UGVA-DPNC 1995/6-166
DESY 95-100
June 1995

# BHAGENE3, A MONTE CARLO EVENT GENERATOR FOR LEPTON PAIR PRODUCTION AND WIDE ANGLE BHABHA SCATTERING IN $e^+e^-$ COLLISIONS NEAR THE Z-PEAK

**J.H.Field** [1]

Département de Physique Nucléaire et Corpusculaire Université de Genève . 24, quai Ernest-Ansermet CH-1211 Genève 4.

**T.Riemann** [2]

Deutsches Elektronen-Synchrotron DESY
Institut für Hochenergiephysik, Platanenallee 6, D–15738 Zeuthen, Germany

**Abstract**

A new Monte Carlo event generator for wide angle Bhabha scattering and muon pair production in $e^+e^-$ collisions is described. The program includes complete one-loop electroweak corrections, and QED radiative corrections. The O($\alpha$) QED correction uses the exact matrix element. Higher order QED corrections are included in an improved soft photon approximation with exponentiation of initial state radiation. Events are generated in the full phase space of the final state including explicit mass effects in the region of collinear mass singularities. The program is intended for centre of mass energies around and above the Z peak and for Bhabha scattering at angles greater than $10^\circ$ .

[1]E-mail JFIELD@CERNVM.CERN.CH
[2]E-mail riemann@ifh.de

## PROGRAM SUMMARY

*Title of the program:* BHAGENE3

*Computer:* IBM 3090

*Operating system:* VM/CMS

*Programming language used:* FORTRAN 77

*High speed storage required:* 678k words

*No. of bits in a word:* 32

*Peripherals used:* Line printer

*Number of cards in combined program and test deck:* about 6900

*Keywords:* Radiative corrections, Monte Carlo simulation, Wide angle Bhabha scattering, Muon pair production, Photons, Quantum electrodynamics, Electroweak theory .

*Nature of physical problem:* Calculation of all 1-loop electroweak corrections, and of QED corrections up to $O(\alpha^3)$ for wide angle Bhabha scattering and muon pair production in the vicinity of the Z peak.

*Method of solution:* The 1-loop electroweak virtual corrections are calculated analytically. The $O(\alpha)$ QED radiative corrections use the exact matrix element. The higher order QED corrections are given by an improved soft photon approximation. Event configurations containing real photons are first generated according to an approximate model, and are then reweighed according to the theoretical distributions. The weight throwing technique is used to produce unit weight events in the full final state phase space.

*Restrictions on the complexity of the problem:* Some terms in $m_l/E$ $(l = e, \mu)$ are neglected, so that fermion threshold effects are not properly taken into account. The approximation used for $O(\alpha^2)$ and higher order real photon radiation may be unreliable in events with multiple hard photon radiation. For cross section accuracy at the % level it is not recommended to use the program for Bhabha scattering angles less than 10°.

*Typical running times:* For Bhabha events: initialisation about 140 sec, generation 709 unit weight events per second. For muon pairs the initialisation time is shorter by a factor of about 3 and the event generation rate is roughly doubled. The time unit is an IBM3090 CPU second.

*Unusual features of the program:* Extensive use is made of one and two dimensional look-up tables for fast and flexible generation of Monte-Carlo variables. The space required for these tables means that the fast memory requirements may be greater than in other comparable programs. These look-up tables as well as average event weights are created in the initialisation phase of the program which is, in consequence, relatively time-consuming.

*References:*

[1 ] J.H.Field, Phys. Lett. B323 (1994) 423

[2 ] M.Böhm, A.Denner and W.Hollik, Nucl. Phys. B304 (1988) 687.

[3 ] F.A.Berends, R.Kleiss and W.Hollik, Nucl. Phys. B304 (1988) 712.

[4 ] D.Bardin, W.Hollik and T.Riemann, Z. Phys. C Particles and Fields 49 (1991) 485.

# 1 Introduction

Data on the leptonic decays of the Z taken since 1990 at LEP and SLC have made possible many precise tests of the Standard Model (SM) of weak and electromagnetic interactions [1]. These electroweak analyses require Monte Carlo event generators, incorporating both the SM predictions and the numerically important pure QED radiative corrections, in order to properly take into account detector acceptance, efficiency and resolution effects, for arbitrary experimental cuts.

For fermion pair production (excluding $e^+e^-$), the most widely used generator is KORALZ [2]. This incorporates one-loop electroweak corrections using the SM, multiple bremsstrahlung photons in the initial state and a single bremsstrahlung photon in the final state as well as a complete simulation of $\tau$ decays, including polarisation effects and QED radiative corrections associated with the $\tau$ decay products.

For wide angle Bhabha scattering two generators have been written prior to that, BHAGENE3, described in the present paper. The first, BABAMC [3, 4] includes one-loop SM electroweak corrections and QED corrections (real and virtual) to O($\alpha$). The second, BHAGEN [5] includes higher order QED corrections (by exponentiation) in both initial and final states, but is limited to 'quasi-elastic' configurations where the photons are soft, or almost collinear with the radiating lepton. This limitation is a severe disadvantage when comparing with actual experimental data, since in general very loose cuts (typically none at all on the lepton-photon angles) are made in order to minimise the size of the QED radiative corrections. More recently a new Monte Carlo generator UNIBAB [6] has been written, which is a general purpose program in many ways comparable to BHAGENE3. Electroweak corrections are included and arbitrary numbers of initial and final state photons generated using a LLA structure function formalism. Unlike BHAGENE3 however, O($\alpha$) initial/final interference effects are not included in the current version.

Bhabha generators also exist [7, 8] for the small angle region of lepton scattering angle $\theta_l$ (dominated by the $t$-channel photon exchange diagram) typical of the angular acceptance of the luminosity monitors of the LEP/SLC experiments.

The generator described in this paper, BHAGENE3, is adapted to the description of wide angle Bhabha scattering ($10^\circ < \theta_l < 170^\circ$), and of $\mu$-pair production, in the region of the Z peak. $\tau$-decays are not incorporated. Electroweak corrections in the SM, including the most important two-loop effects, are implemented as described in Ref.[9]. Multiple hard photon generation, ( $n_\gamma \leq 2(3)$ for initial (final) states) is also included. No restrictions are necessary on the kinematical configuration of the generated events. However, the approximate nature of the hard photon generation algorithm should be borne in mind when multiple, very hard photon configurations are considered [10]. A brief description of the program has been given previously, and comparisons of cross-sections and charge asymmetries made [10] with the analytical or semi- analytical programs: ZFITTER [11], ALIBABA [12] and TOPAZ0 [13].

The approach adopted for the QED radiative corrections is to treat the O($\alpha$) correction exactly (in particular the relevant fermion mass terms are systematically included throughout, so that collinear photon radiation is correctly generated) as in BABAMC [3, 4] and MUSTRAAL [14]. Higher order real photon radiation is treated, not by using exact O($\alpha^2$) amplitudes [15, 16, 17], but by an improved soft photon approximation. This may be justified [10] by the relatively small size of the O($\alpha^2$),O($\alpha^3$), ... hard photon corrections as compared to that at O($\alpha$). As in Ref.[3,4,14] the final state phase space is divided into two regions. A cut ($E_0^\gamma$) is applied on the energy of each photon. For $E^\gamma < E_0^\gamma$ the photon energy and angular distributions are integrated over, so as to yield a Virtual, Soft (VS) corrected cross-section. The corresponding generated events have a 'Born topology'; the outgoing leptons being exactly back-to-back with energy: $E_l = E_{beam} = E$. The generation of such V,S events is described in Section 3 below. For $E^\gamma > E_0^\gamma$ a dilepton event with 1,2,3 hard photons is generated over the full available phase-space. Denoting the number of initial/final state photons by $n_\gamma^I/n_\gamma^F$, the different possibilities are:

$$n_\gamma^I/n_\gamma^F = 1/0, 0/1, 2/0, 1/1, 0/2, 0/3$$

This 'hard photon' generation is described in Section 4 below.

A complete description of the program, its structure and parameters, may be found below in Section 5. In this Introduction only a brief account is given of the most important techniques employed (see also Ref.[10]). The first step in the execution of the program is an initialisation phase where all electroweak parameters are calculated within the SM from standard input parameters such as the fine structure constant, $\alpha$, the Fermi constant G, and the masses of the Z ($M_Z$), the top quark ($M_t$) and the Higgs boson ($M_H$). Parameters needed for the QED radiative corrections, taking into account the lepton flavour (e,$\mu$) selected and the beam energy E are also calculated at this stage. Next the probabilities for VS, 'Initial State hard' (IS) and 'Final State hard' (FS) events are calculated. For the VS events this is done by angular integration of the differential cross-section, including weak and electromagnetic radiative corrections. For IS, FS events approximate factorisable cross-sections are used that may be readily integrated analytically and/or numerically, to give the corresponding probabilities. Also, during the initialisation phase, a number of one or two dimensional Look Up Tables (LUT) for the total photon energy, photon energy splitting fractions, and lepton scattering angle, are produced. For each variable the Integrated Probability Distribution (IPD) is calculated by analytical and/or numerical integration. The IPD is then inverted by linear interpolation (see Appendix C) to yield the LUT. The average weights $\overline{W}_I$, $\overline{W}_F$ of the IS, IF events are also found, during initialisation, by re-weighting events according to the 'exact' hard cross-section, to be described below in Section 4.

The event generation phase of the program execution is now entered. A VS , IS or FS event is first chosen. For IS, FS events $n_\gamma^I$ or $n_\gamma^F$ are then chosen according to a Poisson distribution, and the appropriate sub-generator is entered. The extensive use of LUT rather than a Weight Rejection Procedure (WRP) as in Ref.[4,13] gives a very fast and efficient event generation algorithm. Arbitrary user defined cuts may be applied during the event generation phase. Four-vectors are written out for each unit weight event which is chosen by using a WRP. In the last stage of execution the 'exact' cross section corresponding to the user-supplied cuts is is calculated and printed out as well as the user-defined histograms or other distributions that are up-dated during the generation phase.

Finally in this Introduction a few remarks on the limitations of the program and some recommended restrictions on its use. Only cross-sections summed over final and averaged over initial states of the lepton and photon polarisations are calculated. Only leptonic (not quark anti-quark) final states may be generated. As in Ref.[3,4,14] not all terms $\approx m_l/E$ are included, so that the near-threshold region of lepton pair production is not properly described. As an approximate model is used to generate hard photons at $O(\alpha^2)$ and higher, based on an improved soft photon approximation, cross-sections for configurations with multiple hard photons should be regarded with caution. Some comparisons with exact $O(\alpha^2)$ calculations are given in Ref.[10]. The precision of the LUT's used for the lepton scattering angle requires that $10° < \theta_l < 170°$ in Bhabha scattering, if a cross-section accuracy of $\approx 1\%$ is required.

# 2 Weak Virtual Corrections

In the initialisation, the $W$ mass is determined iteratively from $\alpha, G_\mu, M_Z$ in subroutine SETCON:

$$M_W = M_Z \sqrt{1 - \sqrt{1 - \frac{4\pi\alpha}{\sqrt{2} G_\mu M_Z^2 (1-\Delta r)}}} \qquad (2.1)$$

where $\Delta r$ is calculated in subroutine SEARC1.

The improved Born approximation for the Bhabha differential cross section consists of the sum of contributions with $t$ channel and $s$ channel exchange of the photon and $Z$ boson and their interferences:

$$\frac{d\sigma^{WC}}{dc} = \frac{\pi\alpha^2}{2s} \sum_A \sum_a T_a(A) \quad A = \gamma, \gamma Z, Z, \quad a = s, st, t \qquad (2.2)$$

Here $s$ and $t$ are the usual Mandelstam variables and $c = \cos\theta_l$ where $\theta_l$ is the lepton scattering angle. In Bhabha scattering the final state helicities are not measurable, while the beams may be longitudinally polarized. We introduce the notation:

$$\lambda_1 = 1 - \lambda_+\lambda_- \tag{2.3}$$
$$\lambda_2 = \lambda_+ - \lambda_- \tag{2.4}$$
$$\lambda_3 = 1 + \lambda_+\lambda_- \tag{2.5}$$

where $\lambda_{+(-)}$ is the degree of longitudinal polarisation of the positron (electron) beam.

The s channel cross section contributions are:

$$T_s(\gamma) = |F_A(s)|^2 \left[\lambda_1\left(1+c^2\right)\right] \tag{2.6}$$

$$\begin{aligned} T_s(\gamma Z) = {} & 2\Re e\Big\{\rho(s)\chi(s)F_A^*(s)\Big[[\lambda_1 v_{ee}(s) + \lambda_2 v_e(s)](1+c^2) \\ & + [(\lambda_1+\lambda_2)v_e(s)]2c\Big]\Big\} \end{aligned} \tag{2.7}$$

$$\begin{aligned} T_s(Z) = {} & |\rho(s)\chi(s)|^2 \Big\{\Big[\lambda_1\left(1+2|v_e(s)|^2+|v_{ee}(s)|^2\right) \\ & + 2\lambda_2\Re e\left(v_e(s)[1+v_{ee}(s)^*]\right)\Big](1+c^2) \\ & + 2\Re e\left[\lambda_1\left(|v_e(s)|^2+v_{ee}(s)\right)+\lambda_2 v_e(s)\left(1+v_{ee}^*(s)\right)\right]2c\Big\} \end{aligned} \tag{2.8}$$

Here, we use the following abbreviations:

$$\chi(s) = \kappa\frac{s}{s - M_Z^2 + is\Gamma_Z/M_Z} \tag{2.9}$$
$$\kappa = \frac{G_\mu}{\sqrt{2}}\frac{M_Z^2}{8\pi\alpha} \tag{2.10}$$
$$v_e(s) = 1 - 4s_W^2\kappa_e(s) \tag{2.11}$$
$$v_{ee}(s) = -1 + 2v_e(s) + 16s_W^4\kappa_{ee}(s) \tag{2.12}$$

In our expression (2.2) for $d\sigma^{WC}/dc$ the axial couplings are equal to unity. The effective couplings $v_e(s), v_{ee}(s)$ and the weak form factor $\rho(s)$ are complex valued. They contain virtual weak corrections which are called from the electroweak library `BHASHA` which is part of the program package described here. It has been derived from the electroweak library `DIZET` [18] which is used in the package `ZFITTER` [11] and is intended for the description of $s$ channel fermion pair production in the region of the Z resonance. The complete theoretical description of the renormalisation procedure adopted in the unitary gauge and related topics may be found in [19], the formulae for the $Z$ width in [20] and $s$ channel scattering in [11], and those for Bhabha scattering in [9]. The complex valued $s$ channel corrections from the fermionic vacuum polarisation are contained in $F_A(s)$,

$$F_A(s) = \frac{\alpha(-|s|)}{\alpha} \tag{2.13}$$

and are explained below.

We now describe the t channel contributions:

$$T_t(\gamma) = F_A(t)^2\left[2\lambda_1\frac{(1+c)^2}{(1-c)^2} + 8\lambda_3\frac{1}{(1-c)^2}\right] \tag{2.14}$$

$$
\begin{aligned}
T_t(\gamma Z) &= 2\rho(t)\chi(t)F_A(t) \\
&\times \left\{ 2\left[\lambda_1\left(1+v_{ee}(t)\right) - \lambda_2 v_e(t)\right] \frac{(1+c)^2}{(1-c)^2} - 8\lambda_3\left(1-v_{ee}(t)\right)\frac{1}{(1-c)^2} \right\}
\end{aligned} \tag{2.15}
$$

$$
\begin{aligned}
T_t(Z) &= \left[\rho(t)\chi(t)\right]^2 \left\{ 2\left[\lambda_1\left(1+4v_e(t)^2+2v_{ee}(t)+v_{ee}(t)^2\right)\right.\right. \\
&\left.\left. +4\lambda_2 v_e(t)\left(1+v_{ee}(t)\right)\right]\frac{(1+c)^2}{(1-c)^2} + \ 8\lambda_3\left(1-v_{ee}(t)\right)\frac{1}{(1-c)^2} \right\}
\end{aligned} \tag{2.16}
$$

The following additional abbreviations are used:

$$
\chi(t) = \kappa \frac{t}{t-M_Z^2} \tag{2.17}
$$

$$
t = -\frac{s}{2}(1-c) \tag{2.18}
$$

The form factors $F_A(t), v_e(t), v_{ee}(t), \rho(t)$ are real valued. For wide angle Bhabha scattering, the values of $|t|$ may become substantially smaller than $s$. Nevertheless, the form factors do not vary much since they depend only logarithmically on the scale. There is one exception to this. Bremsstrahlung diagrams with initial state radiation and $t$ channel photon exchange may yield substantial cross section contributions. There, the effective value $|t'|$ of $|t|$ may become extremely small and the different value of the running QED coupling has to be taken into account properly:

$$
F_A(t) = \frac{\alpha(|t|)}{\alpha} \tag{2.19}
$$

For this reason the running alpha correction is included not only in $d\sigma^{WC}/dc$ but also in the cross sections for final states with hard photons (see below).

Finally, we describe in this section the contributions from the $\gamma Z$ interference:

$$
T_{st}(\gamma) = -2\Re e\left[F_A^*(s)F_A(t)\right]\lambda_1 \frac{(1+c)^2}{(1-c)} \tag{2.20}
$$

$$
T_{st}(\gamma Z) = -2\Re e\left\{\chi(t)\rho(t)F_A^*(s)\left[\lambda_1\left(1+v_{ee}(t)\right) - 2\lambda_2 v_e(t)\right] + (t \leftrightarrow s)\right\}\frac{(1+c)^2}{(1-c)} \tag{2.21}
$$

$$
\begin{aligned}
T_{st}(Z) &= -2\Re e\left\{\chi(s)\rho(s)\chi(t)\rho(t)\left[\lambda_1\left([1+v_{ee}(s)][1+v_{ee}(t)]\right.\right.\right. \\
&\left.\left.\left. +4v_e(s)v_e(t)\right) - \lambda_2\left(v_e(s)[1+v_{ee}(t)] + v_e(t)[1+v_{ee}(s)]\right)\right]\right\}\frac{(1+c)^2}{(1-c)}
\end{aligned} \tag{2.22}
$$

The running of the QED coupling $\alpha(Q^2)$ is taken into account as follows:

$$
\alpha(Q^2) = \frac{\alpha}{1-\Delta\alpha} \tag{2.23}
$$

$$
\Delta\alpha = \Delta\alpha_l + \Delta\alpha_{udcsb} + \Delta\alpha_t \tag{2.24}
$$

where we use the convention that in the $s$ channel it is $Q^2 = -s$, and in the $t$ channel $Q^2 = |t|$. The $\Delta\alpha$ is calculated in function XFOTF1.

For leptons:

$$\Delta\alpha_l = \sum_{f=e,\mu,\tau} Q_f^2 N_f \Delta F_f(Q^2) \tag{2.25}$$

$$\Delta F_f(Q^2) = \frac{\alpha}{\pi}\left\{-\frac{5}{9}+\frac{4}{3}\frac{m_f^2}{Q^2}+\frac{1}{3}\beta_f\left(1-\frac{2m_f^2}{Q^2}\right)\left[\ln\left|\frac{\beta_f+1}{\beta_f-1}\right|-i\pi\theta(-Q^2-4m_f^2)\right]\right\} \tag{2.26}$$

$$\beta_f = \sqrt{1+\frac{4m_f^2}{Q^2}} \tag{2.27}$$

and their colour factor $N_f$ and charge $Q_f$ are unity. In the weak library, the $\Delta F_f$ is calculated by function XI3, $\Delta F_f = 2$ XI3.

The contribution from the light quarks has been parametrised in two different ways. Either it is calculated with (2.25) (with flag setting NPAR(2)=2); in this case, we use effective quark masses [21]: $m_u = m_d = 0.041, m_s = 0.15, m_c = 1.5, m_b = 4.5$ GeV. The other, preferred approach (with NPAR(2)=3) uses a parametrisation of the hadronic vacuum polarisation [19], which is contained in function XADRQQ.

The $t$ quark corrections are:

$$\Delta\alpha_t = Q_t^2 N_t \Delta F_t(Q^2) + \Delta\alpha^{2\mathrm{loop},\alpha\alpha_s} \tag{2.28}$$

where the latter term contains higher order corrections and is calculated from functions ALQCDS, ALQCD:

$$\Delta\alpha^{2\mathrm{loop},\alpha\alpha_s} = \frac{\alpha\alpha_s}{3\pi^2}Q_t^2\frac{m_t^2}{Q^2}\mathcal{R}e\left\{\Pi_t^{VF}(Q^2)+\frac{45}{4}\right\} \tag{2.29}$$

$\Pi_t^{VF}(Q^2)$ is a two loop self energy function [22, 23]. Setting the corresponding flag NPAR(3) to zero (default value) results in the neglect of this very small correction. For NPAR(3)=1,2 approximate, exact calculations according to Eqn.(2.29) are made. The first one is not a really good approximation, the second is very time consuming.

For the function $\Delta F_f(Q^2)$ one may derive the following two approximations. For light fermions with $m_f^2 << |Q^2|$, the following approximate formula is valid:

$$\Delta F_f(Q^2) \to \frac{\alpha}{3\pi}\left[\ln\frac{|Q^2|}{m_f^2}-\frac{5}{3}-i\pi\theta(-Q^2)\right] \tag{2.30}$$

While, heavy fermions with $m_f^2 >> |Q^2|$ practically decouple:

$$\Delta F_f(Q^2) \to \frac{\alpha}{3\pi}\left(\frac{4}{15}\frac{-Q^2}{m_f^2}\right) \tag{2.31}$$

At LEP 1, the effective QED coupling may be treated as a constant in the $s$ channel [24]:

$$F_A(M_Z^2) \approx \frac{137.036}{128.87} \tag{2.32}$$

The form factors $\kappa_e, \kappa_{ee}, \rho$ are calculated in subroutine ROKAP(..., s, t, u,..., QE, QF, ..., XFF,...). XFF is a vector of 4 functions[3] which depend on $s$,$t$,$u$ and the charges of the initial and final state fermions (here minus one):

$$u = -t - s \tag{2.33}$$

and

$$\begin{aligned}
\rho(s) &= \mathrm{XFF}(1; u, -s, t; -1, -1) \\
v_e(s) &= \mathrm{XFF}(2; u, -s, t; -1, -1) \\
v_{ee}(s) &= \mathrm{XFF}(4; u, -s, t; -1, -1) \qquad (2.34) \\
\rho(t) &= \mathrm{XFF}(1; s, -t, u; -1, -1) \\
v_e(t) &= \mathrm{XFF}(2; s, -t, u; -1, -1) \\
v_{ee}(t) &= \mathrm{XFF}(4; s, -t, u; -1, -1) \qquad (2.35)
\end{aligned}$$

These corrections are switched on and off with flag NPAR(1); the order $\alpha^2 m_t^4$ contributions to them with NPAR(6), the $\alpha\alpha_s m_t^2$ corrections with flag NPAR(3), and the $ZZ$ and $WW$ box terms with NPAR(4). The latter are negligibly small at LEP 1[4]. They introduce in the weak corrections to the $s$ channel a dependence on the scattering angle. In the $t$ channel, correspondingly, the weak corrections will depend not only on the scattering angle, but also on s.

The total Z width [20] is used in (2.9) and is calculated in subroutine ZWRATE:

$$\begin{aligned}
\Gamma_Z &= \frac{G_\mu M_Z^3}{12\pi\sqrt{2}} \sum_f N_C(f) R_f^{QCD} \sqrt{1 - \frac{4m_f^2}{M_Z^2}} \left(1 + \frac{3}{4}\frac{\alpha}{\pi} Q_f^2\right) \rho_f^Z \\
&\times \left\{ \left(1 + \frac{2m_f^2}{M_Z^2}\right) \left[1 - 4\kappa_f^Z s_W^2 |Q_f| + 8\left(\kappa_f^Z\right)^2 s_W^4 Q_f^2\right] - \frac{3m_f^2}{M_Z^2} \right\} \qquad (2.36)
\end{aligned}$$

The form factors $\rho_f^Z, \kappa_f^Z$ are not identical with the process dependent form factors for the scattering process although the leading terms of the latter, if calculated at the $Z$ peak, yield a good approximation of the first ones. The overall factor $N_C(f)$ is equal to 1 for leptons and 3 for quarks and $R_f^{QCD}$ describes final state QCD corrections in case of quarks: $R_f^{QCD} = 1 + \alpha_s/\pi + \ldots$. The numerical value has to be chosen in accordance with the scheme and order of the QCD corrections in mind and is set by the user with the parameters XPAR(10) for the $b$ quark and XPAR(9) for the other, light quarks. More details on the $Z$ decay rate may be found in [11, 25] and references therein.

# 3 Virtual and Soft Photonic corrections

Three distinct contributions to the virtual and soft radiative correction may be distinguished:

---

[3]The third of these four functions, $v_f$, is for Bhabha scattering equal to the second one, $v_e$.
[4]This statement depends to a certain extent on the calculational scheme chosen.

1) Purely weak virtual corrections as described in the preceeding Section. These include all self-energy and vertex corrections involving W, Z and H (Higgs) bosons, as well as ZZ and WW box diagrams.

2) Electromagnetic virtual corrections: vertex corrections involving only photons, $\gamma\gamma$ and $\gamma$Z box diagrams and the effect of both leptonic and hadronic Vacuum Polarisation Insertions (VPI) in all off-shell photon propagators.

3) Corrections due to real soft photons.

For 1) BHAGENE3 uses the 'Dubna-Zeuthen' (DZ) renormalisation scheme described in Refs.[19] and [9]. The weak virtual corrections modify the vector and axial-vector coupling constants appearing in the $s$, $t$ channel Z-exchange diagrams according to Eqns. 3.15-3.18 of Ref.[9]. For 2), the O($\alpha$) vertex and $\gamma\gamma$ and $\gamma$Z box contributions are taken from Ref.[3], and the VPI contribution from Ref.[27]. Vertex corrections to O($\alpha^2$) and the corresponding exponentiated soft photon correction, 3), (the integral over the angles and energies of all photons with total energy below a fixed cut-off: $y = E^{\gamma}_{tot}/E < y_0$) are given by Ref.[28,29]. The differential cross-section, corrected for weak loop effects 1), and also the VPI part of 2), is denoted by $d^{WC}\sigma/dc$ where $c = \cos\theta_l$ , and $\theta_l$ is the lepton scattering angle. Including the soft real photon, the O($\alpha$) virtual, and the leading log O($\alpha^2$) virtual corrections as well as the $\gamma\gamma$, $\gamma$Z box diagrams, gives the following expression for the Virtual,Soft differential cross-section:

$$\begin{aligned}\frac{d\sigma^{VS}}{dc} &= C_V^i\left\{\frac{d\sigma^{WC}}{dc}+\frac{d\sigma^{EC}}{dc}\left[\exp(\beta_e\ln\frac{1}{y_0})-1\right]\right\}\\ &\quad+\frac{d\sigma^{EC}}{dc}\left[C_V^f+(\beta_{int}+\beta_f)\ln\frac{1}{y_0}\right]+\sum_{i=1}^{N}\frac{d\sigma_i^{EC}}{dc}\delta_i^{EBOX}\end{aligned}\tag{3.1}$$

where

$$\begin{aligned}\beta_e &= \frac{2\alpha}{\pi}\left[\ln\frac{s}{m_e^2}-1\right]\\ \beta_f &= \frac{2\alpha}{\pi}\left[\ln\frac{s}{m_f^2}-1\right]\\ \beta_{int} &= \frac{4\alpha}{\pi}\ln\frac{t}{u}\\ C_V^i &= 1+\frac{\alpha}{\pi}\left[\frac{3}{2}\ln(\frac{s}{m_e^2})+\frac{\pi^2}{3}-2\right]+\frac{9}{8}(\frac{\alpha}{\pi})^2\ln^2(\frac{s}{m_e^2})-\frac{\pi^2}{12}\beta_e^2\\ C_V^f &= 1+\frac{\alpha}{\pi}\left[\frac{3}{2}\ln(\frac{s}{m_f^2})+\frac{\pi^2}{3}-2\right]\\ m_f &= \text{mass of final state leptons}\end{aligned}$$

The differential cross section d$\sigma^{EC}$/dc is corrected for the 'running' of $\alpha$ in the s and t-channel photon exchange diagrams by the replacement:

$$\alpha \to \alpha(u) = \frac{\alpha}{1+\Pi^{\gamma}(u)} \tag{3.2}$$

where $\alpha$ is the on-shell QED fine structure constant ($\alpha^{-1} = 137.036...$) and $\Pi^{\gamma}(u) = -\Delta\alpha$ is the photon proper self energy function, parameterised according to Ref.[27]. The most important virtual weak corrections are also included in $d\sigma^{EC}/dc$ via the replacements [9]

$$G_\mu \to \rho_e^{DZ} G_\mu, \qquad s_W^2 \to \kappa_e^{DZ}(s) s_W^2 \tag{3.3}$$

where $G_\mu$ is the Fermi constant, determined from the muon lifetime, and $s_W = \sin\theta_W$ and $\theta_W$ is the weak mixing angle in the on-shell scheme:

$$s_W^2 \equiv 1 - M_W^2/M_Z^2 \tag{3.4}$$

The replacements (3.3) modify the tree level axial vector and vector coupling constants:

$$g_A = \frac{1}{4}\left[\frac{\sqrt{2}G_\mu M_Z^2}{\pi\alpha}\right] \tag{3.5}$$

$$g_V = g_A\left[1 - 4s_W^2\right] \tag{3.6}$$

Since however $d\sigma^{EC}/dc$ is multiplied, in Eqn.(3.1) by the electromagnetic correction of $O(\alpha)$, the effect of the weak corrections in these terms is quite negligible ( $\ll 0.1\%$ ) and so $d\sigma^{EC}/dc$ may be considered to be simply Eelectromagnetically Corrected. The label i in the last term in Eqn.(3.1) denotes the different partial cross-sections and interference terms (see for example Refs.[3,10,30]) resulting from the two (four) Feynman diagrams that contribute for $f \neq e$ ( $f = e$ ). $\delta_i^{EBOX}$ are the corrections due to $\gamma\gamma$ and $\gamma$ Z box diagrams [3].

In (3.1) initial state radiation is exponentiated whereas final state radiation and initial/final interference effects are calculated only to O($\alpha$). This simplification is justified by the following argument. The large logarithmic terms at O($\alpha^2$), O($\alpha^3$),... associated with initial state radiation do not cancel in the cross-section when the VS and 'hard' contributions, separated by the arbitrary cut $y_0$ on the scaled photon energy, are added, and give a sizeable (several %) correction. On the other hand, for final state radiation, because of the KLN theorem [31] the leading logarithms cancel exactly in the fully integrated cross-section, and cancel almost completely when only loose cuts are applied. For the level of accuracy aimed for in BHAGENE3 ( $\approx 0.5\%$ fractional error in the cross- section) it is then sufficient to consider, in cross-section calculations (both $VS$ and 'hard'), final state radiation and initial/final interference effects [5] effects to O($\alpha$).

The differential cross-section (3.1) is integrated over the angular range $c_{MIN} < c < c_{MAX}$ :

$$\sigma^{VS} = \int_{c_{MIN}}^{c_{MAX}} \frac{d\sigma^{VS}}{dc} dc \tag{3.7}$$

[5] See Ref.[32] for a discussion of the effect of initial/final interference in the charge asymmetry for fermion pair production ($f \neq e$).

The probability that a $VS$ type event is generated is proportional to $\sigma^{VS}$. The lepton scattering angle $\theta_l$ is generated by first calculating an integrated probability distribution in the form of a histogram with content $P_i$ in bin $i$ where:

$$P_i = \left[\int_{c_{MIN}}^{c_i} \frac{d\sigma^{VS}}{dc} dc\right] / \sigma^{VS} \tag{3.8}$$

Inverting this distribution by linear interpolation yields a LUT for $c$ with bin index $j$ :

$$c_j = f(P_j) \tag{3.9}$$

$c$ is then generated using:

$$c_j = f(Rn) \tag{3.10}$$

In Eqn(3.10) and subsequently $Rn$ is a random number uniformly distributed in the interval $0 \leq Rn \leq 1$. $c$ is calculated by linear interpolation in the LUT (3.9) using the bins $k, k+1$, adjacent to $P = Rn$ ($P_k \leq Rn \leq P_{k+1}$).

A similar technique (see also Appendix C) is used throughout the program for the generation of one or two dimensional distributions. The advantages are:

(i) Complete generality.

(ii) 100% efficiency. Multiple trials, as in the weight rejection technique are unnecessary.

(iii) Fast generation. The simple operations needed to perform linear interpolation in a LUT use, in general, much less computing time than the calculation of a weight.

The main disadvantage is a certain overhead in computing time during the initialisation phase, when the LUT's are created. This becomes progressively less important as large samples of events are generated. The number of bins used in the LUT depends on the steepness of the function to be generated and the desired accuracy. Because of the sharp peaking of the angular distribution near $c$=1 in Bhabha scattering, due to $t$-channel photon exchange, a finer binning of the integrated probability distribution (3.8) is used in this region. 200 bins are assigned to each of the angular regions:

$$c_{MIN} < c < 0.8c_{MAX}$$

and

$$0.8c_{MAX} < c < c_{MAX}$$

In the LUT (3.9), 500 uniformly spaced bins are used.

# 4 Hard Photon Corrections

## 4.1 Generation of Hard Photon Events

The 'exact' hard photon cross-section is calculated by first generating events according to simple factorised differential cross-sections, valid for collinear initial state or final state radiation, and then reweighting each event according to the procedure described, for example, in the first of Refs.[4]. 'Exact' is written in quotes because the hard cross section formula used is not the result of an exact, fixed order, matrix element calculation, but rather an improved soft approximation comparable to that of Ref.[33]. From now on this distinction will be understood and the term 'exact' used without quotation marks. The exact cross-section formula (see Appendix B) is derived by exponentiating the initial state radiation terms of the O($\alpha$) hard cross-section [4] in such a way that consistency with Eqn.(3.1) is obtained in the soft photon limit:

$$y = 2\sum_i E_i^\gamma/\sqrt{s} \ll 1 \tag{4.1}$$

The use of such a 'soft' approximation for the generation of 'hard' photons, may be justified, in the region of the Z peak, by the small width of the Z relative to its mass : $\Gamma_Z/M_Z \approx 0.03$ which strongly damps out hard initial state radiation. Typically $y_0$ in Eqn.(3.1) is chosen to be 0.005, corresponding, at the Z peak, to $\sum E_i^\gamma = 225\ MeV$, so that the majority of 'hard' photons are indeed soft at the scale of $\Gamma_Z = 2.7\ GeV$. Exponentiation is not applied to the final state radiation terms, since the KLN Theorem [31] guarantees the smallness of O($\alpha^2$), O($\alpha^3$),... corrections in this case, as already discussed in Section 2 above. Similarly, in accordance with Refs.[11-13,32] the initial/final interference is also treated only to O($\alpha$).

The approximate differential cross-sections $d\sigma_A^I$ , $d\sigma_A^F$ for initial, final state radiation respectively, are:

$$d\sigma_A^I = \frac{\alpha s y}{16\pi^3\kappa_+\kappa_-}\frac{d\sigma_0(s',t^\star)}{d(-t^\star)}\left[(1+\delta_V^i)y^{\beta_e} - y + \frac{y^2}{2}\right] d\Omega_+ d\Omega_\gamma dy \tag{4.2}$$

$$d\sigma_A^F = \frac{\alpha s y}{16\pi^3\kappa'_+\kappa'_-}\frac{d\sigma_0(s,t)}{d(-t)} d\Omega_+ d\Omega_\gamma dy \tag{4.3}$$

The notation closely follows that of Ref.[3,4]. 4-vectors are defined according to:

$$e^+(p_+)e^-(p_-) \to l^+(q_+)l^-(q_-)\gamma(k)$$

and then:

$$\begin{array}{ccc} s = (p_+ + p_-)^2 & t = (p_+ - q_+)^2 & u = (p_+ - q_-)^2 \\ s' = (q_+ + q_-)^2 & t' = (p_- - q_-)^2 & u' = (p_- - q_+)^2 \\ \kappa_\pm = p_\pm \cdot k & \kappa'_\pm = q_\pm \cdot k & \end{array} \tag{4.4}$$

$$d\Omega_+ = d(\cos\theta_+)d\phi_+ \qquad d\Omega_\gamma = d(\cos\theta_\gamma)d\phi_\gamma \tag{4.5}$$

$\theta_+$, $\phi_+$ are the polar and azimuthal angles of the $l^+$ relative to the incoming $e^+$ direction, while $\theta_\gamma$, $\phi_\gamma$ are the polar and azimuthal angles of the photon relative to the incoming $e^+$ ( $l^+$) directions for initial (final) state radiation. Also in Eqns.(4.2,4.3):

$$t^\star = \frac{-s'}{2}(1-\cos\theta^\star) \tag{4.6}$$

$$\delta_V^i = \frac{\alpha}{\pi}\left[\frac{3}{2}\ln(\frac{s}{m_e^2}) + \frac{\pi^2}{3} - 2\right] \tag{4.7}$$

where $\theta^\star$ is the $l^+$ scattering angle relative to the incoming $e^+$ direction in the Outgoing Dilepton Rest Frame (ODLR). $d\sigma_0$ is the Born level (uncorrected) differential cross- section for $e^+e^- \to l^+l^-$. The probabilities to generate 'initial state' (I) or 'final state' (F) events, according to the approximate models are proportional to the cross-sections $\sigma_A^I$, $\sigma_A^F$ given by integrating Eqns.(4.2,4.3) respectively, over the angles and energies of the outgoing leptons and photon:

$$\sigma_A^I = \frac{\alpha}{4\pi}\ln\left(\frac{s}{m_e^2}\right)\int_{y_{MIN}}^{y_{MAX}}\frac{dy}{y}\int_{c^\star_{MIN}}^{c^\star_{MAX}}\frac{d\sigma_0(s',t^\star)}{d(-t^\star)}\left[(1+\delta_V^i)y^{\beta_e} - y + \frac{y^2}{2}\right]dc^\star \tag{4.8}$$

where

$$s' = s(1-y) \qquad t^\star = -\frac{s}{2}(1-y)(1-c^\star) \qquad c^\star = \cos\theta^\star$$

$$\sigma_A^F = \frac{\alpha}{4\pi}\ln\left(\frac{s}{m_f^2}\right)\ln\left(\frac{y_{MAX}}{y_{MIN}}\right)\int_{c^+_{MIN}}^{c^+_{MAX}}\frac{d\sigma_0(s,t)}{d(-t)}dc^+ \tag{4.9}$$

where

$$c^+ = \cos\theta_+$$

The angular integrals over the lepton scattering angle may be performed analytically [4, 34]. The results are given below in Appendix A. The expression for the exact differential cross-section $d\sigma^{EXACT}$ is very lengthy and may be found in Appendix B.

The approximate cross-section $d\sigma_A^I$, and the exact cross- section $d\sigma^{EXACT}$ contain, because they are exponentiated, contributions from 2,3,.. hard photons, even though they are functions of the angular variables of a single 'photon'. In fact the angular variables of all, except one, of the photons are already integrated out. On performing the angular integration for this 'photon' [6] a distribution differential in only the total photon energy (the derivative of Eqn.(3.1) with respect to $y_0$, with the replacement $y_0 \to y$) is obtained. The ansatz employed is to use the same average weight for all pure initial state or all pure final state topologies, and, in the case of one initial and one final state photon in the same event, to use the harmonic mean of the initial and final state weights.

The calculation of the average weights $\overline{W}'_I$, $[\overline{W}'_f]$ for events generated according to Eqn.(4.2), [(4.3)] proceeds as follows. At the end of the initialisation phase of the program $N(1,0)$,and $N(0,1)$ events are generated with relative probabilities proportional to $\sigma_A^I$ and $\sigma_A^F$ according to Eqns.(4.2) and (4.3). The details of the event generation algorithm used in each case are given below in Sections 4.3.1 and 4.4.1. The average weights are then given by the expressions:

$$\overline{W}'_I = \frac{1}{N(1,0)}\sum_{i=1}^{N(1,0)}\frac{d\sigma^{EXACT}(\alpha_k^i)}{d\sigma_A^I(\alpha_k^i) + d\sigma_A^F(\alpha_k^i)} \tag{4.10}$$

$$\overline{W}'_F = \frac{1}{N(0,1)}\sum_{j=1}^{N(0,1)}\frac{d\sigma^{EXACT}(\alpha_k^j)}{d\sigma_A^I(\alpha_k^j) + d\sigma_A^F(\alpha_k^j)} \tag{4.11}$$

[6]Note that the energy of this 'photon' is in fact the summed energy of all real photons.

where $\alpha_k^i$, ($\alpha_k^j$) are the $k$ kinematical variables necessary to completely specify the kinematical configuration of the $i$th I event ($j$th F event). In the subsequent event generation phase weights are assigned to multiphoton events according to Table 1. The relative probabilities of radiating 1,2,3 photons are given by a Poisson distribution function [35]:

$$P_I^{(\geq 2\gamma)}/P_I^{(\geq 1\gamma)} = 1 - e^{-r_e} \tag{4.12}$$
$$P_F^{(\geq 2\gamma)}/P_F^{(\geq 1\gamma)} = 1 - e^{-r_f} \tag{4.13}$$
$$P_F^{(\geq 3\gamma)}/P_F^{(\geq 1\gamma)} = 1 - (1 + r_f)e^{-r_f} \tag{4.14}$$

where

$$r_e = \beta_e \ln\left(\frac{1}{y_0}\right), \qquad r_f = \beta_f \ln\left(\frac{1}{y_0}\right) \tag{4.15}$$

These probabilities are used to construct the 'a priori' event generation probabilities $P(n_\gamma^I, n_\gamma^F)$ presented in Table 2. The following definitions are used in this Table:

$$P_{VS} = \sigma^{VS}/\sigma_{TOT}^A \tag{4.16}$$
$$P_I = \sigma_A^I/\sigma_{TOT}^A \tag{4.17}$$
$$P_F = \sigma_A^F/\sigma_{TOT}^A \tag{4.18}$$

where

$$\sigma_{TOT}^A = \sigma^{VS} + \sigma_A^I + \sigma_A^F$$

and

$$\rho_{IF} = (S_I + S_F)/(S_I + S_F + \sqrt{S_I S_F}) \tag{4.19}$$

where

$$S_I = P_I(1 - e^{-r_e}), \qquad S_F = P_F(1 - e^{-r_f}) \tag{4.20}$$

In Table 2 $P(2,0)$, $P(0,3)$ are actually the probabilities for $\geq 2$, ($\geq 3$) initial, (final) state photons, so so that initial states with 2,3,.. photons and final states with 3,4,.. photons are assigned exactly 2 and 3 photons respectively. $P(1,1)$ is derived from $P(2,0)$ and $P(0,2)$ using a factorisation ansatz. The probabilities in Table 2 sum to unity. For the case of single photon I or F events the weight $W$ is calculated as:

$$W = \frac{d\sigma^{EXACT}(\alpha_k)}{d\sigma_A^I(\alpha_k) + d\sigma_A^F(\alpha_k)} \tag{4.21}$$

The generation of the kinematical variables $\alpha_k$ necessary to completely define the event configuration is described below. After checking for consistency with the imposed kinematical cuts (events failing the cuts are assigned weight zero) events with unit weight are written out by requiring that:

$$W/W_{MAX} > Rn \tag{4.22}$$

$W_{MAX}$ is chosen as small as possible (typically $W_{MAX} = 2$) in order to maximise the efficiency of event generation.

## 4.2 Cross-Section Calculation

The calculation of the cross-section printed out after completion of the event generation loop is now described. Suppose that $N$ event generation trials are made in the loop with

$N(n_\gamma^I, n_\gamma^F)$ trials in each different final state channel chosen according to the probabilities in Table 2. Event weights are defined as follows :

$$\begin{aligned}
W_I^i &= \frac{d\sigma^{EXACT}(\alpha_k^i)}{d\sigma_A^I(\alpha_k^i)+d\sigma_A^F(\alpha_k^i)}, & 1 < i < N(1,0) \qquad (4.23)\\
W_F^j &= \frac{d\sigma^{EXACT}(\alpha_k^j)}{d\sigma_A^I(\alpha_k^j)+d\sigma_A^F(\alpha_k^j)}, & 1 < j < N(0,1) \qquad (4.24)\\
W_{VS}^l &= 1, & 1 < l < N(0,0) \qquad (4.25)
\end{aligned}$$

where the $\alpha_k^{i,j}$ are defined after Eqns.(4.10,4.11). The cross-section $\sigma^{CUT}$ corresponding to the imposed kinematical cuts (trial events that fail the cuts are assigned weight zero) is given by:

$$\sigma^{CUT} = \overline{W}(\sigma^{V,S} + \sigma_A^I + \sigma_A^F) \qquad (4.26)$$

where

$$\begin{aligned}
\overline{W} &= [(N(1,0)+N(2,0)\overline{W}_I + (N(0,2)+N(0,2)+N(0,3))\overline{W}_F \\
&\quad + N(1,1)\sqrt{\overline{W}_I\overline{W}_F} + N(0,0)\overline{W}_{VS}]/N \qquad (4.27)
\end{aligned}$$

$$\begin{aligned}
\overline{W}_I &= \frac{1}{N(1,0)}\sum_{i=1}^{N(1,0)} W_I^i \qquad (4.28)\\
\overline{W}_F &= \frac{1}{N(0,1)}\sum_{j=1}^{N(0,1)} W_F^j \qquad (4.29)\\
\overline{W}_{VS} &= \frac{1}{N(0,0)}\sum_{l=1}^{N(0,0)} W_{VS}^l \qquad (4.30)
\end{aligned}$$

The weights $\overline{W}_I$, $\overline{W}_F$ defined by (3.26), (3.27) are identical to $\overline{W}'_I$, $\overline{W}'_f$ defined by (4.10,4.11). Since however, the former are more precise than the latter, due to the typically smaller statistical errors of the Monte-Carlo integration in the main generation loop, they are preferred for the calculation of $\sigma^{CUT}$. The error on $\sigma^{CUT}$, $\Delta\sigma^{CUT}$ due to the statistical error of the Monte-Carlo integration is [36]:

$$\frac{\Delta\sigma^{CUT}}{\sigma^{CUT}} = [SW2 - (SW)^2/N_{IF}]^{\frac{1}{2}}/SW \qquad (4.31)$$

where

$$\begin{aligned}
SW2 &= \sum_{i=1}^{N(1,0)} (W_I^i)^2 + \sum_{j=1}^{N(0,1)} (W_F^j)^2 \\
SW &= N(1,0)\overline{W}_I + N(0,1)\overline{W}_F \\
N_{IF} &= N(1,0) + N(0,1)
\end{aligned}$$

## 4.3 Initial State Radiation Events

### 4.3.1 One Initial State Photon

The final state 4-vectors of events with a single initial state photon are generated by the subroutine ZINIGB. These events are first generated according to the approximate differential cross-section (4.2) and then re-weighted according to the exact differential cross-section using Eqn.(4.23).

During the initialisation phase a LUT for the scaled photon energy $y = E_\gamma/E$, with 200 uniformly spaced bins from $y_{MIN}$ to $y_{MAX}$ is created, as described in Section 2 above. This is done by integrating Eqn.(4.2) first over $\cos\theta^\star$, then over $y$. The angular integral is done analytically (see Appendix A), and the y integral by numerical integration [7]. At the same time a two dimensional LUT (200 bins in $y$, 500 in $\cos\theta^\star$) is created. Details of the creation and use of such a two dimensional LUT may be found in Appendix C. The subroutines used to create these one dimensional (1-D) and two dimensional (2-D) LUT are SETYD, SETCTD respectively.

The kinematical variables $\alpha_k$ used to completely specify the configuration are, in order of their generation:

$$y, \qquad \cos\theta_\gamma, \qquad \phi_+^\star, \qquad \cos\theta^\star, \qquad \phi_\gamma$$

The scaled photon energy y is chosen from the 1-D LUT. The angle $\theta_\gamma$ between the incoming $e^+$ and the photon is chosen according to the distribution:

$$\frac{dn}{dc_\gamma} \simeq \frac{1}{1-\beta_{IN}^2 c_\gamma^2} \tag{4.32}$$

where

$$c_\gamma = \cos\theta_\gamma, \qquad\qquad \beta_{IN} = \sqrt{1-\frac{4m_e^2}{s}}$$

In this case the integrated probability distribution is calculated analytically and inverted to give:

$$c_\gamma = (a-1)/[\beta_{IN}(a+1)] \tag{4.33}$$

where

$$a = [(1-\beta_{IN})/(1+\beta_{IN})]\exp[2Rn\ln\frac{1+\beta_{IN}}{1-\beta_{IN}}]$$

The azimuthal angle $\phi_+^\star$ between the planes P1 (defined by photon and the incoming $e^+$), and P2 (defined by the outgoing $l^+$ and the incoming $e^+$), in the outgoing dilepton rest frame (ODLR), is generated uniformly. Next the 2-D LUT is used, together with the previously generated value of y to give $\cos\theta^\star$. Finally the angle $\phi_\gamma$, giving the orientation of the plane defined by the photon and the incoming $e^+$ in the incoming $e^+, e^-$ (LAB)

[7] Using the CERN Library program DGAUSS

frame, is generated uniformly. To obtain the 4-vectors in the LAB system of the outgoing $l^+, l^-, \gamma$ a Lorentz transformation between the ODLR and the LAB frames is required. The corresponding boost direction is parallel to the photon direction. Denoting by $\alpha_B$ the angle between the photon and the incoming $e^+$ in the ODLR frame, the transformation gives:

$$
\begin{aligned}
\sin\alpha_B &= \frac{\sin\theta_\gamma}{1-\beta^\star\cos\theta_\gamma} && (4.34)\\
\cos\alpha_B &= \frac{\cos\theta_\gamma-\beta^\star}{1-\beta^\star\cos\theta_\gamma} && (4.35)
\end{aligned}
$$

where

$$\beta^\star = y/(2-y) \qquad (4.36)$$

The 3-momentum components of the $l^+$ in the ODLR frame:

$$
\begin{aligned}
q_+^{(1)\star} &= q^\star s^\star \sin\phi_+^\star && (4.37)\\
q_+^{(2)\star} &= q^\star s^\star \cos\phi_+^\star && (4.38)\\
q_+^{(3)\star} &= q^\star c^\star && (4.39)
\end{aligned}
$$

where

$$s^\star = \sin\theta^\star, \qquad c^\star = \cos\theta^\star, \qquad q^\star = \sqrt{\frac{s(1-y)}{4} - m_l^2}$$

are transformed into the LAB frame using the Lorentz transformation defined by Eqns.(4.34-4.36). Conservation of 3-momentum in the LAB frame (the 3-momenta of the $l^+$ and photon being known) then determines the 3-momentum of the $l^-$. Energy conservation of the complete event is now checked. In the case of a deviation of more than $\pm 0.5\%$ the event is rejected and a new one is generated. A similar check is made for all other hard photon event topologies described in the following Sections. If the event is accepted, the 3-vectors of all particles are rotated about the incoming $e^+$ axis, so that the azimuthal angle of the photon becomes $\phi_\gamma$.

### 4.3.2 Two Initial State Photons

Events with two initial state photons are generated by subroutine ZINI2G according to an improved soft photon approximation [3, 14, 37]. The differential cross-section is:

$$
\begin{aligned}
d\sigma^I_{2\gamma} &= C_n d\sigma_0(s')\frac{p_+\cdot p_-}{(p_+\cdot k_1)(p_-\cdot k_1)}\frac{p_+\cdot p_-}{(p_+\cdot k_2)(p_-\cdot k_2)}\\
&\quad\times(1-\frac{k_1}{E}+\frac{k_1^2}{2E^2})(1-\frac{k_2}{E}+\frac{k_2^2}{2E^2})\frac{d^3k_1}{k_1}\frac{d^3k_2}{k_2}\\
&= 4C_n\frac{d\sigma_0(s')d(y_1)d(y_2)dy_1d\Omega_1dy_2d\Omega_2}{(1-\beta_{IN}^2\cos^2\Theta_{\gamma1})(1-\beta_{IN}^2\cos^2\Theta_{\gamma2})} && (4.40)
\end{aligned}
$$

where

$$s' = M^2_{l^+l^-}, \quad E = \sqrt{s}/2, \quad y_i = 2k_i/\sqrt{s}, \quad d(y) = (1 - y + \frac{y^2}{2})/y$$

Here $d\sigma_0(s')$ is the Born differential cross-section and $k_1$, $k_2$;$d\Omega_1$,$d\Omega_2$ are, respectively, the energies and solid angle elements of the radiated photons. $\theta_{\gamma 1}$, $\theta_{\gamma 2}$ are the angles between the photons and the incoming $e^+$. The factors $d(y_i)$ are the Gribov-Lipatov [38] kernels. In Eqn.(4.40) recoil effects are neglected, whereas in the event generation algorithm, the change in direction of the radiator due to the recoil from the first radiated photon (in the case that both photons are radiated from the same incoming $e^\pm$) is allowed for. The photon angles relative to the radiator are still however generated according to Eqn.(4.40), so that effects due to the virtuality of the first recoiling $e^\pm$ are not taken into account. Also symmetrisation effects due to different possible time ordering in the radiation of two photons from the same incoming line are neglected.

The kinematical variables $\alpha_k$ used to define the configuration of an event with two initial state photons are, in order of generation:

$$y, \quad \cos\theta_{\gamma 1}, \quad \cos\theta_{\gamma 2}, \quad u, \quad \phi_{\gamma 12}, \quad \phi^\star_+, \quad \cos\theta^\star, \quad \phi_{\gamma 1}$$

Here $y$ is the scaled total photon energy:

$$2y_{MIN} \leq y = 2(k_1 + k_2)/\sqrt{s} \leq 2y_{MAX} \tag{4.41}$$

$u$ is defined as $\ln(y_1/y_2)$ where $y_1$ and $y_2$ are the scaled energies of the photons:

$$y_{MIN} \leq y_i = 2k_i/\sqrt{s} \leq y_{MAX} \tag{4.42}$$

and $y_1 > y_2$. The angles $\theta_{\gamma 1}$, $\theta_{\gamma 2}$ are generated according to the same distribution (Eqn.(4.32)) as the angle $\theta_\gamma$ for the case of single initial state photon events. $\phi_{\gamma 12}$ is the angle between the planes defined by the incoming beam direction and each of the photons. The angle $\phi^\star_+$ is that between the plane formed by the summed momentum vector of the two photons and the beam direction, and the plane formed by the $l^+$ and the beam direction, all in the ODLR frame. The other variables are defined in the same way as for single initial state radiation events. The variables $y$ and $u$ are chosen using LUT generated at initialisation. The joint distribution in $y$ and $u$ is derived from that for $y_1$ and $y_2$ (Eqn.(4.40)):

$$\frac{d^2 n}{dy_1 dy_2} \simeq \int_{c^\star_{MIN}}^{c^\star_{MAX}} d\sigma_0(\tilde{s}', c^\star) dc^\star d(y_1) d(y_2) \tag{4.43}$$

where:

$$\tilde{s}' = s(1 - y_1 - y_2 + y_1 y_2) \simeq s' \tag{4.44}$$

In general the outgoing dilepton effective mass $\sqrt{s'}$ depends also on the angles of the radiated photons [8]. The expression (4.44) holds for back-to-back photons, the configuration corresponding to the maximum value of $y$ in (4.40), and therefore spanning the full

[8]The exact expression, in the present case, is given after Eqn.(4.57) below

possible phase space. For collinear photons, where $y_{MAX} = 1$, the $y_1 y_2$ term in (4.44) is absent. Separate LUT are thus generated for events with (approximately) back-to-back or collinear photons. If $\cos\theta_{\gamma 1}$, $\cos\theta_{\gamma 2}$ have opposite signs (back-to-back configuration) then Eqn.4.44 is used to define $\tilde{s}'$ in Eqn.4.44 and $y_{MAX} \simeq 2$. If $\cos\theta_{\gamma 1}$ , $\cos\theta_{\gamma 2}$ have the same sign (collinear configuration), the $y_1 y_2$ term in Eqn 4.44 is omitted and $y_{MAX} \simeq 1$.

By the change of variables:

$$y_1 = \frac{y}{2}(1+R) \quad (4.45)$$
$$y_2 = \frac{y}{2}(1-R) \quad (4.46)$$
$$R = (e^u - 1)/(e^u + 1) \quad (4.47)$$
$$W = \ln(1/y) \quad (4.48)$$

(4.43) may be re-written as:

$$\frac{d^2 n}{dW du} \simeq \int_{c^\star_{MIN}}^{c^\star_{MAX}} d\sigma_0(\tilde{s}', c^\star) dc^\star d_1 d_2 \quad (4.49)$$

where:

$$d_i = 1 - y_i + \frac{y_i^2}{2}, \qquad y_1 = e^{u-W}/(e^u + 1), \qquad y_2 = e^{-W}/(e^u + 1)$$

1-D LUTs for $y = e^{-W}$ are created at initialisation by subroutines SETYDJ,SETYDL for back-to-back, collinear photons respectively using Eqn.(4.49). The integral over $c^\star$ is performed analytically, and those over $W$ and $u$ numerically, using, respectively Simpson's rule and Gaussian integration. At the same time 2-D LUTs in the variables $W$ and $u$ are created by the subroutines SETRI, SETRL for back-to-back, collinear photons respectively. The integration limits on $W = \ln(1/y)$ are derived from Eqn.(4.41). Overall energy conservation leads to the following limits for the variable $u$ :

$$u_{MAX} = \ln[(y + y_1^{MAX} - y_2^{MIN})/(y - y_1^{MAX} + y_2^{MIN})] \quad (4.50)$$
$$u_{MIN} = -u_{MAX} \quad (4.51)$$

where:

$$y_1^{MAX} = \min[y_{MAX}, y - y_{MIN}]$$
$$y_2^{MIN} = \max[y_{MIN}, y - y_{MAX}]$$

Because the factorised formula (4.49) used to generate the photon energies does not take into account the angles of the radiated photons, overall overall energy conservation of the event is checked, after generation of the variables $\alpha_k$ and before construction of 4-vectors. In the LAB system, the 3-momentum of the outgoing dilepton is equal to the total 3-momentum of the two photons. Energy conservation then requires that:

$$\sqrt{s} = k_1 + k_2 + [(\vec{k_1} + \vec{k_2})^2 + M^2_{l^+l^-}]^{\frac{1}{2}} \quad (4.52)$$
$$> k_1 + k_2 + |\vec{k_1} + \vec{k_2}| \quad (4.53)$$

The inequality (4.53) follows from the condition that the dilepton effective mass $M_{l^+l^-}$ is $> 0$. (4.53) may also be written as:

$$2 > \quad y_1 + y_2 + [y_1^2 + y_2^2 + 2y_1y_2(\sin\theta_{\gamma 1}\sin\theta_{\gamma 2}\sin\phi_{\gamma 12} + \cos\theta_{\gamma 1}\theta_{\gamma 2}]^{\frac{1}{2}} \qquad (4.54)$$

If the R.H.S. of (4.54) is found to be $> 1.99$ the current event is rejected and a new one generated.

In constructing the 4-vectors of the photons, the recoil of the radiating $e^\pm$ is allowed for. The recoil angle, $\alpha_R$ of the $e^\pm$, after photon 1 has been radiated, is:

$$\alpha_R = \arctan\left(\frac{y_1 \sin\theta_{\gamma 1}}{1 - y_1|\cos\theta_{\gamma 1}|}\right) \qquad (4.55)$$

In the case of radiation of the two photons from the same incoming line, given by the condition:

$$\cos\theta_{\gamma 1}\cos\theta_{\gamma 2} > 0$$

the angle $\theta_{\gamma 2}$ is chosen relative to the direction of the recoiling $e^\pm$, rather than the beam direction.

The remainder of the event generation follows closely the procedure for the case of one initial state photon, described above. The lepton scattering angle is chosen using the same 2-D LUT as in the one photon case. The boost between th LAB and the ODLR frames is now along the direction of the vector sum ($\vec{K}$) of the 3-momenta of the two photons. The angle $\theta_\gamma$ in Eqns.(4.34,4.35) is replaced by $\theta_K$, the angle between the incoming $e^+$ direction and $\vec{K}$. The Lorentz transformation parameter $\beta^\star$ of Eqn.(4.36) is replaced by the expression:

$$\beta^\star = \frac{2|\vec{K}|}{\sqrt{s}(2-y)} \qquad (4.56)$$

and the lepton momentum in the ODLR frame used in Eqns.(4.37-4.39) is now given by:

$$q^\star = \sqrt{\frac{s'}{4} - m_l^2} \qquad (4.57)$$

where

$$s' = \frac{s}{4}(2-y)^2 - (\vec{K})^2$$

Finally, since events were generated using the approximate value of $s'$, $\tilde{s}'$ given by Eqns.(4.43) and (4.44) (with the appropriate modification for collinear photons) the events are re-weighted using a WRP, to take into account the exact value of $s'$ as given above after Eqn.(4.57). The weight used is:

$$d\sigma_0(s', c^*)/d\sigma_0(\tilde{s}', c^*)$$

## 4.4 Final State Radiation Events

### 4.4.1 One Final State Photon

Events with a single final state photon are generated by the subroutine ZFINGB, according to the differential cross-section Eqn.(4.3). They are subsequently re-weighted according to the exact differential cross-section using Eqn.(4.24)

The kinematical variables $\alpha_k$ used to define the event configuration are, in order of their generation:

$$y, \qquad \cos\theta^\star_\gamma, \qquad \cos\theta_\pm, \qquad \phi'_\gamma, \qquad \phi_\pm$$

Following (4.8) the scaled photon energy, $y$, is given by:

$$y = y_{MIN}\exp[Rn\ln\left(\frac{y_{MAX}}{y_{MIN}}\right)] \tag{4.58}$$

The angle, $\theta^\star_\gamma$, between the $l^+$ and the photon in the ODLR frame, is chosen according to the distribution:

$$\frac{dn}{dc^\star_\gamma} \simeq \frac{1}{1-\beta^2_{FI}(c^\star_\gamma)^2} \tag{4.59}$$

where

$$c^\star_\gamma = \cos\theta^\star_\gamma, \qquad \beta_{FI} = \sqrt{1-\frac{4m_l^2}{s}}$$

The scaled energies of the $l^+, l^-$ in the LAB frame, $y_+, y_-$, are obtained by a Lorentz transformation:

$$y_\pm = \frac{2}{\sqrt{s}}\gamma^\star[q^{(0)\star} \pm \beta^\star q^\star c^\star_\gamma] \tag{4.60}$$

where

$$q^{(0)\star} = \frac{\sqrt{s(1-y)}}{2}, \qquad q^\star = \sqrt{(q^{(0)\star})^2 - m_l^2}$$

$$\beta^\star = y/(2-y), \qquad \gamma^\star = (1-\frac{y}{2})/\sqrt{1-y}$$

The LAB scattering angles $\theta_\pm$ of the $l^\pm$ are found using the same 2-D LUT in $y$ and $\cos\theta^\star$ as used for initial state radiation. Here $y$ is set to zero so that $s' \to s$ and $\theta^\star \to \theta_+$. Following Ref[14], the angle $\theta_+$ is assigned to the $l^+$, or $\pi - \theta_+$ to the $l^-$, according to whether, or not :

$$P_s > Rn \tag{4.61}$$

where

$$P_s = y_+^2/(y_+^2 + y_-^2)$$

This has the effect that, if the photon is radiated from the $l^\pm$ then the scattering angle is most likely assigned to the other lepton $l^\mp$, whose direction is unaffected by recoil effects. In the case that the $\theta_+$ is assigned to the $l^+$, the 3-axis is chosen along the $l^+$ direction and the 3-momentum components of the $l^+$,$\gamma$ are given, in the LAB system, by the expressions:

$$\begin{aligned} q_+^{(1)} &= 0 \\ q_+^{(2)} &= 0 \\ q_+^{(3)} &= \frac{\sqrt{s}}{2} y_+ \beta_+ \end{aligned} \tag{4.62}$$

$$\begin{aligned} k^{(1)} &= \frac{\sqrt{s}}{2} y s'_\gamma \cos\phi'_\gamma \\ k^{(2)} &= \frac{\sqrt{s}}{2} y s'_\gamma \sin\phi'_\gamma \\ k^{(3)} &= \frac{\sqrt{s}}{2} y c'_\gamma \end{aligned} \tag{4.63}$$

where

$$c'_\gamma = 1 - 2[1-(1-y)/y_+]/y, \qquad s'_\gamma = \sqrt{1-(c'_\gamma)^2}, \qquad \beta_+ = \sqrt{1 - \frac{4m_l^2}{sy_+^2}}$$

Here $\phi'_\gamma$ is the azimuthal angle of the photon about the $l^+$ direction. If $\pi - \theta_+$ is assigned to the $l^-$, the 3-axis is chosen along the $l^-$ direction, and the 3-momentum components of the $l^-$, $\gamma$ are given in the LAB system as:

$$\begin{aligned} q_-^{(1)} &= 0 \\ q_-^{(2)} &= 0 \\ q_-^{(3)} &= \frac{\sqrt{s}}{2} y_- \beta_- \end{aligned} \tag{4.64}$$

$$\begin{aligned} k^{(1)} &= \frac{\sqrt{s}}{2} y s''_\gamma \cos\phi'_\gamma \\ k^{(2)} &= \frac{\sqrt{s}}{2} y s''_\gamma \sin\phi'_\gamma \\ k^{(3)} &= \frac{\sqrt{s}}{2} y c''_\gamma \end{aligned} \tag{4.65}$$

where

$$c''_\gamma = 1 - 2[1-(1-y)/y_-]/y, \qquad s''_\gamma = \sqrt{1-(c''_\gamma)^2}, \qquad \beta_- = \sqrt{1 - \frac{4m_l^2}{sy_-^2}}$$

The momentum components in the standard LAB system with 3-axis along the incoming $e^+$ direction, are found by rotating the above 3-vectors by $\theta_+$ (or $\pi - \theta_+$) about the 1-axis, as appropriate. The azimuthal angle $\phi_+$,($\phi_-$) of the $l^+$, ($l^-$) about the 3-axis is then generated uniformly. Finally the 3-momentum of the remaining lepton is given by momentum conservation in the LAB frame.

### 4.4.2 Two Final State Photons

Events with two final state photons are generated by the subroutine ZFIN2G according to the factorised differential cross-section [3, 14, 37]:

$$\begin{aligned} d\sigma_{2\gamma}^F &= C_n d\sigma_0(s) \frac{q_+ \cdot q_-}{(q_+ \cdot k_1)(q_- \cdot k_1)} \frac{q_+ \cdot q_-}{(q_+ \cdot k_2)(q_- \cdot k_2)} \\ &\quad \times (1 - \frac{k_1}{E} + \frac{k_1^2}{2E^2})(1 - \frac{k_2}{E} + \frac{k_2^2}{2E^2}) \frac{d^3k_1}{k_1} \frac{d^3k_2}{k_2} \\ &= 4C_n \frac{d\sigma_0(s) d(y_1) d(y_2) dy_1 d\Omega_1 dy_2 d\Omega_2}{(1 - \beta_{FI}^2 \cos^2 \Theta'_{\gamma 1})(1 - \beta_{FI}^2 \cos^2 \Theta'_{\gamma 2})} \end{aligned} \tag{4.66}$$

The variables in (4.66) are defined after Eqns.(4.40) and (4.59), with the exception of $\theta'_{\gamma 1}$, $\theta'_{\gamma 2}$, which are the angles between the photons 1,2 and the $l^+$ in the LAB system. To take into account recoil effects however, when $\cos\theta'_{\gamma 2} < 0$ the photon 2 is assigned the angle $\pi - \theta'_{\gamma 2}$ relative to the $l^-$ direction rather than $\theta'_{\gamma 2}$ relative to the $l^+$ direction.

The kinematical variables $\alpha_k$ in the order of generation are:

$$y, \quad cos\theta'_{\gamma 1}, \quad cos\theta'_{\gamma 2}, \quad u, \quad cos\theta_{\pm}, \quad \phi'_{\gamma 12}, \quad \phi'_{\gamma T}, \quad \phi_{\pm}$$

Here $\theta_\pm$ are the scattering angles of the $l^\pm$ relative to the incoming $e^+$ in the LAB frame. The azimuthal angle $\phi'_{\gamma 12}$ is chosen to be that between the planes defined by : $(\vec{k_1}, \vec{q_+})$; $(\vec{k_2}, \vec{q_+})$ for $\cos\theta'_{\gamma 2} > 0$ and between the planes: $(\vec{k_1}, \vec{q_-})$; $(\vec{k_2}, \vec{q_-})$ for $\cos\theta'_{\gamma 2} < 0$. $\phi'_{\gamma T}$ is the azimuthal angle about the $l^+$ direction of the total momentum vector $\vec{K_2}$ of the two photons. $\phi_\pm$ are the azimuthal angles of the $l^\pm$ relative to the incoming $e^+$. All three angles are generated uniformly in the interval 0 to $2\pi$ radians. The polar angles $\theta'_{\gamma 1}$, $\theta'_{\gamma 2}$ are generated according to Eqn.(4.32) with the replacements $\theta_\gamma \to \theta'_\gamma$, $\beta_{IN} \to \beta_{FI}$. The variables $y$ and $u$ ($y = y_1 + y_2$, $u = \ln(y_1/y_2)$) are chosen from, respectively, 1-D and 2-D LUT's generated at initialisation. The distribution (4.49) is here replaced by:

$$\frac{d^2n}{dW du} \simeq d_1 d_2 \tag{4.67}$$

where $W = \ln(1/y)$. The subroutines that generate the 1-D LUT for $y$ and the 2-D LUT for $W$ are SETYDK, SETRK respectively. The angle $\theta_+$ is generated as described above for single final state photon events.

The construction of the 3-vectors of the final state particles is now described. Unlike for the case of single final state photon radiation it is here more convenient to work entirely in the LAB frame. Choosing the 3-axis along the $l^+$ direction, with the 2-axis in the plane containing $\vec{q_+}$ and $\vec{k_1}$ the momentum components of photon 1 are:

$$\begin{aligned}
k_1^{(1)} &= 0 \\
k_1^{(2)} &= \frac{\sqrt{s}}{2} y_1 s'_{\gamma 1} \\
k_1^{(3)} &= \frac{\sqrt{s}}{2} y_1 c'_{\gamma 1}
\end{aligned} \tag{4.68}$$

where

$$s'_{\gamma 1} = \sin\theta'_{\gamma 1}, \qquad c'_{\gamma 1} = \cos\theta'_{\gamma 1}$$

In the case that $c'_{\gamma 2} > 0$ (photon 2 radiated from the $l^+$) the momentum components of photon 2 are given by:

$$\begin{aligned}
k_2^{(1)} &= \frac{\sqrt{s}}{2} y_2 s'_{\gamma 2} \cos\phi'_{\gamma 12} \\
k_2^{(2)} &= \frac{\sqrt{s}}{2} y_2 s'_{\gamma 2} \sin\phi'_{\gamma 12} \\
k_2^{(3)} &= \frac{\sqrt{s}}{2} y_2 c'_{\gamma 2}
\end{aligned} \tag{4.69}$$

For $c'_{\gamma 2} < 0$ (photon 2 radiated from the $l^-$) the direction of the 3-axis in Eqn.(4.69) is chosen opposite to the direction of the $l^-$. In this case the momentum components are still constructed according to Eqn.(4.69), but they are subsequently rotated about the 1-axis through the acollinearity angle $\alpha_A$ between the $l^+$, $l^-$ directions. This angle is calculated from the equations:

$$\begin{aligned}
\sin\alpha_A &= y_1 s'_{\gamma 1}/y_- \\
y_- &= 2 - y_1 - (1-y_1)/[1 - 0.5(1 - c'_{\gamma 1})y_1] \\
\cos\alpha_A &= \sqrt{1 - \sin^2\alpha_A}
\end{aligned} \tag{4.70}$$

By this procedure the strong peaking of the radiated photons along the $l^\pm$ directions is properly accounted for, even in the case of recoils generated by the first radiated photon [9]. Before proceeding to construct the 3-vectors of the outgoing leptons energy conservation is checked using Eqn.(3.51). If the R.H.S. is $> 0.99\sqrt{s}$ the event is rejected and a new one is generated.

The scaled energies $y_\pm$ of the $l^\pm$ are next calculated:

$$y_+ = 2[1 - y(1 - \frac{y}{4}) - \frac{(\vec{K_2})^2}{s}]/(2 - y + \frac{2K_2^{(3)}}{\sqrt{s}}) \tag{4.71}$$

$$y_- = 2 - y - y_+ \tag{4.72}$$

[9] In Eqn.(4.69) the angle $\theta'_{\gamma 2}$ is that between the photon and the radiating (virtual) $l^-$, rather than that between the photon and the outgoing $l^-$ as in Eqn.(4.66). This difference is of little importance for soft or nearly collinear photons. For hard non-collinear photons however the rate will be over-estimated as compared to Eqn.(4.66). In the case that both photons are radiated from the $l^+$ the ansatz used tends, on the contrary to underestimate the rate of such photons as compared to Eqn.(4.66)

As in Eqns.(4.68,4.69) the 3-axis is along the $l^+$ direction. The angle $\theta_+$ is assigned to $l^+$ or $\pi - \theta_+$ to the $l^-$, as described above for single photon final state radiation. The azimuthal angle of the total photon momentum $\vec{K}_2$ is then generated uniformly by assigning the angle $\phi'_{\gamma T}$. The 3-axis is now rotated into the direction of the incoming $e^+$ and the angle $\phi_+$ (or $\phi_-$) is assigned. Finally the 3-momentum of the remaining lepton is found using momentum conservation in the LAB system.

### 4.4.3 Three Final State Photons

Events with three final state photons are generated, according to the obvious generalisation of Eqn.(4.66), by the subroutine ZFIN3G. The event configuration is defined by the following kinematical variables:

$$y, \quad \cos\theta'_{\gamma 1}, \quad \cos\theta'_{\gamma 2}, \quad \cos\theta'_{\gamma 3}, \quad u', \quad u, \quad \cos\theta_{\pm}, \quad \phi'_{12}, \quad \phi'_{\gamma 3}, \quad \phi'_{\gamma T}, \quad \phi_{\pm}$$

$\theta'_{\gamma 3}$ is similarly defined to $\theta'_{\gamma 1}$, $\theta'_{\gamma 2}$ but refers to the third (least energetic) photon. The variable $u'$ is given by:

$$u' = \ln\left(\frac{y_1 + y_2}{y_3}\right) \tag{4.73}$$

The definition of $\phi'_{\gamma 3}$ is similar to that given above for the case of 2 final state photons except that the total photon momentum vector $\vec{K}_3$ corresponds to three photons. The angle $\phi'_{\gamma 3}$ is that between the plane defined by the total momentum of the first two (most energetic) photons $\vec{K}_2$ and the $l^+$ (or $l^-$), and that defined by the direction of photon 3 and $l^+$ (or $l^-$). $l^+$ ($l^-$) is taken for $\cos\theta'_{\gamma 3} > 0 (< 0)$. All other variables have been defined previously.

Following the soft photon limit of the generalisation of Eqn.(4.66) the photon energies are first generated according to the distribution:

$$\frac{d^3 n}{dy_1 dy_2 dy_3} \simeq \frac{1}{y_1 y_2 y_3} \tag{4.74}$$

By a change of variables:

$$W = \ln[1/(y_1 + y_2 + y_3)], \qquad u = \ln(y_1/y_2), \qquad u' = \ln[(y_1 + y_2)/y_3]$$

(4.74) simplifies to

$$\frac{d^3 n}{dW du du'} \simeq constant \tag{4.75}$$

A 1-D LUT for the variable $W$, and a 2-D LUT for $u'$ (for a given $W$) are created at initialisation by the subroutines SETYD3, SETRK3 respectively. The limits on the variable $u$ in the nested integration:

$$I = \int dW \left\{ \int du' \left[ \int du \right] \right\}$$

are

$$u_{MIN} = \ln\left(\frac{y' - y_1^{MAX}}{y_1^{MAX}}\right) \tag{4.76}$$

$$u_{MAX} = \ln\left(\frac{y' - y_1^{MIN}}{y_1^{MIN}}\right) \tag{4.77}$$

where

$$y_1^{MIN} = \max[y' - y_{MAX}, y_{MIN}]$$
$$y_1^{MAX} = \min[y' - y_{MIN}, y_{MAX}]$$

and

$$y' = y_1 + y_2 = ye^{u'}/(1 + e^{u'})$$

The limits on the $u'$ integration are given by Eqns.(4.50,4.51) with the replacements $u \to u'$ and

$$y_1^{MIN} \to \max[2y_{MIN}, y - y_{MAX}]$$
$$y_1^{MAX} \to \min[y_{MAX}, y - 2y_{MIN}]$$

The integral over $u$ is done analytically, that over $u'$ by Gaussian integration, and that over $W$ by Simpson's rule. During the event generation phase the value of $W$ is chosen from the 1-D LUT, The value of $u'$ is then chosen using the 2-D LUT. Finally, $u$ is given by:

$$u = u_{MIN} + Rn(u_{MIN} - u_{MAX}) \tag{4.78}$$

where $u_{MIN}$, $u_{MAX}$ are given by Eqns.(4.76,4.77). $y_1$, $y_2$, $y_3$ are then derived from $W$,$u$, $u'$ via the equations:

$$\begin{aligned} y &= e^W \\ y' &= ye^{u'}/(1 + e^{u'}) \\ y_1 &= y'e^u/(1 + e^u) \\ y_2 &= y' - y \\ y_3 &= y - y' \end{aligned} \tag{4.79}$$

The photon spectra are then modified by weight rejection to take into account the hard photon corrections given by the Gribov-Lipatov [38] kernels.

Defining the weight function:

$$\mathcal{W} = d(y_1)d(y_2)d(y_3)y_1y_2y_3 \tag{4.80}$$

where $d(y)$ is defined after Eqn.(4.40), the event is rejected if:

$$\mathcal{W} < Rn$$

The construction of the momentum vectors of the final state particles follows closely that described in the previous section for the case of two photons. The 3-vectors of the two most energetic photons are given by Eqns(4.68,4.69) above, and the angle $\theta'_{\gamma 2}$ is chosen relative to the $l^+$ direction (opposite of the $l^-$ direction ) according to whether $\cos\theta'_{\gamma 2}$ is $> 0$ or ($< 0$). Energy conservation is then checked using Eqn.(4.54). To take into account recoil effects in the radiation of the third photon the acollinearity angle between the $l^+$ and $l^-$ after the first two photons have been radiated is calculated, and $\theta'_{\gamma 3}$ is chosen relative to the $l^+$ direction (opposite of the $l^-$ direction, for $\cos\theta'_{\gamma 3} > 0$ ($< 0$). That is, the procedure used above for the second photon is iterated. The azimuthal angle $\phi'_{\gamma 3}$ is chosen to be that between the planes defined by $(\vec{K_2}, \vec{q_+})$ ; $(\vec{k_3}, \vec{q_+})$ for $\cos\theta'_{\gamma 3} > 0$, and that between $(\vec{K_2}, \vec{q_-})$ ; $(\vec{k_3}, \vec{q_-})$ for $\cos\theta'_{\gamma 3} < 0$. Since the relative directions of all three photons are now fixed energy conservation is again checked using the generalisation of Eqn.(4.53):

$$\sqrt{s} > k_1 + k_2 + k_3 + |\vec{k_1} + \vec{k_2} + \vec{k_3}| \tag{4.81}$$

If the RHS of Eqn(4.81) is $> 0.99\sqrt{s}$, the event is rejected. Eqns.(4.71,4.72) with the replacement $\vec{K_2} \to \vec{K_3}$ is now used to calculate $y_+$, $y_-$. The scattering angle $\theta_+$ is assigned to the $l^+$, or $\pi - \theta_+$ to the $l^-$, according to Eqns.(4.61). The angles $\phi'_{\gamma T}$ and $\phi_+$ or $\phi_-$ are assigned as described above for the two photon case. Finally the 3-momentum of the remaining lepton is calculated from overall momentum conservation.

### 4.4.4 Events with One Initial State and One Final State Photon

Events with one initial state and one final state photon are generated by the subroutine ZINF2G according to the differential cross-section (c.f Eqns.(4.40,4.66)):

$$\begin{aligned} d\sigma^I F_{2\gamma} &= C_n d\sigma_0(s'') \frac{p_+ \cdot p_-}{(p_+ \cdot k_1)(p_- \cdot k_1)} \frac{q_+ \cdot q_-}{(q_+ \cdot k_2)(q_- \cdot k_2)} \\ &\quad \times (1 - \frac{k_1}{E} + \frac{k_1^2}{2E^2})(1 - \frac{k_2}{E} + \frac{k_2^2}{2E^2}) \frac{d^3 k_1}{k_1} \frac{d^3 k_2}{k_2} \\ &= 4C_n \frac{d\sigma_0(s'') d(y_1) d(y_2) dy_1 d\Omega_1 dy_2 d\Omega_2}{(1 - \beta_{IN}^2 \cos^2\Theta_{\gamma 1})(1 - \beta_{FI}^2 \cos^2\Theta'_{\gamma 2})} \end{aligned} \tag{4.82}$$

where

$$s'' = s(1 - y_1)$$

and photons 1,(2) are radiated in the initial,(final) state. The kinematical variables used, in this case, to define the event configuration are:

$$y, \quad cos\theta_{\gamma 1}, \quad cos\theta'_{\gamma 2}, \quad u, \quad cos\theta''_+, \quad \phi''_+, \quad \phi''_{\gamma 2}, \quad \phi_{\gamma 1}$$

Here $\theta''_{\gamma 2}$ is the angle between photon 2 and the $l^+$ in the rest frame of $l^+ l^- \gamma 2$ (ODLGR frame). In this frame the 1-axis is chosen perpendicular to the plane defined by the incoming $e^+$ and the direction of photon 1. $\theta''_+$, $\phi''_+$ are the polar and azimuthal angles of the $l^+$ relative to the incoming $e^+$ direction (allowing, if necessary, for the recoil generated by photon 1) in the ODLGR frame. The other variables have been previously defined.

The variables $y$, $u$ are generated using 1-D, 2-D LUT created at initialisation by the subroutines SETYDI, SETR respectively. The procedure is the same as that described in

Section 4.3.2 above. The 2-D LUT is created according to Eqn.(4.49) with the replacement: $\tilde{s}' \to s''$. The configuration of $l^+l^-\gamma 2$ is generated as described in Section 4.4.1, for single final state radiation, but in the ODLGR frame rather than the LAB frame. Recoil effects from the initial state photon are accounted for by rotation by the angle $\alpha_B$ given by Eqns.(4.34-4.35) with the replacement $y \to y_1$ in the formula (4.36) for $\beta^\star$. Since the scaled photon energies $y_1$, $y_2$ are defined in the LAB frame, the energy of photon 2 must first be calculated, by Lorentz transformation, in the ODLGR frame, from its angles in this frame (specified by $\theta''_{\gamma 2}$, $\phi''_{\gamma 2}$, $\theta''_+$, $\phi''_+$ ) and its LAB energy $\frac{\sqrt{s}}{2}y_2$. For simplicity, in this case, $\theta''_+$ is always assigned to the $l^+$, rather than $\theta''_+$ to the $l^+$ or $\pi - \theta''_+$ to the $l^-$ according to Eqn(4.61) [10]. Finally the 4-vectors of $l^+$, $l^-$ and photon 2 are tranformed back into the lab. frame and the event is rotated about the incoming $e^+$ direction so that photon 1 has azimuthal angle $\phi_{\gamma 1}$.

# 5 Program Structure and Performance

## 5.1 General Organisation. How to use the Program

The program has a very short main program containing definitions of the most important input parameters, which are stored in the labelled common block ICOM. These variables are described in Table 3. The execution of the program has three distinct phases:

- Initialisation
- Generation of a single unit weight event
- Termination

Each of these phases is entered via a call to subroutine BHAGENE3 in the main program:

CALL BHAGENE3(MODE,CTP1,CTP2,CTM1,CTM2,CTAC,EP0,EM0)

MODE is set to $-1, 0, 1$ for the initialisation, generation and termination phases respectively. The other parameters of BHAGENE3 define the kinematical cuts to be applied to the generated events:

CTP1 = minimum value of $\cos\theta_{l^+}$
CTP2 = maximum value of $\cos\theta_{l^+}$
CTM1 = minimum value of $\cos\theta_{l^-}$

[10] The procedures described in Ref.[14] for assigning the scattering angle in the case of single initial or final state photons are not directly applicable in this case

CTM2 = maximum value of $\cos\theta_{l^-}$

CTAC = maximum value of $\cos\phi_{col}$

EP0 = minimum energy of $l^+$ $(GeV)$

EM0 = minimum energy of $l^-$ $(GeV)$

All these cuts are applied in the laboratory (incoming $e^+, e^-$ centre of mass) system. The angle $\phi_{col}$ is the collinearity angle between the $l^+$ and the $l^-$ (CATC = -1 for a back-to-back configuration). In the calls of BHAGENE3 with MODE = 0,1 only this parameter need be specified. A typical main program to generate 5000 unit weight events might be:

```
PROGRAM BHAMAIN
IMPLICIT REAL*8 (A-H,O-Z))
COMMON/ICOM/OIMZ,OIMT,OIMH,OMAS,IOCHA,IOEXP,OW,OCTC1,OCTC2,IOXI
OIMZ=91.18D0            ! Z Mass (GeV)
OIMT=150.0D0            ! Top quark mass (GeV)
OIMH=100.0D0            ! Higgs boson mass (GeV)
OMAS=0.12D0             ! Alphas
IOCHA=1                 ! 0 for Muon pairs, 1 for electron pairs
IOEXP=1                 ! 0 for O(Alpha), 1 for exponentiation
OW=91.00D0              ! Collision energy (GeV)
OCTC1=-0.8D0            ! Lower cos(theta) limit in ODLR frame
OCTC2=0.8D0             ! Upper cos(theta) limit in ODLR frame
IOEXI=713883717         ! Intial random number

             ! KINEMATICAL CUTS FOR GENERATED EVENTS

CTP1=-0.7D0             ! Lower cos(theta) for l+ in the lab frame
CTP2=0.7D0              ! Upper cos(theta) for l+ in the lab frame
CTM1=-0.7D0             ! Lower cos(theta) for l- in the lab frame
CTM2=0.7D0              ! Upper cos(theta) for l- in the lab frame
CTAC=-0.9D0             ! Upper cosine of collinearity angle
EP0=2.0D0               ! Minimum energy of l+ (GeV)
EM0=2.0D0               ! Minimum energy of l- (GeV)

             ! INITIALISATION PHASE

CALL BHAGENE3(-1,CTP1,CTP2,CTM1,CTM2,CTAC,EP0,EM0)
```

```
                    ! GENERATION LOOP

DO J=1,5000
CALL BHAGENE3(0)
ENDDO

                    ! TERMINATION PHASE

CALL BHAGENE3(1)
STOP
END
```

Other initialisation parameters of interest to users are defined in BHAGENE3 itself. A list of the most important of these can be found in Table 4.

## 5.2 Initialisation Phase

A flow chart of the initialisation phase of the program is shown in Fig 1. The functionality of the different subprograms shown there has been described above. The main physical quantities calculated are also indicated. In order to estimate the average weights of events with $\geq$ two hard photons the initialisation phase actually includes the generation of $4 \times 10^4$ events, which is relatively time consuming. Users should be aware of this.

## 5.3 Generation Phase

As shown in the flow chart in Fig 2. each of the different event topologies is generated by a different subprogram. The calculations performed by the five different hard photon subgenerators : ZINIGB, ZFINGB, ZFIN2G, ZFIN3G, and ZINF2G are described above in Section 4. The 4-vectors of the generated events are written in the labelled common block C4VEC :

```
COMMON/C4VEC/PPV(4),PMV(4), QPV(4),QMV(4),GAM1V(4),GAM2V(4),GAM3V(4),
WEIGHT,NPHOT,ISG
```

The 4-vectors (defined as: $p_x, p_y, p_z, E$ ) are in the order: $(e^+, e^-, l^+, l^-, \gamma_1, \gamma_2, \gamma_3)$. WEIGHT is the event weight, NPHOT the number of filled photon 4-vectors and ISG a flag indicating the subgenerator that produced the event (see Table 2.). The photons are ordered in energy, the first photon being the most energetic.

## 5.4 Termination Phase

A flow chart of the termination phase is shown in Fig 3. The exact cross-section ($\sigma^{CUT}$) and its error ($\Delta\sigma^{CUT}$) are printed out, together with the input parameters (including those calculated in the initialisation phase). Other cross-sections used to calculate the event generation probabilities $P(n_\gamma^I, n_\gamma^F)$ are also printed out. A sample output is shown in Fig 4.

## 5.5 Program Performance

As mentioned above, after a relatively lengthy initialisation procedure, during which all look-up tables are created and average weights are calculated for multiphoton events, the event generation procedure is itself fast. Typical times for the initialisation phase are 140, 48 IBM 3090 CPU seconds for $e^+e^-$, $\mu^+\mu^-$ pair generation, respectively. Average times to generate a single unit weight $e^+e^-$, ($\mu^+\mu^-$) event are $1.41 \times 10^{-3}$, ($0.64 \times 10^{-3}$) IBM 3090 CPU seconds. The combined weight distribution for initial and final state radiation $e^+e^-$ events for parameters and cuts as in the sample main program given above, (whose output is shown in Fig. 4) is presented in Fig. 5. The weight distribution can be seen to be well centered around 1.0, resulting in an efficient generation of unit weight events by the weight throwing procedure [4]. The fractions of events with weights $> 2.0$, (3.0) is $\simeq 6 \times 10^{-3}$, ($2 \times 10^{-4}$) respectively. The maximum weight may then be chosen to be 2.0, allowing efficient generation of event samples with cross-sections known at, or below the % level [10].

## Appendix A

Following Ref.[39], the Born differential cross-section for Bhabha scattering, including both $s$ and $t$ channel Z exchange may be written as:

$$\frac{d\sigma_0}{dt} = \frac{2\pi\alpha^2}{s^2}[B_0 + B_2(\frac{t}{s})^2 + B_3(1+\frac{t}{s})^2] \qquad (A1)$$

where

$$\begin{aligned} t &= -\frac{s}{2}(1-\cos\theta_+) \\ B_0 &= [\frac{s}{t} + c_-\chi_t(t)]^2 \\ B_2 &= |1 + c_-\chi_s(s)|^2 \end{aligned}$$

$$B_3 = \frac{1}{2}\{|1 + \frac{s}{t} + a_+[\chi_t(t) + \chi_s(s)]|^2 + (a_+ \to a_-)\}$$

$$c_- = g_V^2 - g_A^2 \quad , \quad a_\pm = (g_V \pm g_A)^2$$
$$\chi_t(t) = \frac{s}{t - M_Z^2} \quad , \quad \chi_s(s) = \frac{s}{s - M_Z^2 + is(\Gamma_Z/M_Z)}$$

$\theta_+$ is the scattering angle between the incoming and outgoing $e^\pm$. $M_Z$, $\Gamma_Z$ are the mass and width of the Z. The Z width is neglected in the t-channel exchange amplitude. The vector and axial-vector coupling constants $g_A$, $g_V$ are given, in the Standard Model by Eqns. 3.3-3.6. The cross-section for $\mu$-pair production is recovered on setting $s/t$ to zero in $B_0$, $B_3$.

Analytical integration of $A1$ over $t$ yields the result [34] ( see also the first of Ref.[4]):

$$\int_{t_{MIN}}^{t_{MAX}} d\sigma_0 = \frac{2\pi\alpha^2}{s^2}[S(t_{MIN}) - S(t_{MAX})] \qquad (A2)$$

The function $S(t)$ is the sum of the following 5 terms derived from Eqn. $A1$ :

$$\int B_0 dt = s^2[-\frac{1}{t} + \frac{2c_-}{M_Z^2}\ln\frac{t-M_Z^2}{t} - \frac{c_-^2}{t-M_Z^2}]$$

$$\begin{aligned} \int B_3 dt = & \frac{1}{2}\{s^2[-\frac{2}{t} + \frac{2(a_+ + a_-)}{M_Z^2}\ln\frac{t-M_Z^2}{t} - \frac{(a_+^2 + a_-^2)}{t-M_Z^2}] \\ & +2s[(b_+ + b_-)\ln t + (a_+b_+ + a_-b_-)\ln(t-M_Z^2)] + (b'_+ + b'_-)t\} \end{aligned}$$

$$\begin{aligned} \int \frac{2B_3 t}{s} dt = & s\{2\ln t + 2(a_+ + a_-)\ln(t-M_Z^2) + (a_+^2 + a_-^2)[\ln(t-M_Z^2) - \frac{M_Z^2}{t-M_Z^2}]\} \\ & +2\{(b_+ + b_-)t + (a_+b_+ + a_-b_-)[t + M_Z^2\ln(t-M_Z^2]\} + (b'_+ + b'_-)\frac{t^2}{2s} \end{aligned}$$

$$\int B_2 \frac{t^2}{s^2} dt = \frac{B_2 t^3}{3s^2}$$

$$
\begin{aligned}
\int B_3 \frac{t^2}{s^2} dt &= \frac{1}{2}\{[2t + 2(a_+ + a_-)(t + M_Z^2 \ln(t - M_Z^2)) + (a_+^2 + a_-^2)(t + 2M_Z^2 \ln(t - M_Z^2) \\
&\quad - \frac{M_Z^4}{t - M_Z^2})] + \frac{2}{s}[(b_+ + b_-)\frac{t^2}{2} + (a_+ b_+ + a_- b_-)(\frac{t^2}{2} + tM_Z^2 + M_Z^4 \ln(t - M_Z^2))] \\
&\quad + (b'_+ + b'_-)\frac{t^3}{3s^2}\}
\end{aligned}
$$

where

$$
\begin{aligned}
b_\pm &= 1 + a_\pm Re\chi_s \\
b'_\pm &= 1 + a_\pm 2Re\chi_s + a_\pm^2 |\chi_s|^2
\end{aligned}
$$

## Appendix B

The exact hard photon cross-section, with exponentiated initial state radiation, is derived from formulae given in the second of Refs.[4] :

$$
\frac{d\sigma^{EXACT}}{d\Omega_+ d\Omega_\gamma dy} = \frac{\alpha^3 y}{16\pi^2 s} X(\alpha_k) \qquad (B1)
$$

The function $X(\alpha_k)$ of the kinematical variables $\alpha_k$ specifying the event configuration, (defined in Sections 4.3.1 to 4.4.4), has contributions from $s$-channel exchanges ($ss$), $t$-channel exchanges ($tt$) and $s - t$ interference ($st$). In each of these cases separate contributions from initial state radiation (INI), final state radiation (FIN), and initial/final interference (INT) may be distinguished. Thus :

$$
X = \sum_{i,j} X_i^j \quad i = ss, tt, st; \quad j = INI, FIN, INT \qquad (B2)
$$

where

$$
\begin{aligned}
X_{ss}^{INI} &= \frac{1}{s'\kappa_+\kappa_-}\left[B_1(c_-, s')[f(s)t^2 + t'^2] + B_5(s')[f(s)u^2 + u'^2]\right] \\
&\quad - m_e^2 f(s)\left[\frac{B_{ss}(s', t, u)}{\kappa_-^2} + \frac{B_{ss}(s', t', u')}{\kappa_+^2}\right] \\
X_{ss}^{FIN} &= \frac{1}{s\kappa'_+\kappa'_-}\left[B_1(c_-, s)[t^2 + t'^2] + B_5(s)[u^2 + u'^2]\right] \\
&\quad - m_l^2\left[\frac{B_{ss}(s, t, u')}{\kappa_-'^2} + \frac{B_{ss}(s, t', u)}{\kappa_+'^2}\right] \\
X_{ss}^{INT} &= \frac{1}{ss'}\left[\frac{u}{\kappa_+\kappa'_-} + \frac{u'}{\kappa_-\kappa'_+} - \frac{t}{\kappa_+\kappa'_+} - \frac{t'}{\kappa_-\kappa'_-}\right]
\end{aligned}
$$

$$
\begin{aligned}
& \times \left[B_4(s,s')(t^2+t'^2)+B_6(s,s')(u^2+u'^2)\right] \\
& + \frac{\vec{p_+}\cdot(\vec{q_+}\times\vec{q_-})(s-s')}{2\sqrt{s}s'\kappa_+\kappa_-\kappa'_+\kappa'_-}B_7(s,s')(u^2-u'^2)
\end{aligned}
$$

$$
\begin{aligned}
X_{tt}^{INI} &= \frac{s}{\kappa_+\kappa_-}\left[B_6(t,t')[\frac{f(s)u^2+u'^2}{tt'}]+B_4(t,t')[\frac{f(s)s^2+s'^2}{tt'}]\right] \\
& -m_e^2 f(s)\left[\frac{B_{tt}(s',t,u)}{\kappa_-^2}+\frac{B_{tt}(s',t',u')}{\kappa_+^2}\right] \\
X_{tt}^{FIN} &= \frac{s'}{\kappa'_+\kappa'_-}\left[B_6(t,t')[\frac{u^2+u'^2}{tt'}]+B_4(t,t')[\frac{s^2+s'^2}{tt'}]\right] \\
& -m_l^2\left[\frac{B_{tt}(s,t,u')}{\kappa'^2_-}+\frac{B_{tt}(s,t',u)}{\kappa'^2_+}\right] \\
X_{tt}^{INT} &= \frac{s^2+s'^2}{tt'}\left[-\frac{t}{\kappa_+\kappa'_+}B_1(c_-,t')-\frac{t'}{\kappa_-\kappa'_-}B_(c_-,t)\right] \\
& +\frac{u^2+u'^2}{tt'}\left[-\frac{t}{\kappa_+\kappa'_+}B_5(t')-\frac{t'}{\kappa_-\kappa'_-}B_5(t)\right] \\
& +\left(\frac{u}{\kappa_+\kappa'_-}+\frac{u'}{\kappa_-\kappa'_+}\right)\left[B_6(t,t')[\frac{u^2+u'^2}{tt'}]+B_4(t,t')[\frac{s^2+s'^2}{tt'}]\right] \\
X_{st}^{INI} &= \frac{f(s)u^2+u'^2}{s'\kappa_+\kappa_-}\left[-\left(\frac{u'+t'}{t'}\right)B_6(s',t')-\left(\frac{u+t}{t}\right)B_6(s',t)\right] \\
& -m_e^2 f(s)\left[\frac{B_{st}(s',t,u)}{\kappa_-^2}+\frac{B_{st}(s',t',u')}{\kappa_+^2}\right] \\
X_{st}^{FIN} &= \frac{u^2+u'^2}{s\kappa'_+\kappa'_-}\left[-\left(\frac{u'+t}{t}\right)B_6(s,t)-\left(\frac{u+t'}{t'}\right)B_6(s,t')\right] \\
& -m_l^2 f(s)\left[\frac{B_{st}(s,t,u')}{\kappa'^2_-}+\frac{B_{st}(s,t',u)}{\kappa'^2_+}\right]
\end{aligned}
$$

$$
\begin{aligned}
X_{st}^{INT} &= \left[\frac{u'}{\kappa_-\kappa'_+}+\frac{u'+s'}{\kappa_+\kappa'_+}\right]\left(\frac{u^2+u'^2}{s't'}\right)B_6(s',t') \\
& +\left[\frac{u}{\kappa_+\kappa'_-}+\frac{u+s'}{\kappa_-\kappa'_-}\right]\left(\frac{u^2+u'^2}{s't}\right)B_6(s',t) \\
& +\left[\frac{u'}{\kappa'_+\kappa_-}+\frac{u'+s}{\kappa'_-\kappa_-}\right]\left(\frac{u^2+u'^2}{st}\right)B_6(s,t) \\
& +\left[\frac{u}{\kappa_+\kappa'_-}+\frac{u+s}{\kappa'_+\kappa_+}\right]\left(\frac{u^2+u'^2}{st'}\right)B_6(s,t') \\
& +\frac{\sqrt{s}[\vec{p_+}\cdot(\vec{q_+}\times\vec{q_-})](u^2-u'^2)}{2\kappa_+\kappa_-\kappa'_+\kappa'_-}[-\left(\frac{t-t'}{tt'}\right)B_7(t,t')+2\frac{B_7(s,t)}{st(\kappa_-\kappa'_+\kappa'_-)} \\
& +2\frac{B_7(s,t')}{st'(\kappa_+\kappa'_+\kappa'_-)}-2\frac{B_7(s',t)}{s't(\kappa_+\kappa_-\kappa'_-)}-2\frac{B_7(s',t')}{s't'(\kappa_+\kappa_-\kappa'_+)}]
\end{aligned}
$$

$$B_1(c,s) = 1+[\frac{2c}{s}(s-M_Z^2)+c^2]|\chi_s(s)|^2$$

$$\begin{aligned}B_4(s,s') &= 1+c_-[(1-\frac{M_Z^2}{s})|\chi_s(s)|^2+(1-\frac{M_Z^2}{s'})|\chi_s(s')|^2] \\ &\quad +c_-^2[(1-\frac{M_Z^2}{s})(1-\frac{M_Z^2}{s'})+\frac{M_Z^2\Gamma_Z^2}{ss'}]|\chi_s(s)|^2|\chi_s(s')|^2\end{aligned}$$

$$B_5(s) = 1+[2(g_V^2+g_A^2)(1-\frac{M_Z^2}{s})+(g_V^4+g_A^4+6g_V^2g_A^2)]|\chi_s(s)|^2$$

$$\begin{aligned}B_6(s,s') &= 1+(g_V^2+g_A^2)\left[(1-\frac{M_Z^2}{s})|\chi_s(s)|^2+(1+\frac{M_Z^2}{s'})|\chi_s(s')|^2\right] \\ &= +(g_V^4+g_A^4+6g_V^2g_A^2)\left[(1-\frac{M_Z^2}{s})(1-\frac{M_Z^2}{s'})+\frac{M_Z^2\Gamma_Z^2}{ss'}\right]|\chi_s(s)|^2|\chi_s(s')|^2 \\ B_7(s,s') &= -4M_Z\Gamma_Z\left\{g_Ag_V[\frac{|\chi_s(s)|^2}{s}-\frac{|\chi_s(s')|^2}{s'}]\right. \\ &\quad \left.+2g_Vg_A(g_V^2+g_A^2)(\frac{1}{s}-\frac{1}{s'})|\chi_s(s)|^2|\chi_s(s')|^2\right\} \\ B_{ss}(s,t,u) &= \left[B_1(a_+^2,s)+B_1(a_-^2,s)\right](\frac{u}{s})^2+2B_1(c_-,s)(\frac{t}{s})^2 \\ B_{tt}(s,t,u) &= \left[B_1(a_+^2,t)+B_1(a_-^2,t)\right](\frac{u}{t})^2+2B_1(c_-,t)(\frac{s}{t})^2 \\ B_{st}(s,t,u) &= 4B_6(s,t)\frac{u^2}{st}\end{aligned}$$

The constants $c_-$, $a_\pm$ are defined in Appendix A and $g_V$, $g_A$ in Eqns 2.3-2.6. Note that the functions $\chi_s$, $\chi_t$ are assigned according to the arguments of $B_1$, $B_4$ ... . For example $B_4(s,t')$ is given by the replacement $\chi_s(s')\to\chi_t(t')$ in the expresssion above for $B_4(s,s')$.

Exponentiation of the initial state radiation is included via the function:

$$f(s) = 2C_V^i \exp\beta_e \ln(y) - 1 \qquad (B3)$$

where $\beta_e$, $C_V^i$ are defined after Eqn 2.1. This ensures that the $y\to 0$ limit of Eqn. $B1$ is identical to the derivative of Eqn 2.1 with respect to $y_0$, in which the replacement $y_0\to y$ is made. That is 'soft' and 'hard' photons are treated in a consistent way.

## Appendix C

Two dimensional look up tables are generated according to a generalisation of the procedure described in the text (Eqns 3.7-3.10) for a one dimensional look up table. Consider for example the case of the Born differential cross-section $d\sigma_0/dc$ for a range of different CM energies : $s_{min}<s<s_{max}$. Let $k$ be the bin index for $s$. The equations 2.7, 2.8 generalise to:

$$\sigma_0^k = \int_{c_{min}}^{c_{max}}\frac{d\sigma_0^k}{dc}dc \qquad (C1)$$

where

$$d\sigma_0^k \equiv d\sigma_0(s_k)$$

$$P_i^k = \frac{1}{\sigma_0^k} \int_{c_{min}}^{c_i} \frac{d\sigma_0^k}{dc} dc \qquad (C2)$$

For each value of $k$ the distribution $P_i^k$ is inverted by linear interpolation to yield a look up table with bin index $j$ :

$$c_i^j = f^k(P_i^k) \qquad (C3)$$

To generate the angular distribution for a given value of $s$, the adjacent bins in $s$ of index $k$, $k+1$ are first located :

$$s_k < s < s_{k+1}$$

The closest bins in the look up tables of index $k$, $k+1$ to the random number $Rn$ are the found:

$$P_i^k < Rn < P_{i+1}^k$$

$$P_{i'}^{k+1} < Rn < P_{i'+1}^{k+1}$$

With:

$$\delta_i = (Rn - P_i^k)/(P_{i+1}^k - P_i^k)$$

$$\delta_{i'} = (Rn - P_{i'}^{k+1})/(P_{i'+1}^{k+1} - P_{i'}^{k+1})$$

Two values of $c$ are now calculated according to :

$$c^k = f^k(P_i^k) + [f^k(P_{i+1}^k) - f^k(P_i^k)]\delta_i$$

$$c^{k+1} = f^{k+1}(P_{i'}^k) + [f^{k+1}(P_{i'+1}^{k+1}) - f^{k+1}(P_{i'}^{k+1})]\delta_{i'}$$

with

$$\Delta_k = (s - s_k)/(s_{k+1} - s_k)$$

the generated value of $c$ is, finally, given by the equation :

$$c = c^k + (c^{k+1} - c^k)\Delta_k \qquad (C4)$$

# References


[1] S.Glashow, Nucl. Phys. 22 (1961) 579.

S.Weinberg, Phys. Rev. Lett. 19 (1967) 1264.

A.Salam in : Elementary Particle Theory. Ed H.Svartholm (Almquist, Stockholm 1968)

[2] S.Jadach, B.F.L.Ward and Z.Wąs, Comp. Phys. Comm. 66 (1991) 276.

[3] M.Böhm, A.Denner and W.Hollik, Nucl. Phys. B304 (1988) 687.

[4] F.A.Berends, R.Kleiss and W.Hollik, Nucl. Phys. B304 (1988) 712.

F.A.Berends et al., Nucl. Phys. B206 (1982) 53.

[5] M.Caffo, H.Czyz and E.Remiddi, Il Nuovo Cimento Vol 105A No 2 (1992) 277, J. Mod. Phys. C4 (1993) 591.

[6] H.Anlauf et al., Comp. Phys. Comm. 79 (1994) 466.

[7] F.A.Berends and R.Kleiss, Nucl. Phys. B228 (1983) 537.

[8] S.Jadach et al., Phys. Lett. B268 (1991) 253, Comp. Phys. Comm. 70 (1992) 305.

[9] D.Bardin, W.Hollik and T.Riemann, Z. Phys. C Particles and Fields 49 (1991) 485.

[10] J.H.Field, Phys. Lett. B323 (1994) 432.

[11] D.Bardin et al., FORTRAN program ZFITTER, and CERN-TH. 66443/92 [hep-ph/9412201]; Z. Phys. C Particles and Fields 44 (1989) 493, Nucl. Phys. B351 (1991) 1, Phys. Lett. B255 (1991) 290.

[12] W.Beenakker, F.A. Berends and S.C. van der Marck, Nucl. Phys. B349 (1991) 323.

[13] M.Cacciari et al., Phys. Lett. B268 (1991) 441, B279 (1992) 384, B286 (1992) 387.

G.Montagna et al., Nucl. Phys. B401 (1993) 3.

G.Montagna et al., Comp. Physics Commun. 76 (1993) 328.

[14] F.A.Berends, R.Kleiss and S.Jadach, Comp. Phys. Comm. 29 (1983) 185.

[15] D.J. Summers, Phys. Lett. B302 (1993) 326.

[16] M.Martinez and R.Miquel, CERN pre-print CERN-PPE/92-11 (1992). Submitted to Phys. Lett. B.

[17] J.Fujimoto et al., Phys. Lett. B305 (1993) 189.

[18] D. Bardin et al., Computer Physics Commun. 59 (1990) 303.

[19] D. Bardin, P. Christova and O. Fedorenko, Nucl. Phys. B175 (1980) 235; B197 (1982) 1.

[20] A. Akhundov, D. Bardin and T. Riemann, Nucl. Phys. B276 (1986) 1.

[21] W. Hollik, Fortschritte d. Physik 38 (1990) 165.

[22] A. Djouadi and C. Verzegnassi, Phys. Lett. B195 (1987) 265;
A. Djouadi, Nuovo Cim. 100A (1988) 357. See also Ref[25] P95.

[23] D. Bardin and A. Chizhov, in: JINR Dubna E2–89–525 (1989) P42.

[24] F. Jegerlehner in Proc. of the 1990 TASI in Elem. Particle Physics, ed. by P. Langacker and M. Cvetic (World Scientific, Singapore, 1991) P476.

[25] D. Bardin, W. Hollik, G. Passarino (eds.) Reports on precision calculations of $Z$ resonance observables, CERN Yellow Report: CERN 95-03.

[26] A.Ballestrero, E.Maina and S.Moretti, Phys. Lett. B305 (1993) 312.

[27] H.Burkhardt, F.Jegerlehner, G.Penso and C.Verzegnassi, Z. Phys. C Particles and Fields 42 (1989) 697.

[28] F.A. Berends, W.L. van Neerven and G.J.H. Burgers, Nucl. Phys. B297 (1988) 429.

[29] F.Aversa and M.Greco, Phys. Lett. 228B (1989) 134.

[30] M.Greco, Rivista del Nuovo Cimento Vol 1 N5 (1988) 1.

[31] T.Kinoshita, J. Math. Phys. 3 (1962) 650.
T.D.Lee and M.Nauenberg, Phys. Rev. B133 (1964) 1549.

[32] S.Jadach and Z.Wąs, Phys. Lett. B219 (1989) 103.

[33] S.Jadach and B.F.L.Ward, Phys. Lett. B274 (1989) 103.

[34] J.H.Field and B.Richter in 'Physics with Very High Energy Colliding Beams' CERN Yellow Report CERN 76-18 Nov 1976 P95.

[35] F.Bloch and A.Nordsieck, Phys. Rev. 52 (1937) 54.

[36] F.James, Reports on Progress in Physics 43 (1980) 1145.

[37] V.N.Baier et al., Phys. Reports 78 (1981) 390.

[38] V.N.Gribov and L.N.Lipatov, Sov. J. Nucl. Phys. 15 (1972) 438.

[39] R.Budny, Phys. Lett. 55B (1975) 227.

| $n_\gamma^I$ | $n_\gamma^F$ | Assigned Weight |
|---|---|---|
| 2 | 0 | $\overline{W}'_I$ |
| 0 | 2 | $\overline{W}'_F$ |
| 0 | 3 | $\overline{W}'_I$ |
| 1 | 1 | $\sqrt{\overline{W}'_I \overline{W}'_F}$ |

Table 1: Weights assigned to multiphoton events

| $n_\gamma^I$ | $n_\gamma^F$ | P( $n_\gamma^I$ , $n_\gamma^F$ ) | ISG |
|---|---|---|---|
| 0 | 0 | $P_{V,S}$ | 0 |
| 1 | 0 | $P_I e^{-r_e}$ | 1 |
| 0 | 1 | $P_F e^{-r_f}$ | 2 |
| 2 | 0 | $P_I(1-e^{-r_e})\rho_{IF}$ | 3 |
| 0 | 2 | $P_F r_f e^{-r_f})\rho_{IF}$ | 4 |
| 0 | 3 | $P_F[1-(1+r_f)e^{-r_f}]\rho_{IF}$ | 6 |
| 1 | 1 | $[P_I P_F(1-e^{-r_e})(1-e^{-r_f})]^{\frac{1}{2}}\rho_{IF}$ | 5 |

Table 2: 'A priori' probabilities for different hard photon multiplicities

| | |
|---|---|
| OIMZ | Z mass (GeV) |
| OIMT | Top quark mass (GeV) |
| OIMH | Higgs boson mass (GeV) |
| OMAS | $\alpha_s(M_Z)$ |
| IOCH | $= 0(\mu^+\mu^-), = 1(e^+e^-)$ |
| IOEXP | $= 1$ exponentiated , $= 0$ $O(\alpha)$ |
| OW | collision energy (GeV) |
| OCTC1 | lower $\cos\theta_{l^+}$ in the ODLR frame |
| OCTC2 | lower $\cos\theta_{l^-}$ in the ODLR frame |
| IOXI | initial random number |

Table 3: Variables of the labelled common ICOM. OCTC1,OCTC2 are used in setting up the LUT of the lepton scattering angles. To allow for the effects of the Lorentz boost the angular range should be chosen somewhat wider than that defined by the cuts in the LAB system.

| | |
|---|---|
| NPAR(1) | <u>1</u>, 0 weak loop corrections ON, OFF |
| NPAR(2) | 2,<u>3</u> parameterisations of had. vac. pol. |
| NPAR(3) | <u>0</u>,1,2 two-loop $\alpha\alpha_s m_t^2$ correction |
| NPAR(4) | <u>1</u>,0 weak box diagrams ON, OFF |
| NPAR(6) | <u>1</u>, 0 two-loop terms $\propto m_t^4$ ON,OFF |
| XPAR(1) | initial lepton charge (-1.D0) |
| XPAR(2) | final lepton charge (-1.D0) |
| XPAR(3) | final lepton colour (1.D0) |
| XPAR(4) | final lepton mass (GeV) |
| XPAR(9) | QCD correction to $\Gamma_q^Z$ (non-$b$ quarks) |
| XPAR(10) | QCD correction to $\Gamma_b^Z$ |
| YMA | maximum value of $\sum E_\gamma/E_{beam}$ (0.99D0) |
| YMI | minimum value of $E_\gamma/E_{beam}$ (0.005D0) |
| WTMAX | maximum value of the event weight (2.0D) |

Table 4: Control parameters defined in SUBROUTINE BHAGENE3. Default values are underlined or given in parentheses.

## FIGURE CAPTIONS

**Fig 1** Flow Chart of the Initialisation Phase.

**Fig 2** Flow Chart of the Event Generation Phase.

**Fig 3** Flow Chart of the Termination Phase.

**Fig 4** A typical line-printer output from BHAGENE3.

**Fig 5** Distribution of weights for events with hard initial or final state photons.