irreducible representation of the Poincaré group (the wavefunctions include both signs of the energy in their manifestly covariant formulation). In the scalar Klein-Gordon case, the constraint is just the mass shell condition  $p^2 = m^2$ . In the  $q$ -deformed case the description of the one particle states [26] and of the invariant equations is more complicated. It is possible to define the  $q$ -operators corresponding to the kernels of the  $q=1$  relativistic equations, like the Klein-Gordon, Dirac or even Weinberg-Joos [8, 58, 27] or Bargmann-Wigner ones in the arbitrary spin case by using their covariance as their defining property as it may be done in  $q=1$  case (see, e.g., [59]). We may then look to the solutions of the  $q$ -relativistic equations as the kernels of the corresponding covariant  $q$ -operators. However, as the previous formalism clearly shows, we cannot have hermitean/antihermitean coordinates and derivatives simultaneously; if the coordinates are taken to be hermitean, the linear conjugation structure of the derivatives is lost [9, 31], a fact that goes against the linearity principle of quantum mechanics. Moreover, the space of solutions ( $q$ -wavefunctions') is mathematically not well defined, i.e., it is not clear what kind of subspace or subalgebra of  $\mathcal{M}_q$  they define, nor their relation to the irreducible representations of the quantum Poincaré group. All these problems require further study.

## 5 Non-commutative differential calculus and invariant operators

In the previous sections, the coordinates  $K$ , the  $q$ -derivatives  $Y$  and the  $q$ -differentials  $dK$  for the  $q$ -Minkowski space have been introduced. Using these basic elements, higher rank tensors and invariant differential operators can be constructed respectively by tensoring or by contraction. The contraction (scalar product) of a covariant vector with a contravariant one is given by the  $q$ -trace of the product of the corresponding matrices. Examples of this procedure are the invariant operators  $l_{q,q}$  and  $d$ , already introduced. Another one is the  $q$ -analogue of the dilatation operator,

$$s = tr_q(KY) . \quad (5.1)$$

Using the flexibility of RE formalism, we shall now derive a complete list of the relations involving all these invariant operators and the generators of the  $\mathcal{M}_q, \mathcal{D}_q$  and  $\Lambda_q$  algebras.

First of all, since the time variable  $x^0$  and the corresponding derivative  $\partial_0$  are singularized by being the linear central elements of  $\mathcal{M}_q$  and  $\mathcal{D}_q$ , let us deduce their commutation relations with the generators of  $\mathcal{D}_q$  and  $\mathcal{M}_q$ . For those between  $x^0=q^{-1}[2]^{-1}tr_q K$  and  $Y$  it is enough to take the  $q$ -trace of the eq. before the last in (3.19),

$$Y_2 R_{12} K_1 R_{21} = R_{12} K_1 R_{12}^{-1} Y_2 + q^2 R_{12} \mathcal{P} \quad , \quad (5.2)$$

with respect to the first space. For  $\partial_0=q[2]^{-1}tr_q Y$  and  $K$ , eq. (5.2) is multiplied by  $R_{12}^{-1}$  from the left and by  $R_{21}^{-1}$  from the right before taking the  $q$ -trace with respect to the second space. In this way we get

$$Y c_1 = c_1 Y + q^4 I - q^2 \lambda Y K \quad , \quad (5.3)$$

$$\partial_0 K = K \partial_0 + I - q^2 \lambda K Y \quad , \quad (5.4)$$

where the invariance of the  $q$ -trace and the first two of the relations (see (A.9), (A.10))

$$R_{21}^{-1} = R_{12} - \lambda \mathcal{P}_{12} = \mathcal{P}_{12}(q^{-1}I - [2]P_{-12}) \quad , \quad (5.5)$$

$$tr_{q(1)}(R_{12}\mathcal{P}_{12})^{\pm 1} = q^{\pm 2}I_2 \quad , \quad tr_{q(2)}(\mathcal{P}_{12}R_{12})^{\pm 1} = q^{\pm 2}I_1 \quad , \quad (5.6)$$

were used. Inspection of formulae (5.3) and (5.4) demonstrates that setting  $c_1=0$ ,  $tr_q Y \propto \partial_0=0$ , does not produce a consistent reduction to a three dimensional space algebra since  $tr_q Y c_1 = c_1 tr_q Y + [2] - q^{-2} \lambda s$ .

Next, we compute the commutation relations involving the quadratic central elements with generators. The  $q$ -Minkowski length  $l_q$  satisfies the following relations with the generators of  $\mathcal{D}_q$  and  $\Lambda_q$ :

$$Y l_q = q^{-2} l_q Y - q^2 K^\epsilon \quad , \quad (5.7)$$

$$l_q dK = q^{-2} dK l_q \quad . \quad (5.8)$$

To obtain (5.7),  $Y$  must go through  $l_q P_{-12} = -q P_{-12} K_1 \hat{R}_{12} K_1$ , what may be done multiplying eq. (5.2) by  $R_{31} R_{32} K_3 R_{21}^{-1} \mathcal{P}_{13}$  from the right and by  $P_{-13}$  from the left. Using then (5.2) once more in the r.h.s., the Yang-Baxter equation (1.4) to reorder the  $R$ -matrices, the  $R$ -matrix  $q$ -determinant  $P_{-13} R_{12} R_{32} = q P_{-13}$  and finally the expression (B.6) for the covariant vector  $K^\epsilon$ , eq. (5.7) is obtained. In a similar way, eq. (5.8) follows from (3.22) multiplying it by  $R_{32} K_3 R_{23} R_{13}$  from the left, using (1.4) and (3.22) for  $dK_2, K_3$  and finally multiplying by  $P_{-31}$  from the left.

Iterating (5.7) any power of  $l_q$  can be differentiated

$$Y l_q^n = q^{-2n} l_q^n Y - q^2 [n; q^{-2}] K^\epsilon l_q^{n-1} \quad , \quad (5.9)$$

where

$$[n; q] \equiv \frac{q^n - 1}{q - 1} \quad , \quad [1; q] = 1 \quad . \quad (5.10)$$

For the relations involving the D'Alembertian  ${}_q$  and the generators of  $\mathcal{M}_q$  and  $\Lambda_q$ , we obtain

$${}_q K = q^{-2} K_q - Y^\epsilon, \quad (5.11)$$

$${}_q dK = q^2 dK_q. \quad (5.12)$$

Expression (5.11) (which can be seen as the 'dual' analog of (5.7)) follows from eq. (5.2) multiplying it by  $Y_3 R_{32}^{-1} R_{13}$  from the left, using (1.4), and again (5.2) in the r.h.s., and finally multiplying the resulting equation by  $R_{21}^{-1} \mathcal{P}_{23}$  from the left and by  $P_{-(32)}$  from the right; for the contravariant vector  $Y^\epsilon$ , eq. (B.7) is used. Eq. (5.12) follows from (3.27) in a similar way.

Let us now look at  ${}_q$  and  $l_q$ , eqs. (3.9) and (2.35). Applying  ${}_q$  to the product  $K^\epsilon K$  leads to

$${}_q K^\epsilon z K = q^{-4} K^\epsilon z K_q - q^{-2} K^\epsilon z Y^\epsilon - Y z K \quad (5.13)$$

by using the transformed (5.11),

$${}_q K^\epsilon = q^{-2} K_q^\epsilon - Y. \quad (5.14)$$

To relate  ${}_q l_q$  to  $l_{qq}$  we need the intermediate expression

$$Y_1 K_1 = q^{-2} tr_{q(2)}(R_{21} K_2 \hat{R}_{21}^{-1} Y_2 R_{21}^{-1}) + [2] I_1, \quad (5.15)$$

which follows from (5.2) by using the second equality in (5.6). Its  $q$ -trace is

$$tr_q(Y K) = q^{-4} s + [2]^2. \quad (5.16)$$

Then, using (5.16) and  ${}^3 tr_q(K^\epsilon Y^\epsilon) = tr_q(K Y) \equiv s$ , the  $q$ -trace of (5.13) gives final expression

$${}_q l_q = q^{-4} l_{qq} + q^{-2} s + (q^2 + 1). \quad (5.17)$$

Using the relation between  $sl_q$  and  $l_q s$  to be proved below [(5.36)] and iterating (5.17) we find

$${}_q l_q^n = q^{-4n} l_{qq}^n + a_n l_q^{n-1} s + (q^2 + 1) b_n l_q^{n-1}, \quad (5.18)$$

where the coefficients  $a_n$  and  $b_n$  are determined by the recurrence equations

$$a_{n+1} = q^{-2}(q^{-4n} + a_n), \quad b_{n+1} = q^{-4n} + a_n + b_n, \quad n \geq 1, \quad (5.19)$$

( $a_1 = q^{-2}$ ,  $b_1 = 1$ ) which give (see (5.10))

$$a_n = q^{-2n} [n; q^{-2}], \quad b_n = (q^{-2} + 1)^{-1} [n + 1; q^{-2}] [n; q^{-2}]. \quad (5.20)$$

Let us now consider the exterior derivative  $d = tr_q(dK Y)$ . Its action on coordinates,  $q$ -derivatives and  $q$ -differentials is given by

$$d \cdot K = (dK) + K d, \quad (5.21)$$

---

<sup>3</sup>The equality  $tr_q(AB) = tr_q(A^\epsilon B^\epsilon)$  obviously holds for any pair  $A, B$ , if one is covariant and the other contravariant.



$$Y d = q^2 d Y + q \lambda d K_q^\epsilon, \quad (5.22)$$

$$d(dK) = -(dK) d. \quad (5.23)$$

The first one is the expression of the Leibniz rule and (5.23) reflects the nilpotency of  $d$ ,  $d^2 = 0$ . These relations are easily obtained using the invariance of the  $q$ -trace, the appropriate RE and the properties of the  $R$ -matrix. For example, for (5.21) we write

$$d \cdot K_2 = tr_{q(1)}(dK_1 Y_1) K_2 = tr_{q(1)}(R_{21}^{-1} dK_1 Y_1 R_{21}) K_2 \quad (5.24)$$

and, using (5.2) and (3.22) in the form  $dK_1 R_{21} K_2 R_{21}^{-1} = R_{21} K_2 R_{12} dK_1$ , we obtain

$$\begin{aligned} d \cdot K_2 &= tr_{q(1)}(K_2 R_{12} dK_1 Y_1 R_{12}^{-1}) + q^2 tr_{q(1)}(R_{21}^{-1} \mathcal{P}_{12} dK_2) \\ &= K_2 tr_{q(1)}(R_{12} dK_1 Y_1 R_{12}^{-1}) + q^2 tr_{q(1)}(R_{21}^{-1} \mathcal{P}_{12}) dK_2, \end{aligned} \quad (5.25)$$

from which and from (5.6) eq. (5.21) immediately follows.

The relation (5.22) is obtained in a similar way. The RE (3.3), (3.27), and the relations (cf. (5.5))

$$R_{21} R_{12} = q^2 I - \lambda [2] P_{-12}, \quad (5.26)$$

$$dK_1^\epsilon = [2] tr_{q(2)}(\hat{R}_{12} dK_1 P_{-12}), \quad (5.27)$$

(cf. (B.6)) are enough to get (5.22), starting from

$$Y_2 d = tr_{q(1)}(Y_2 R_{21}^{-1} dK_1 Y_1 R_{21}) \quad (5.28)$$

Eq. (5.23) is obtained from  $d dK_2 = tr_{q(1)}(R_{12} dK_1 Y_1 R_{12}^{-1} dK_2)$ , using only (3.27) and (3.23).

Relations (5.22) and (5.23) can be used to show explicitly that  $d^2 = 0$ , since

$$d^2 = d d = d tr_q(dK Y) = tr_q(d dK Y) \quad (5.29)$$

Using now (5.22) and (5.23) to move  $d$  through  $dK Y$

$$d^2 = -q^{-2} tr_q(dK Y) d + q^{-1} \lambda tr_q(dK dK^\epsilon)_q, \quad (5.30)$$

and  $tr_q(dK dK^\epsilon) = 0$  [(3.25)], one finds that  $d^2 = -q^{-2} d^2$ , or that  $d^2 = 0$  ( $q$  is real).

The previous set of relations allow us to compute in a direct way the commuting properties of  $d$  with any invariant operator. The action of  $d$  on the quadratic central elements  $q$  and  $l_q$  gives

$$d_q = q_q^{-2} d, \quad (5.31)$$

$$dl_q = l_q d - q^2 W, \quad W \equiv tr_q(dK K^\epsilon) \quad (5.32)$$

Eq. (5.31) is obtained from the expression (3.9) for  $q$  and by using twice (5.22) (a simpler possibility is to apply  $d = tr_q(dK zY)$  to (5.12)). In the same way, to find (5.32) we may use (5.21) twice and that

$$tr_q(K dK^\epsilon) = q^2 tr_q(dK K^\epsilon) \equiv q^2 W \ , \quad (5.33)$$

(a property which follows from (3.22)) or, equivalently, eqs. (5.7) and (5.8) to move  $l_q$  through  $d$ .

The relation of  $d$  with the  $q$ -analogue [(5.1)] of the invariant dilatation operator is easily obtained using (5.21) and (5.22):

$$\begin{aligned} d s &= d tr_q(K Y) = tr_q(dK Y) + tr_q(K dY) \\ &= tr_q(dK Y) + q^{-2} tr_q(K Y) d - q^{-1} \lambda tr_q(K z dK^\epsilon)_q \end{aligned} \quad (5.34)$$

so that, recalling (5.33) we get

$$d s = q^{-2} s d - q \lambda W_q + d \ . \quad (5.35)$$

The operator  $s = tr_q(KY)$  is related (see below) to a central element for the complete algebra  $\mathcal{M}_q \times \mathcal{D}_q \times \Lambda_q$ , and it is useful in the irreducible representation description (cf. the  $q$ -oscillator case [60, 61]). Therefore, it is important to have the complete set of relations of  $s$  with the generators of  $\mathcal{M}_q$ ,  $\mathcal{D}_q$  and  $\Lambda_q$ , and with the central elements  $l_q$  and  ${}_q$ . The latter ones may be obtained in a simple way. Using eq. (5.7) for  $l_q$  and (5.11) for  ${}_q$ , the resulting relations are given by

$$s l_q = q^{-2} l_q s + (q^2 + 1) l_q \ , \quad (5.36)$$

$${}_q z s = q^{-2} s {}_q + (1 + q^2) {}_q \ . \quad (5.37)$$

The operator  $s$  commutes with the elements of  $\Lambda_q$ , for

$$s z dK = dK z s \ . \quad (5.38)$$

Indeed, the invariance of the  $q$ -trace permits us to write

$$s z dK_2 = tr_{q(1)}(R_{12} K_1 Y_1 R_{12}^{-1}) dK_2 \ , \quad (5.39)$$

and eqs. (3.27) and (3.22) transform the r.h.s. of (5.39) into  $tr_{q(1)}(dK_2 R_{12} K_1 Y_1 R_{12}^{-1}) = dK z s$  again by the invariance of the  $q$ -trace.

The relations among  $s$  and the coordinates  $K$  and the  $q$ -derivatives  $Y$  are more complicated. Multiplying eq. (5.2) by  $R_{21} K_2$  from the left and by  $R_{21}^{-1}$  from the right, taking the  $q$ -trace in the second space and using (1.6) and the identities (5.5), (5.6) one gets

$$s K = K s - q^2 \lambda K Y K + q^4 K \ . \quad (5.40)$$

For the covariant combination  $KYK$ , it follows using eqs. (5.15), (1.6), (5.2), (B.7) and the defining relation (2.24) for  $l_q$  are used, that

$$K Y K = q^{-1} s K + q K + q^{-1} l_q Y^\epsilon \ . \quad (5.41)$$

Thus, it follows from (5.40) and (5.41) that

$$sK = q^{-2}Ks - q^{-1}\lambda l_q Y^\epsilon + K . \quad (5.42)$$

The relation between  $s$  and  $Y$  is found in the same way. Multiplying eq. (5.2) by  $R_{21}^{-1}Y_1R_{21}$  from the right, using (3.3) and taking the  $q$ -trace in the first space with the help of (5.5), (5.6), one gets

$$sY = Ys + \lambda q^2 YKY - q^4 Y . \quad (5.43)$$

To compute  $YKY$  we use (5.15),  ${}_q[(3.6)]$ , (5.5), (5.6) and the expression (B.6) for  $K^\epsilon$  with the result

$$YKY = q^{-3}sY + q^{-1}K^\epsilon{}_q + [2]Y ; \quad (5.44)$$

then, from (5.43) and (5.44) we obtain

$$Ys = q^{-2}sY - q\lambda K^\epsilon{}_q + Y . \quad (5.45)$$

Finally, we mention that it is possible to find a central element in  $\mathcal{M}_q \times \mathcal{D}_q \times \Lambda_q$  using the operators  $s$ ,  $l_q$ ,  ${}_q$  and a grading operator  $N$  defined by the relations

$$[N, K] = K \quad , \quad [N, dK] = 0 \quad , \quad [N, Y] = -Y \quad , \quad (5.46)$$

which may be introduced since  $\mathcal{M}_q$  and  $\mathcal{D}_q$  are graded algebras. Indeed, the element  $z$  (cf [9])

$$z = q^{2N}\tilde{s} \quad , \quad \tilde{s} \equiv (q^{-2} - 1)s + (q^{-2} - 1)^2 l_{qq} + 1 \quad , \quad (5.47)$$

is central in  $\mathcal{M}_q \times \mathcal{D}_q \times \Lambda_q$ . Using the set of relations among  $s$ ,  $l_q$  and  ${}_q$  and the generators  $K$ ,  $Y$  and  $dK$ , i.e., (5.7), (5.8), (5.11), (5.12), (5.38), (5.42) and (5.45), it is found that  $\tilde{s}$  is a scaling operator

$$\tilde{s}K = q^{-2}K\tilde{s} \quad , \quad \tilde{s}Y = q^2Y\tilde{s} \quad (5.48)$$

and  $\tilde{s}dK = dK\tilde{s}$ . Since, in contrast, eq. (5.46) gives  $q^{2N}K = q^2Kq^{2N}$  and  $q^{2N}Y = q^{-2}Yq^{2N}$  the centrality of  $z$  follows.

## 6 Comments on representations of quantum space-time algebras and other problems

In classical and quantum relativistic theory the Poincaré group transformations may be realized in terms of a complete set of observables. The generators of the infinitesimal Lorentz transformations are functions of  $x^\mu$  and  $p_\nu$ ,

$$M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu \quad , \quad (6.1)$$

so that  $M_{\mu\nu}$  and  $p_\nu$  define a basis of the ten dimensional Lie algebra  $\mathcal{P}$  of the Poincaré group  $P$ . In the  $q$ -deformed case we have encountered the analogue of the momenta  $p_\mu$ , provided by the four quantum derivatives  $Y_{ij}$  giving

the  $q$ -translation algebra  $\mathcal{D}_q$ , and the quantum Lorentz group transformations, described by the six independent entries of  $M$ ,  $\tilde{M} = (M^{-1})^\dagger$  and the corresponding coaction (2.9), (3.1). Due to the quantum group structure, the relation between the quantum group and the corresponding quantum algebra is expressed by the duality among these Hopf algebras: the elements of one are linear functionals for the other. The duality relation is an abstract one, but its explicit realization requires selecting a basis of the Hopf algebras in question, which (in our case) are constructed from a finite number of generators. Once we change the generators (often nonlinearly) of a Hopf algebra  $\mathcal{A}$  there is no easy and canonical way to find the corresponding change in the dual Hopf algebra  $\mathcal{A}^*$  generators. If the quantum Poincaré group  $P_q$  and the corresponding quantum algebra  $(P_q)^* = U_q(\mathcal{P})$  must have ten generators each, then from the previous sections we can extract six generators of  $L_q \subset P_q$  and four generators of  $\mathcal{D}_q \subset U_q(\mathcal{P})$  (the subindex in  $U_q$  denotes quantization of the universal enveloping algebra). A set of generators of the quantum Poincaré algebra was found in [62, 9] and it was transformed into another set in [28]. However, the duality relation with the  $P_q$  and  $U_q(\mathcal{P})$  basis remains to be clarified.

Once the complete set of the  $q$ -Poincaré algebra generators is obtained, one may adopt the Newton-Wigner's point of view and identify the  $q$ -deformed elementary systems (particles) with the unitary irreducible representations of  $U_q(\mathcal{P})$ . These were constructed in [26] for the scalar particle. Nevertheless, there are still some difficulties that require further study. To illustrate them, it is sufficient to consider the representations of the  $q$ -translation algebra  $\mathcal{D}_q$  as a subalgebra of  $U_q(\mathcal{P})$ . Since  $\mathcal{D}_q$  is isomorphic to  $\mathcal{M}_q$ , eq. (3.10), we may consider the irreducible unitary representations of this algebra [63]. In the non-deformed case ( $q=1$ ) this problem is trivial: since the algebra of translations or coordinates is commutative, its irreducible representations are one-dimensional. For  $q \neq 1$ , however, this is no longer the case.

Due to the fact that one central element of  $\mathcal{M}_q$  is linear in its generators one can change the basis (2.10) to  $\alpha, \beta, \gamma$  and  $\tau$ ,  $q\tau \equiv c_1 = q^{-1}\alpha + q\delta$ . Then, there are three non-trivial commutation relations ( $\hat{\lambda} \equiv \lambda/q \equiv (1 - q^{-2})$ ;  $\hat{\lambda} > 0$  for  $q > 1$ )

$$\begin{aligned}\alpha\gamma &= q^2\gamma\alpha, & \alpha\beta &= q^{-2}\beta\alpha, \\ \beta\gamma &= q^2\gamma\beta + \hat{\lambda}(l_q - \alpha^2),\end{aligned}\tag{6.2}$$

which follow from (2.13) plus the centrality of  $\tau$  and of (cf. (2.27))

$$l_q = \alpha\tau - \alpha^2/q^2 - q^2\gamma\beta = q^2(\alpha\tau - \alpha^2 - \beta\gamma).\tag{6.3}$$

To analyze the irreducible representations in a Hilbert space with positive metric the following consequences of (6.2) are useful:

$$\beta\gamma^n = (q^2\gamma)^n\beta + \hat{\lambda}[n; q^2]\gamma^{n-1}(l_q - q^{2(n-1)}\alpha^2),\tag{6.4}$$

$$\gamma\beta^n = (q^{-2}\beta)^n\gamma - \hat{\lambda}/q^2[n; q^{-2}]\beta^{n-1}(l_q - q^{-2(n-1)}\alpha^2).\tag{6.5}$$

These relations may be proved by induction or by assuming, looking at (6.2) that, e.g.,

$$\beta\gamma^n = (q^2\gamma)^n\beta + \hat{\lambda}\gamma^{n-1}(a_n l_q - b_n\alpha^2), \quad (6.6)$$

which gives the recurrence relations  $a_{n+1} = q^{2n} + a_n$  and  $b_{n+1} = q^{2n} + q^4 b_n$ , which have the solutions  $a_n = [n; q^2]$  and  $b_n = q^{2(n-1)}[n; q^2]$ .

Since  $\mathcal{M}_q \approx \mathcal{M}_{q^{-1}}$  [(3.10)] we may assume  $q > 1$ . The irreducible representations (irreps) are parametrized by the different values of the central elements  $l_q$  (denoted by  $l$ ) and  $\tau$ . The representations fall into two broad categories,  $l \leq 0$  or  $l > 0$ , but there are subclasses in each of them, which are listed below.

**1.**  $\alpha = 0$ , then the other three generators  $\beta, \gamma, \delta$  commute among themselves as it follows from (2.13). Hence,  $\delta$  is real and arbitrary while  $\beta = \gamma^*$  is an arbitrary complex number,  $\tau = \delta, l = -q^2|\gamma|^2$ . This irrep is not faithful. It gives a one-dimensional irrep of (2.13).

**2.**  $l - \alpha^2 = 0$ ,  $\delta = \alpha, \beta = \gamma = 0, \tau = \alpha(1 + q^{-2})$ . This is also a one-dimensional representation, which is not faithful and corresponds to the stationary point  $\alpha I$  of the coaction  $K \mapsto UKU^\dagger$  of the quantum ‘subgroup’  $SU_q(2)$  of  $L_q$ .

**3.**  $l > \alpha_0^2 > 0$ , where  $\alpha_0$  is the vacuum eigenvalue of  $\alpha$  and  $\beta|0\rangle = 0$ . Then from (6.4) for unnormalized eigenvectors  $|n\rangle = \gamma^n|0\rangle$  of  $\alpha$  one gets

$$\langle n|n\rangle = \langle 0|(\gamma^*)^n\gamma^n|0\rangle = \langle 0|\beta^n\gamma^n|0\rangle = (\hat{\lambda})^n[n; q^2]!\Pi_{k=1}^n(l - q^{2(k-1)}\alpha_0^2), \quad (6.7)$$

where  $[n; q]! = [n; q][n-1; q]\dots 1$ . Clearly,  $\langle 1|1\rangle = \hat{\lambda}(l - \alpha_0^2) > 0$ , but for  $\alpha_0 \neq 0$  and  $q > 1$  the norm will be negative if the integer  $n$  is sufficiently large. Because we are looking for irreps in a Hilbert space with states of positive norm there must be some  $n, N = d-1$ , such that  $\| |N+1\rangle \| \sim (l - q^{2N}\alpha_0^2) = 0$ , or

$$\gamma|N\rangle = \gamma|d-1\rangle = 0, \quad (6.8)$$

where  $d$  is the dimension of the irrep and hence

$$l = q^{2d}\alpha_0^2/q^2, \quad \tau = (q^{2d} + 1)\alpha_0/q^2, \quad (6.9)$$

where the last expression follows from computing  $l_q|N\rangle = q^2(q^{2N}\tau\alpha_0 - q^{4N}\alpha_0^2)|N\rangle = l|N\rangle$  using (6.3).

**4.**  $\alpha_0^2 > l > 0$ , hence  $(l - \alpha_0^2) < 0$ . From (6.4) one now concludes that  $\beta$  cannot be an annihilation operator. So we have to use (6.5) supposing that  $\gamma$  is now the annihilation operator,  $\gamma|0\rangle = 0$ . Then for  $|n\rangle = \beta^n|0\rangle$  one gets

$$\langle n|n\rangle = (\hat{\lambda}/q^2)^n[n; q^{-2}]!\Pi_{k=1}^n(q^{-2(k-1)}\alpha_0^2 - l) \quad (6.10)$$

Using the same positivity arguments of the previous case we now obtain, with  $d = N + 1$  as before,

$$l = q^{-2(d-1)}\alpha_0^2 \quad , \quad \tau = (q^{-2d} + 1)\alpha_0 \quad . \quad (6.11)$$

**5.**  $l \leq 0$  ,  $\alpha \neq 0$ , hence  $(l - \alpha^2) < 0$  and one has to use (6.5) with  $\gamma$  as the annihilation operator  $\gamma|0\rangle = 0$ . In this way one obtains for  $|n\rangle = \beta^n|0\rangle$  again (6.10), which is now positive for any integer  $n$ . This irrep is infinite dimensional.

There are some relations of the  $q$ -Minkowski algebra  $\mathcal{M}_q$  with other  $q$ -algebras:  $su_q(2)$ , the  $q$ -oscillator algebra  $\mathcal{A}(q)$  and the  $q$ -sphere  $S_q^2$ . For instance, once  $\tau$  and  $l$  are fixed the relations (6.2), (6.3) coincide with the defining relations of the quantum sphere algebra [64, 65].

The algebra  $\mathcal{M}_q$  is isomorphic to the  $q$ -derivative or  $q$ -momentum algebra  $\mathcal{D}_q$ , hence the irreducible representations coincide with those found in [26] for the  $q$ -deformed Poincaré algebra, which has the algebra  $\mathcal{D}_q$  as a subalgebra. Once we identify  $\mathcal{M}_q \approx \mathcal{D}_q$ , we can set  $m_q^2 = l_q$  and  $\tau = [2]p_0$ ; the energy is a central element. However, although unitarity permits any real value for  $m_q^2$  and  $p_0$ , the physical meaning of the central elements of  $\mathcal{D}_q$  requires that the eigenvalues of the energy  $p_0$  and the square of mass  $m_q^2$  satisfy the obvious physical restrictions  $p_0^2 \geq m_q^2 \geq 0$ . If  $m_q^2 = l > 0$ , one may eliminate  $\alpha_0$  from (6.9) or (6.11). In both cases we obtain

$$p_0^2 = m_q^2 \frac{(q^d + q^{-d})^2}{q^2[2]^2} . \quad (6.12)$$

In the classical  $q \rightarrow 1$  limit, the values of all generators are fixed while the dimension  $d \rightarrow \infty$  in such a way that the factor  $(q^d + q^{-d})^2$  is also fixed and the usual formula  $p_0^2 = m^2 + \vec{p}^2$  is reproduced.

The next step in the representation theory is related to the construction of a representation in the tensor product of two irreducible representations. It depends on existence of a bialgebra (or a Hopf algebra) structure for the algebra  $\mathcal{M}_q$  or a homomorphism from  $\mathcal{M}_q$  to  $\mathcal{M}_q \otimes \mathcal{M}_q$ . The existence of such a map could be interpreted physically as the  $q$ -Lorentz group covariance for two- (in general multi-) particle system. There are a few proposals for a possible ‘coproduct’  $\Delta : \mathcal{M}_q \rightarrow \mathcal{M}_q \otimes \mathcal{M}_q$ . These proposals use:

**1.** the relation of  $\mathcal{M}_q$  to the quantum algebra  $su_q(2)$ , extending it to isomorphism (modulo some additional requirements) and introducing the bialgebra structure through the factorization [66] (here and below the indices (1), (2) refer to the factors in  $\mathcal{M}_q \otimes \mathcal{M}_q$ )

$$K = L^{(+)}(L^{(-)})^{-1} = L_{(1)}^{(+)}K_{(2)}(L_{(1)}^{(-)})^{-1} ; \quad (6.13)$$

**2.** the appropriate non-commutativity (‘braid statistics’) of the factors in a ‘tensor product’  $\mathcal{M}_q \otimes \mathcal{M}_q$  [20, 25, 67]. It is not difficult to check that the matrix product of two matrices

$$\Delta(K) = K \dot{\otimes} K = K^{(1)}K^{(2)} \quad , \quad \Delta(K)_{ij} = K_{il} \otimes K_{lj} \quad , \quad (6.14)$$

satisfying the commutation relations given by

$$\hat{R}^{-1} K_1^{(2)} \hat{R} K_1^{(1)} = K_1^{(1)} \hat{R} K_1^{(2)} \hat{R} \quad , \quad (6.15)$$

satisfy the RE (1.6) and that its entries generate an algebra isomorphic to  $\mathcal{M}_q$ ;

**3.** the coaddition of the two  $2 \times 2$  matrices expressed by

$$\Delta(K) = K^{(1)} + K^{(2)} \equiv K \otimes 1 + 1 \otimes K \quad . \quad (6.16)$$

If the non-commutativity between the generators of  $K^{(1)}$  and  $K^{(2)}$  is given e.g., by (cf. [44])

$$\hat{R} K_1^{(2)} \hat{R} K_1^{(1)} = K_1^{(1)} \hat{R} K_1^{(2)} \hat{R}^{-1} \quad (6.17)$$

(eq. (6.15) corresponds to interchanging (1) and (2)) the corresponding set of commutation relations permits that  $\Delta(K)$  satisfies (1.6). Eq. (6.17) is easily obtained by imposing that the matrix  $K_1^{(1)} + K_1^{(2)}$  satisfies (2.11).

**4.** an additional matrix  $\mathcal{O}$  including the  $q$ -Lorentz algebra generators acting on the first factor of  $\mathcal{M}_q \otimes \mathcal{M}_q$  such that the matrix

$$\Delta(K) = K^{(1)} + \mathcal{O}^{(1)} \otimes K^{(2)} \quad (6.18)$$

will satisfy the RE (1.6), while the entries of  $K^{(1)}$  and  $K^{(2)}$  commute [9] (a kind of ‘undressing’ of the preceding case).

Though last two cases look physically reasonable, they, together with **2**, are not symmetric with respect to the permutation of factors (notice that (6.17) is not symmetric) and not all irreps of  $K^{(1)}$  and  $K^{(2)}$  are compatible.

The form of the coproduct depends on the chosen basis. The coproduct of the  $q$ -Lorentz algebra part of  $U_q(\mathcal{P})$  looks simple in the  $sl_q(2)$  basis [3]  $L^\pm, \tilde{L}^\pm$  related to the matrices  $M, \tilde{M}$ . However, the basis of [9] is related to the  $SU_q(2)$  ‘subgroup’ of  $L_q$ . The coproduct for the  $q$ -derivatives subalgebra  $\mathcal{D}_q$  in this basis looks like [9]

$$\Delta(Y_{ij}) = Y_{ij} \otimes I + \sum_{kl} l_{ij,kl} \otimes Y_{kl} \quad , \quad (6.19)$$

where the  $l_{ij,kl}$  are made up of the  $q$ -Lorentz algebra generators and the scaling operator (which is outside the  $q$ -Lorentz algebra). One of the obvious property of this set of elements of the dual algebra  $(L_q)^*$  of  $L_q$  follows from the covariance requirement, which must be preserved by the coproduct:

$$Y \mapsto \tilde{M} Y M^{-1} \Rightarrow \Delta(Y) \mapsto \tilde{M} \Delta(Y) M^{-1} \quad . \quad (6.20)$$

This form of the coproduct demonstrates that the representation theory of the  $q$ -momentum subalgebra  $\mathcal{D}_q \subset U_q(\mathcal{P})$  is not closed: one cannot consider the tensor product of arbitrary irreducible representations  $V_1, V_2$  of  $\mathcal{D}_q$ , related to  $Y \otimes I$  and  $I \otimes Y$ . It is necessary to take the  $V_1$ , reducible generally speaking, that permits an irreducible representation of the whole set,  $Y_{ij}$  and  $l_{ij,kl}$ . This raises the question of the physical interpretation and its consequences: the observables of two particles enter into the coproduct in an asymmetric way.

If we introduce two sets of the triangular  $L^\pm, \tilde{L}^\pm$  matrices corresponding to  $M$  and  $\tilde{M}$ , then their entries (six of them) will define a basis of generators of the quantum Lorentz algebra  $(L_q)^*$ . The commutation relations of the entries of  $L^\pm$  are the standard ones ( $sl_q(2)$ , see Appendix A) and the same is true for  $\tilde{L}^\pm$ . They could commute in between  $[L_{ij}^\pm, \tilde{L}_{kl}^\pm]$ , for the coproducts of  $M$  and  $\tilde{M}$  are the usual ones,  $\Delta(M) = M \otimes M$ . However, in order to have the usual coproduct for  $L^\pm, \tilde{L}^\pm$  due to the non-commutativity of  $M$  and  $\tilde{M}$  (2.7) one can introduce a ‘mild’ non-commutativity between the entries of  $L^\pm$  and  $\tilde{L}^\pm$  too. Then, the corresponding duality relations are:

$$\begin{aligned} \langle L_1^\pm, M_2 \rangle &= R_{12}^\pm, & \langle \tilde{L}_1^\pm, \tilde{M}_2 \rangle &= R_{12}^\pm, \\ \langle L_1^\pm, \tilde{M}_2 \rangle &= A_{12}, & \langle \tilde{L}_1^\pm, M_2 \rangle &= \tilde{A}_{12}. \end{aligned} \quad (6.21)$$

Then, the action of such operators on the algebra  $\mathcal{M}_q$  (the entries of  $L^\pm$  and  $\tilde{L}^\pm$  are now operators on  $\mathcal{M}_q$ ) could also be written in matrix form. Adding a hat to stress the operator character of  $\hat{L}^\pm$  we find

$$\begin{aligned} (\hat{L}_1^\pm K_2) &= \langle L_1^\pm, \cdot \rangle \phi(K_2) = \langle L_1^\pm, M_2 K_2 \tilde{M}_2^{-1} \rangle \\ &= \langle \Delta(L_1^\pm), M_2 K_2 \tilde{M}_2^{-1} \rangle = \langle L_1^\pm, M_2 \rangle K_2 \langle L_1^\pm, \tilde{M}_2^{-1} \rangle \\ &= R_{12}^\pm K_2 A_{12}^{-1}, \end{aligned} \quad (6.22)$$

where we have used the duality between product and coproduct in the third equal sign, after which  $M_2 K_2 \tilde{M}_2^{-1}$  really means  $(M_2)_{ij} \otimes (\tilde{M}_2^{-1})_{kl} K_{jk}$ . Thus, we obtain

$$\hat{L}_1^\pm K_2 = R_{12}^\pm K_2 A_{12}^{-1} \hat{L}_1^\pm. \quad (6.23)$$

It would be interesting to study which of the above constructed representations may be extended to representations of a larger algebra  $\mathcal{M}_q \times \mathcal{D}_q$  including  $\mathcal{D}_q$  as well as  $\mathcal{M}_q$ . This algebra is defined by 8 generators (the entries of  $K, Y$ ) and the relations (1.6), (3.3) and (5.2). Introducing explicitly the matrix elements  $\partial_I$  [(4.18)] of  $Y$ , one may check that  $\partial_B$  and  $\partial_C$  together with  $\mathcal{M}_q$  generate a closed subalgebra. Most of the constructed irreps can be easily extended to this subalgebra. However, these extensions usually have a singular  $q$  dependence for  $q \rightarrow 1$ . For instance, for the one-dimensional representation  $\alpha = 0$  (in which the representation is, in fact, of the whole  $\mathcal{M}_q \times \mathcal{D}_q$  algebra) one finds

$$\partial_C = 0, \quad \partial_A = q^4/\gamma(q^2-1), \quad \partial_B = q^2/\beta(q^2-1), \quad \partial_D = -q\delta/(q^2-1). \quad (6.24)$$

It is well established that covariant algebras such as  $SU_q(2)$  or the  $q$ -sphere  $S_q^2$  can be represented as a direct sum of subspaces invariant with respect



to the corresponding quantum groups coactions:  $U' \rightarrow UU'$  for  $SU_q(2)$  or  $K \rightarrow UKU^t$  for the  $q$ -sphere (cf. [64, 65]). This expansion is related to the fusion procedure of the quantum inverse scattering method. These invariant subspaces in the case of the  $SL_q(2)$  covariant RE algebra are generated by the entries of the  $(2j+1) \times (2j+1)$  matrices  $K^{(j)}$ , which correspond to the higher spin representations of the quantum group from the coaction, e.g.

$$K^{(1)} = P_+ K_1 R_{21} K_2 \mathcal{P}_{12} = K_1 \hat{R}_{12} K_1 P_+. \quad (6.25)$$

This is associated with the construction of  $q$ -deformed relativistic wave equations [27]. There are even some universal  $K$  matrices [68] related to a peculiar representation of  $\mathcal{M}_q$ .

## 7 Concluding remarks

The main aim of this paper was to analyze the  $q$ -deformed Minkowski space-time and the associated non-commutative differential calculus by using the  $R$ -matrix and the reflection equation formalism. This permitted us to establish in a systematic and economic way many features of the quantum Minkowski space algebras including the complete definition of  $\mathcal{M}_q$ , the corresponding De Rham complex  $\Lambda_q$  and the algebra of  $q$ -derivatives  $\mathcal{D}_q$ ; the covariance properties of these algebras under the quantum Lorentz group transformation (coaction), and the action of the quantum Lorentz algebra (by duality). A special basis of generators of  $\mathcal{M}_q$  was defined by using the  $q$ -adjoint (co)action of the quantum (group) algebra; this allowed us to introduce  $q$ -sigma and  $q$ -gamma matrices as appropriate  $q$ -tensors. The possible ambiguities in the definition of  $q$ -Minkowski space and consequently in its differential calculus as well as some important  $q$ -algebra isomorphism were discussed and the irreducible representations of  $\mathcal{M}_q$  algebra were found. Also, in the course of the discussion, a few invariant (scalar) operators were defined by means of the  $q$ -trace, and their commutation relations among themselves and with the generators of  $\mathcal{M}_q$  and  $\mathcal{D}_q$  were established.

All the previous discussions of the  $q$ -deformed space-time were directly formulated in the  $q \neq 1$  framework, without considering a classical counterpart. The exact  $q$ -relations may be used, however, for some constructions in the classical theory. For instance, it is known that the quasiclassical limit of the main ingredients of the quantum inverse scattering method gives rise to the classical  $r$ -matrix and the classical Yang-Baxter equation. If in the present case we introduced Planck's constant just by multiplying the defining relations of the  $q$ -algebras by  $\hbar$ , and then we took the independent limits  $q \rightarrow 1$ ,  $\hbar \rightarrow 0$ , the resulting relations would be nothing but the standard Poisson brackets for the commuting coordinates and momenta of the scalar relativistic particle,  $\{x_\mu, p_\nu\} = g_{\mu\nu}$ . If, on the other hand, the Planck's constant and the deformation parameter are directly related e.g. by setting  $q = \exp(\gamma\hbar)$  (which requires

having an additional dimensional constant in the theory), the Poisson brackets in the quasiclassical limit are highly nontrivial [19]. For instance, writing  $R \simeq I + \gamma hr$  in the quasiclassical limit, the RE (1.6) gives, neglecting the terms in  $\hbar^2$ ,

$$-\frac{1}{\hbar}[K_1, K_2] = \gamma(K_1 r_{21} K_2 - K_2 r_{12} K_1) + \gamma(r_{12} K_1 K_2 - K_2 K_1 r_{21})$$

so that the Poisson brackets for the classical entries of  $K$  (coordinates) read

$$\{K_1, K_2\} = \gamma([K_1 \hat{r}_{12} K_1, \mathcal{P}_{12}] + [\hat{r}_{21}, \mathcal{P}_{12} K_1 K_2]) \quad . \quad (7.1)$$

In this case the Poincaré group would not be purely geometrical: it would be dynamical (a Lie-Poisson group [69, 1, 70, 13]) because its parameters would have nontrivial Poisson brackets. An application of the Dirac theory for constrained systems results in non-autonomous equations, though with conserved momentum. This gives rise to additional questions of interpretation if one wished to preserve the usual mathematical structure of a physical theory.

To conclude, we wish to come back to other topics that were not discussed in the paper in detail. The covariance of a two(multi-)particle system requires a map from  $\mathcal{M}_q$  to  $\mathcal{M}_q \otimes \mathcal{M}_q$  which is an algebra homomorphism ('coproduct'). Two variants among those given in the text have reasonable physical behaviour, reproducing in the classical limit the sum of the particle coordinates, although these coordinates do not commute in between in the case  $q \neq 1$ . Both recipes have the drawback of being asymmetric under interchange of two particles. These properties result in a more complicated representation theory of these quantum algebras. Another subject just mentioned in the text is the construction of (free)  $q$ -relativistic wave equations. To discuss a physical meaning for the formal solutions of the  $q$ -Klein-Gordon and/or  $q$ -Dirac equations in  $\mathcal{M}_q$  we have to construct irreducible representations of the whole algebra  $\mathcal{M}_q \otimes \mathcal{D}_q$  (coordinates and momenta). Then, the  $q$ -Dirac equation could be defined as an operator in the corresponding Hilbert space with an orthodox interpretation of its spectrum and eigenvectors (wavefunctions). Another point which requires clarification is the relation between the relativistic wave equations in configuration and momentum space which in the classical theory are connected by the Fourier transform. A possibility to discuss the quantum Fourier transform is by using the  $*$ -quantization which operates in algebra of functions on a phase space (cf. [71] and refs. therein); another one is along the lines of [72]. Nevertheless, the difficulties already mentioned prevent us, at present, to speculate on a possible quantum field theory having  $\mathcal{M}_q$  as a base space-time.

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## Appendices

### A Some facts and formulae on quantum groups

#### A1 Notation and useful expressions

We list here some expressions and conventions that are useful in the main text. ‘*RTT*’ relations as those in (1.3), (2.7) follow the usual conventions i.e., the  $4 \times 4$  matrices  $T_1, T_2$  are the tensor products

$$T_1 = T \otimes I \quad , \quad T_2 = I \otimes T \quad . \quad (\text{A.1})$$

The tensor product of two matrices,  $C = A \otimes B$ , reads in components

$$C_{ij,kl} = A_{ik}B_{jl} \quad , \quad (\text{A.2})$$

so that the comma separates the row and column indices of the two matrices. Thus,  $(A_1)_{ij,kl} = A_{ik}\delta_{jl}$ ;  $(A_2)_{ij,kl} = A_{jl}\delta_{ik}$ . The transposition in the first and second spaces is given by

$$C_{ij,kl}^{t_1} = C_{kj,il} \quad , \quad C_{ij,kl}^{t_2} = C_{il,kj} \quad , \quad (\text{A.3})$$

i.e.,  $C^{t_1} = A^t \otimes B$  (resp.  $C^{t_2} = A \otimes B^t$ ) is given by a matrix in which the blocks 12 and 21 are interchanged (each of the four blocks is replaced by its transpose). Of course,  $C_{ij,kl}^{t_1 t_2} = C_{ij,kl}^t = C_{kl,ij}$  is the ordinary transposition. Similarly, the traces in the first and second spaces are given by

$$(tr_{(1)}C)_{jl} = C_{ij,il} \quad , \quad (tr_{(2)}C)_{ik} = C_{ij,kj} \quad . \quad (\text{A.4})$$

They correspond, respectively, to replacing the  $4 \times 4$  matrix  $C$  by the  $2 \times 2$  matrix resulting from adding its two diagonal boxes or by the  $2 \times 2$  matrix obtained by taking the trace of each of its four boxes. If  $C = A \otimes B$ ,  $tr_{(1)}C = (trA)B$  and  $tr_{(2)}C = A(trB)$ .

The action of the permutation matrix  $\mathcal{P}_{12} \equiv \mathcal{P}$  is defined by  $(\mathcal{P}C\mathcal{P})_{ij,kl} = C_{ji,lk}$  ( $\mathcal{P}(A \otimes B)\mathcal{P} = B \otimes A$  if the entries of  $A$  and  $B$  commute); thus

$$(\mathcal{P}A_1\mathcal{P})_{ij,kl} = (A_1)_{ji,lk} = A_{jl}\delta_{ik} = (A_2)_{ij,kl} \quad ; \quad (\text{A.5})$$

$(\mathcal{P}C)_{ij,kl} = C_{ji,kl}$ ,  $(C\mathcal{P})_{ij,kl} = C_{ij,kl}$ . Explicitly,  $\mathcal{P}=\mathcal{P}^{-1}$  is given by

$$\mathcal{P} = \begin{bmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{bmatrix} \quad , \quad \mathcal{P}_{ij,kl} = \delta_{il}\delta_{jk} \quad ; \quad (\text{A.6})$$

acting from the left (right) it interchanges the second and third rows (columns).

For  $SL_q(2)$ , the  $R_{12}(q) \equiv R_{12} \equiv R$  and  $\mathcal{P}R_{12} \equiv \hat{R}_{12} \equiv \hat{R}$  matrices are given by

$$R = \begin{bmatrix} q & & & \\ & 1 & 0 & \\ & \lambda & 1 & \\ & & & q \end{bmatrix}, \quad \hat{R} = \begin{bmatrix} q & & & \\ & \lambda & 1 & \\ & 1 & 0 & \\ & & & q \end{bmatrix}, \quad (\text{A.7})$$

$$R_{12}(q^{-1}) = R_{12}^{-1}(q), \quad \hat{R}_{12}^{-1}(q) = \hat{R}_{21}(q^{-1}); \quad (\text{A.8})$$

where  $\lambda \equiv q - q^{-1}$ ;  $\hat{R}_{21} = \mathcal{P}\hat{R}_{12}\mathcal{P}$ . Due to the special form of  $R$ ,  $\mathcal{P}R_{12}\mathcal{P} = R_{21} = R_{12}^t$ , but the last equality does not hold for a general  $4 \times 4$  matrix.  $\hat{R}$  satisfies Hecke's condition

$$\hat{R}^2 - \lambda\hat{R} - I = 0, \quad (\hat{R} - q)(\hat{R} + q^{-1}) = 0 \quad (\text{A.9})$$

and

$$\hat{R} = qP_+ - q^{-1}P_-, \quad \hat{R}^{-1} = q^{-1}P_+ - qP_-, \quad [\hat{R}, P_{\pm}] = 0, \quad P_{\pm}\hat{R}P_{\mp} = 0, \quad (\text{A.10})$$

where the projectors  $P_{\pm 12} \equiv P_{\pm}$  are given by

$$P_+ = \frac{1}{[2]} \begin{bmatrix} [2] & 0 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 1 & q^{-1} & 0 \\ 0 & 0 & 0 & [2] \end{bmatrix}, \quad P_- = \frac{1}{[2]} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (\text{A.11})$$

with  $[2] \equiv (q + q^{-1})$ . It is often convenient to express  $P_-$  in the form

$$(P_-)_{ij,kl} = \frac{1}{[2]} \epsilon_{ij}^q \epsilon_{kl}^q, \quad ([x] \equiv \frac{q^x - q^{-x}}{q - q^{-1}}), \quad (\text{A.12})$$

where  $\epsilon^a = -(\epsilon^q)^{-1} \neq (\epsilon^q)^t$  is given in (2.16). The determinant of an ordinary  $2 \times 2$  matrix may be defined as the proportionality coefficient in  $(\det M)P_- = P_- M_1 M_2$  where  $P_-$  is obtained from (A.11) setting  $q=1$ . The analogous definition in the  $q \neq 1$  case

$$(\det_q M)P_- := P_- M_1 M_2, \quad (\det_q M^{-1})P_- = M_2^{-1} M_1^{-1} P_-, \quad (\text{A.13})$$

$(\det_q M^{-1} = (\det_q M)^{-1})$  leads to the expression for  $\det_q M$  given in (1.2). For the  $K$  matrix, the definition of  $\det_q K$  is given by (2.24), (2.25).

## A2 Hopf algebra duality

We recall here some expressions on the quantum groups and quantum algebras duality. Let  $H$  and  $H^*$  be a pair of dual Hopf algebras. Then, there is a *pairing* map  $\langle \cdot, \cdot \rangle: H^* \times H \mapsto C$ , consistent with the commutation relations in each algebra, which satisfies:

$$\begin{aligned}
1) \quad & \langle XY, a \rangle = \langle X \otimes Y, \Delta(a) \rangle \quad , \quad \langle 1, a \rangle = \varepsilon(a) \quad , \\
2) \quad & \langle X, ab \rangle = \langle \Delta(X), a \otimes b \rangle \quad , \quad \langle X, 1 \rangle = \varepsilon(X) \quad , \\
3) \quad & \langle S(X), a \rangle = \langle X, S(a) \rangle \quad ,
\end{aligned} \tag{A.14}$$

where  $X, Y \in H^*$ ,  $a, b \in H$  and  $\Delta, \varepsilon, S$  are the usual notations for the co-product, counit and antipode in Hopf algebras (see, e.g., [13, 14, 15]). In particular, if  $H$  and  $H^*$  are respectively a quantum group *à la* FRT [3] and its quantum dual algebra, the generators of  $H$  are the entries of a matrix  $T$  satisfying a ‘RTT’ relation and the generators of  $H^*$  are arranged in two triangular matrices  $L^\pm$  satisfying

$$R^+ L_1^\pm L_2^\pm = L_2^\pm L_1^\pm R^+ \quad , \quad R^+ L_1^+ L_2^- = L_2^- L_1^+ R^+ \tag{A.15}$$

where  $R^+ = \mathcal{P}R\mathcal{P} = R_{21}$ . In this case, the pairing is defined in terms of the corresponding quantum group  $R$ -matrix by the expressions

$$\langle L^\pm, T \rangle = R^\pm \quad , \quad \langle L^\pm, 1 \rangle = I = \langle 1, T \rangle \quad , \tag{A.16}$$

where  $R^- = R^{-1}$ . This definition is extended to higher order monomials by

$$\begin{aligned}
\langle 1, T_1 T_2 \dots T_k \rangle &= I^{\otimes k} = \langle L_1^\pm L_2^\pm \dots L_k^\pm, 1 \rangle \quad , \\
\langle L_1^\pm L_2^\pm \dots L_k^\pm, T_{k+1} \rangle &= R_{1\ k+1}^\pm R_{2\ k+1}^\pm \dots R_{k\ k+1}^\pm \quad , \\
\langle L_1^\pm, T_2 T_3 \dots T_{k+1} \rangle &= R_{12}^\pm R_{13}^\pm \dots R_{1\ k+1}^\pm \quad .
\end{aligned} \tag{A.17}$$

This is consistent with the commutation relations defining the quantum group and its dual algebra and satisfies the properties (A.14).

The pairing may be used to define the fundamental representation of the generators of the quantum algebra; for each  $l_{ij}^\pm$  (entries of  $L^\pm$ ) the following map is defined

$$\langle l_{ij}^\pm, \cdot \rangle : H \longmapsto C \quad , \quad \langle l_{ij}^\pm, t_{kl} \rangle = R_{ik, jl}^\pm \quad , \tag{A.18}$$

then,  $\langle l_{ij}^\pm, T \rangle$  is a matrix which constitutes the fundamental representation of the generators  $l_{ij}^\pm$ ; in general,  $\langle X, T \rangle$  is the representation of  $X$ . This representation for the generators  $J_\pm, J_3$  of  $su_q(2)$  was used in Sec. 4 to check that the given  $q$ -sigmas constitute a  $q$ -tensor operator according to (4.5).

## B Proof of some properties of $q$ -Minkowski algebras

### B1 The algebras defined by eq. (2.43), (3.4)

We now analyze the algebra (2.43), and why it may be discarded. First, it is easy to explain the presence of the factor  $q^2$ , which is due to the imbalance

of the  $\hat{R}$  and  $\hat{R}^{-1}$  factors; notice that eqs. (2.40) allow for a proportionality constant in the definition of  $R^{(i)}$ . Consider (2.43) written in the form

$$\hat{R}K_1\hat{R}K_1 = \rho K_1\hat{R}K_1\hat{R}^{-1} \quad , \quad (\text{B.1})$$

where  $\rho$  is a constant to be determined. We may now multiply this equation by  $P_{\pm}(\ )P_{\pm}$  and by  $P_{\pm}(\ )P_{\mp}$ . Using eq. (A.10) we obtain that  $P_+(\ )P_+$  gives

$$qP_+K_1\hat{R}K_1P_+ = \rho P_+K_1\hat{R}K_1q^{-1}P_+ \quad (\text{B.2})$$

which fixes  $\rho = q^2$ . Then,  $P_-(\ )P_-$  gives

$$-q^{-1}P_-K_1\hat{R}K_1P_- = q^2P_-K_1\hat{R}K_1P_-(-q) \quad , \quad (\text{B.3})$$

which implies

$$(q^3 - q^{-1})P_-det_qK = 0 \quad (\text{B.4})$$

so that for this algebra ( $q^4 \neq 1$ )  $det_qK = 0$ . The commutation properties of the entries of  $K$ , however, are the same as for (1.6). Using  $q^2\hat{R}^{-1} = \hat{R} - q\lambda[2]P_-$ , eq. (2.43) reproduces (2.11),

$$\hat{R}K_1\hat{R}K_1 = K_1\hat{R}K_1\hat{R} - q\lambda[2]K_1\hat{R}K_1P_- = K_1\hat{R}K_1\hat{R} \quad , \quad (\text{B.5})$$

since  $det_qK = 0$ . This means that the algebra generated by the entries of  $K$  in (2.43) may be obtained by restricting  $\mathcal{M}_q$  (eq. (1.6)) by the condition  $det_qK = 0$  so that nothing is gained by considering (2.43) as a separate case. Notice that the same reasonings applied to (1.6) do not give any condition for  $P_{\pm}(\ )P_{\pm}$ , which is satisfied identically, and  $P_{\pm}(\ )P_{\mp}$  give, as for (2.11),  $P_{\pm}K_1\hat{R}K_1P_{\mp} = 0$ .

The factor  $q^2$  in (3.4) is explained in a similar way; again, this algebra corresponds to the too restrictive condition  $det_qY = 0$ .

We now derive here the expressions for  $K^\epsilon$  and  $Y^\epsilon$  used in Sec. 5,

$$K_1^\epsilon = [2]tr_{q(2)}(\hat{R}_{12}K_1P_{-12}) \quad , \quad (\text{B.6})$$

$$Y_1^\epsilon = [2]tr_{q(2)}(P_{-12}Y_1\hat{R}_{12}^{-1}) \quad . \quad (\text{B.7})$$

Obviously, the covariant vector  $dK^\epsilon$  has an expression (5.27) analogue (B.6). For instance, the r.h.s. of (B.6) reads in explicit component notation

$$[2](tr_{q(2)}(\hat{R}_{12}K_1P_{-12}))_{ik} = [2]D_{jb}\hat{R}_{ib,am}K_{ac}P_{-cm,kj} \quad .$$

Using the expression (A.12) and  $D = \epsilon^q z \epsilon^{qt}$ , the above expressions become equal to

$$- \epsilon_{bk}^q \hat{R}_{ib,am} \epsilon_{cm}^q K_{ac} = (\epsilon^q)_{kb}^t \hat{R}_{ib,am} (\epsilon^q)_{mc}^{t-1} K_{ac} = \hat{R}_{ik,ac}^\epsilon K_{ac} = K_{ik}^\epsilon \quad (\text{B.8})$$

by (2.28) and (2.31). Eq. (B.7) is checked similarly by using  $\hat{R}_{12}^\epsilon = \hat{R}_{21}^\epsilon$ .

## B2 Algebra isomorphisms

Let us first check that

$$K(q^{-1}) \equiv K' \approx K^\epsilon, \quad K' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}, \quad (\text{B.9})$$

i.e., that the commutation properties of the entries of  $K'$  (which generate the same algebra as those of  $K$  but with  $q$  replaced by  $q^{-1}$ ) are those of the elements of  $K^\epsilon$ . The reflection equation for  $K' \equiv K(q^{-1})$  is given by (cf. eq. (1.6))

$$\hat{R}_{12}(q^{-1})K'_1\hat{R}_{12}(q^{-1})K'_1 = K'_1\hat{R}_{12}(q^{-1})K'_1\hat{R}_{12}(q^{-1}), \quad (\text{B.10})$$

and since  $\hat{R}_{12}(q^{-1}) = \hat{R}_{21}^{-1}(q)$ , this is equivalent to

$$\hat{R}_{21}K'_1\hat{R}_{21}^{-1}K'_1 = K'_1\hat{R}_{21}^{-1}K'_1\hat{R}_{21}. \quad (\text{B.11})$$

Now we notice that, due to the specific form of the  $\hat{R}$  matrices, a similarity transformation with  $(\sigma^1 \otimes \sigma^1)$  is equivalent to the action of the permutation operator  $\mathcal{P}$ ,  $(\sigma^1 \otimes \sigma^1)\hat{R}_{12}(\sigma^1 \otimes \sigma^1) = \hat{R}_{21}$ . Also,  $(\sigma^1 \otimes \sigma^1)K_i(\sigma^1 \otimes \sigma^1) = (\sigma^1 K \sigma^1)_i, i = 1, 2$ . Thus, eq. (B.11) gives

$$\hat{R}_{12}(\sigma^1 K' \sigma^1)_1 \hat{R}_{12}^{-1}(\sigma^1 K' \sigma^1)_1 = (\sigma^1 K' \sigma^1)_1 \hat{R}_{12}^{-1}(\sigma^1 K' \sigma^1)_1 \hat{R}_{12}. \quad (\text{B.12})$$

Comparing with (3.3), which as we know is the same reflection equation satisfied by  $K^\epsilon$  because of its transformation properties (2.32), we find that there is an isomorphism among the algebras generated by  $K$  and  $K'$  given by

$$K \longmapsto K' = \sigma^1 K^\epsilon \sigma^1 \equiv \sigma^1 (\hat{R}^\epsilon K) \sigma^1. \quad (\text{B.13})$$

Since we also know that  $K^\epsilon \approx Y$ , we see that the linear mappings which relate the entries of  $K', K$  and  $Y$  define isomorphisms between the algebras generated by their entries. Specifically, the elements of

$$K' = \begin{bmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{bmatrix}, \quad \sigma^1 K^\epsilon \sigma^1 = \begin{bmatrix} -q^{-1}\alpha & q\gamma \\ q\beta & \lambda\alpha - q\delta \end{bmatrix}, \quad \sigma^1 Y \sigma^1 = \begin{bmatrix} z & w \\ v & u \end{bmatrix},$$

$$\mathcal{M}_{q^{-1}} \qquad \qquad \qquad \mathcal{M}_q \qquad \qquad \qquad \mathcal{D}_q \quad (\text{B.14})$$

have the same commutation relations. An analogous argument shows the matrix elements of

$$Y' = \begin{bmatrix} u' & v' \\ w' & z' \end{bmatrix}, \quad \sigma^1 Y^\epsilon \sigma^1 = \begin{bmatrix} -q^{-1}u - \lambda z & q^{-1}w \\ q^{-1}v & -qz \end{bmatrix}, \quad \sigma^1 K \sigma^1 = \begin{bmatrix} \delta & \gamma \\ \beta & \alpha \end{bmatrix},$$

$$\mathcal{D}_{q^{-1}} \qquad \qquad \qquad \mathcal{D}_q \qquad \qquad \qquad \mathcal{M}_q \quad (\text{B.15})$$

where  $Y' \equiv Y(q^{-1})$ , also have the same commutation properties. Identifying  $K$  with the  $q$ -Minkowski coordinates and  $Y$  with the derivatives, eqs. (B.14) and (B.15) show that  $\mathcal{M}_{q^{-1}} \approx \mathcal{M}_q \approx \mathcal{D}_q \approx \mathcal{D}_{q^{-1}}$ .

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