GRAVITATIONAL PHASE OPERATOR AND COSMIC STRINGS

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Abstract

A quantum equivalence principle is formulated by means of a gravitational phase operator which is an element of the Poincare group. This is applied to the spinning cosmic string which suggests that it may contain gravitational torsion. A new exact solution of the Einstein- Cartan-Sciama-Kibble equations for the gravitational field with torsion is obtained everywhere for a cosmic string with uniform energy density, spin density and flux. A novel effect due to the quantized gravitational field of the cosmic string on the wave function of a particle outside the string is used to show that spacetime points are not meaningful in quantum gravity.

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0. INTRODUCTION: RELATIVIZING AND QUANTIZING GRAVITY

After the discovery of special relativity by Lorentz, Poincare, and Einstein, there was the problem of "relativizing gravity", analogous to the problem of "quantizing gravity" which exists today. It was clear that Newtonian gravity was incompatible with special relativity and it was necessary to replace it with a relativistic theory of gravity. While several attempts were made to do this, Einstein succeeded in constructing such a theory because he used i) the *geometrical* reformulation of *special relativity* by Minkowski, and ii) the *operational* approach of asking what may be learned by probing gravity using *classical* particles.

An important ingredient in (i) was Einstein's realization that the times in the different inertial frames, t and t', in the Lorentz transformation were on the same footing. I.e. the *interpretation* Einstein gave to special relativity, whose basic equations were already known to Lorentz and Poincare, was crucial to the subsequent work of Minkwoski. It enabled Einstein to get rid of the three dimensional ether, and thereby pave the way for the introduction of the four dimensional 'ether', called space-time, by Minkowski. By means of (ii), Einstein concluded that the aspect of Newtonian gravity which should be retained when this theory is modified is the equivalence principle. This principle is compatible with special relativity locally. This may be seen from the physical formulation of the strong equivalence principle according to which in the Einstein elevator that is freely falling in a gravitational field the laws of special relativity are approximately valid. But this principle allowed for the modification of special relativity to incorporate gravity as curvature of space-time.

Today we find that general relativity, the beautiful theory of gravity which Einstein discovered in this way, is incompatible with quantum theory. Can we then adopt a similar approach? This would mean that we should use 1) a *geometrical* reformulation of *quantum theory*, and 2) an *operational* approach of asking what may be learned by probing gravity using *quantum* particles.

As for (1), the possibility of using group elements as 'distances' in quantum theory, analogous to space-time distances in classical physics, was studied previously [1]. For a particular quantum system, the corresponding representations of these group elements may be used to relate points of the projective Hilbert space, i.e. the set of rays of the Hilbert space, which is the quantum generalization of the classical phase space [2]. Recent work on protective observation of the quantum state has shown that the points of the projective Hilbert space are real, in the sense that they could be observed by measurements on an individual system, instead of using an ensemble of identical systems [3].

As for (2), the question is whether the motion of a quantum system in a gravitational field enables us to identify the aspect of general relativity which must be preserved when this theory is replaced by a quantum theory of gravity, i.e. the quantum analog of the equivalence principle. In section 1, I shall formulate such a principle. This will be applied to cosmic strings, in section 2, because of their interesting topological, geometrical, and quantum gravitational aspects. I shall present an exact solution of the Einstein-Cartan-Sciama-Kibble gravitational field equations, valid in the interior as well as the exterior of the cosmic string, which depends on three parameters.

It is shown in section 3 that when the gravitational field of the string is quantized so that different geometries may be superposed, the wave function of a test particle even in a simply connected region is affected although each of the superposed geometries is flat in this region. But a special case of this effect is invariant under a quantum diffeomorphism that transforms different geometries differently, as discussed in section 4. This freedom suggests that the points of space-time have no invariant meaning. So, there seems to be a need to get rid of the four dimensional 'ether', namely space-time, in order to incorporate the quantum diffeomorphism symmetry into quantum gravity.

1. THE EQUIVALENCE PRINCIPLE IN CLASSICAL AND QUANTUM PHYSICS

First, consider the classical weak equivalence principle (WEP), due to Galileo and

Einstein. This has two aspects to it: In a space-time manifold with a pure gravitational field, a) the possible motions of all freely falling test particles are the same, and b) at any point p in space-time, there exists a neighborhood $U(p)$ of p and a coordinate system $\{x^\mu, \mu = 0, 1, 2, 3\}$, such that the trajectories of *every* freely falling test particle through p satisfies [4]

$$\frac{d^2x^\mu}{d\lambda^2} = 0, \tag{1.1}$$

for a suitable parameter λ along the trajectory. This is the local form of the law of inertia and the above coordinate system is said to be locally inertial at p. The condition (b) is a special property of the gravitational field, not shared by any other field. For example, in an electromagnetic field test particles with the same charge to mass ratio would satisfy (a) but not (b). (The Lorentz 4-force is proportional to the electromagnetic field strength which, being a tensor, cannot be coordinate transformed away unlike the connection coefficients.)

It may be shown using (b) that there exists an affine connection ω such that the trajectories of freely falling test particles are affinely parametrized geodesics with respect to it. Suppose $\epsilon = \frac{d}{L}$, where $d \sim$ linear dimensions of $U(p)$ and $L \sim$ radius of curvature obtained from the curvature components of this connection, all lengths being measured in the above coordinate system, and we can neglect second orders in ϵ. Such a neighborhood will be called a first order infinitesimal neighborhood of p, and denoted by $U_\epsilon(p)$. Using the geodesic deviation equation, it may be shown that the velocities of the freely falling test particles in $U_\epsilon(p)$ are constant in an appropriately chosen coordinate system. This is a stronger form of the WEP than its usual statement given above, and will be called the modified classical weak equivalence principle. It is valid in Newtonian gravity as well as Einsteinian gravity.

The above formulations of WEPs require for their statements only an affine connection and no metric. Indeed, in U_ϵ the freely falling particle trajectories have as their symmetry group the affine group $A(4)$ that is generated by the general linear transformations and translations in a 4 dimensional real vector space. In classical physics, the *interactions*

between the particles restrict the symmetry group in U_ϵ to the inhomogeneous Galilei group (non relativistic physics), or the Poincare group P (relativistic physics), which are both subgroups of $A(4)$. This is a form of the classical strong equivalence principle (SEP) valid for relativistic and non relativistic gravity. In this way, non flat space-time geometry may also in some sense be brought into the frame-work of Felix Klein's Erlanger program according to which a geometry is determined as the set of properties invariant under a symmetry group [1].

What fundamental aspects about the gravitational field may be learned if it is probed with quantum particles, instead of with classical particles as in the above treatment? It was shown that the evolution of a freely falling wave function is given, in the WKB approximation, by the action on the initial wave function by the operator [5]

$$\Phi_\gamma = Pexp[-i\int_\gamma \Gamma_\mu dx^\mu], \tag{1.2}$$

where

$$\Gamma_\mu = \theta_\mu{}^a P_a + \frac{1}{2}\omega_\mu{}^a{}_b M^b{}_a. \tag{1.3}$$

which will be called the gravitational phase operator. Here the energy-momentum operators P_a and the angular momentum operators $M^b{}_a, a, b = 0, 1, 2, 3$ generate the covering group of the Poincare group $\tilde{P}$ that is a semi-direct product of $SL(2, C)$ and space-time translations $R(4)$. The fact that mass m is a good quantum number in curved space-time and m^2 is a Casimir operator of P already suggests that P is relevant in the presence of gravity.

For every space-time point p, let $H_\epsilon(p)$ be the Hilbert space of wave functions in $U_\epsilon(p)$ in which $\tilde{P}$ acts. Owing to the linearity of the action of (1.2), it determines also the evolution of any freely falling wave packet which can be expanded as a linear combination of WKB wave functions, provided the size of the wave packet is small compared to the radius of curvature, i. e. it is contained primarily inside U_ϵ at each point along γ which may be chosen to be along the center of the wave packet. This will be called the quantum

weak equivalence principle, because (1.2) is a Poincare group element independent of the freely falling wave packet. In this respect, it is like the classical WEP according to which the affine connection determined is independent of the test particle used.

In quantum physics, because the wave packet must necessarily have some spread, the WEP cannot be formulated by particle trajectories as in conditions (a) and (b) above, and it is necessary to use at least the neighborhood U_ϵ. Indeed (1.2) was obtained [5] using the Klein-Gordon [6] and Dirac equations [7] which are covariant under $\tilde{P}$ in U_ϵ. So, in quantum physics there is a close connection between the WEP, as formulated above, and SEP according to which $\tilde{P}$ is the symmetry group of all laws of physics in U_ϵ. It is well known that (a) cannot be valid in quantum physics, because the motions of wave functions depend on their masses [8]. But the modified classical WEP and the classical SEP as stated above have the advantage that they have a smooth transition to quantum physics.

The above approximate concepts may be made mathematically precise as follows: Each neighborhood $U_\epsilon(p)$ may be identified with the tangent space at p regarded as an affine space. The motions of freely falling test particles relate affine spaces associated with two neighboring points by a linear transformation and a translation, generated by P_a. This gives a natural connection on the affine bundle[9] over spacetime which is a principal fiber bundle with $A(4)$ as the structure group. This is the connection used above to express the modified classical WEP. The quantum WEP requires the Poincare subbundle with $\tilde{P}$ (to admit Fermions) as the structure group. Then (1.3) defines a connection in this principal fiber bundle. The gravitational phase operator (1.2) parallel transports with respect to this connection along the curve γ. The above Hilbert space bundle, that is the union of $H_\epsilon(p)$ for all space-time points p, is a vector bundle associated to this principal fiber bundle with a connection that is the representation (1.3) in this Hilbert space.

The curvature of the above connection is the Poincare Lie algebra valued 2-form

$$F = d\Gamma + \Gamma \wedge \Gamma = Q^a P_a + \frac{1}{2} R^a{}_b M^b{}_a, \tag{1.4}$$

where, on using (1.3) and the Lie algebra of the Poincare group,

$$Q^a = d\theta^a + \omega^a{}_b \wedge \theta^b, R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{1.5}$$

which are called respectively the torsion and the linear curvature. If the wave equation used to obtain (1.2) did not contain torsion, then the torsion in (1.4), of course, is also zero. However, the above modified classical WEP and the quantum WEP make it natural to have torsion and suggest that if the torsion is zero then there should be a good physical reason for it.

Suppose γ is a closed curve. Then (1.2) is a holonomy transformation determined by the above affine connection. It may then be transformed to an appropriate integral over a 2-surface Σ spanned by γ as follows. Let O be a fixed point in Σ. Foliate Σ by a 1-parameter family of curves $\lambda(s,t)$, where $s\epsilon[0,1]$ labels the curves and $t\epsilon[0,1]$ is the parameter along each curve. All curves originate at O, which corresponds to $t = 0$. The (s,t) are smooth coordinates on Σ excluding O. Suppose γ begins and ends at $(0,1)$. Let $\Lambda(s,t) = \Phi_{\lambda(s,t)}$. Then

$$\Lambda^{-1}(0,1)\Phi_\gamma\Lambda(0,1) = P_{st}exp[-i\int_0^1 ds\int_0^1 dt\Lambda^{-1}(s,t)F_{\mu\nu}(s,t)\Lambda(s,t)\ell^\mu m^\nu], \tag{1.6}$$

where $\ell^\mu = \frac{\partial x^\mu}{\partial s}, m^\mu = \frac{\partial x^\mu}{\partial t}$ and P_{st} means surface ordering, i. e. in the expansion of (1.6) terms with greater value of s precede terms with smaller value of s, and for equal values of s terms with greater value of t precede terms with smaller value of t. In (1.6) all field variables are transported to the common point O so that the integrals are meaningfully performed in the affine space at O.

To prove (1.6), note that the LHS of (1.6) is a holonomy transformation which begins and ends at O, and may be expressed as a product of holonomy transformations Φ_s over triangles whose sides are two $s = constant$ curves and an infinitesimal segment of γ. Each Φ_s may be written as a product of Φ_{st} over infinitesimal "rectangles", bounded by $s =$ const. , $t =$ const. curves, which are transported to O, which yields (1.6). This extends a known result for Yang-Mills field and linear curvature [10] to include torsion.

It follows from (1.6) that in the absence of gravity in a simply connected region (1.2) is path independent. I shall take the equivalent statement that the path dependence of (1.2) implies gravity as the *definition* of the gravitational field even when the region is not simply connected. This definition makes the converse of this statement also valid. So, by probing gravity using quantum mechanical systems, without paying any attention to gauge fields, gravity may be obtained naturally as a Poincare gauge field in the sense of Yang's integral formulation of gauge field [11].

An advantage of this point of view is that it also provides a unified description of gravity and gauge fields. If a wave function is interacting not only with the gravitational field but also other gauge fields, then its propagation in the WKB approximation is given by the action of an operator of the form (1.2) with

$$\Gamma_\mu = \theta_\mu{}^a P_a + \frac{1}{2}\omega_\mu{}^a{}_b M^b{}_a + A_\mu{}^j T_j, \tag{1.7}$$

where $A_\mu{}^j$ is the Yang-Mills vector potential and T_j generate the gauge group G. So, (1.2) now is an element of the entire symmetry group, namely $\tilde{P} \times G$. Thus, unlike the classical WEP, the quantum WEP naturally extends to incorporate all gauge fields.

The above fact that the observation of all the fundamental interactions in nature is via elements of the symmetry group suggest a symmetry ontology. By this I mean that the elements of symmetry group are observable and therefore real. Moreover, the observables such as energy, momentum, angular momentum and charge, which are usually observed in quantum theory are some of the generators of the above symmetry group. Observation always requires interaction between the observed system and the apparatus. Ultimately, these interactions are mediated by gravity and gauge fields. I therefore postulate that the only observables which can actually be observed are formed from the generators of symmetry group, which according to our current understanding of physics are generators of $\tilde{P} \times G$. Symmetry is destiny.

2. COSMIC STRING - AN EXACT SOLUTION

As an example, consider cosmic strings, which are predicted by gauge theories [12] and are of astrophysical interest because of their possible role in galaxy formation [13] and as gravitational lenses [14,15]. Consider a cosmic string whose axis is along the $z-$axis. Since the torsion and curvature outside the string are zero, its exterior geometry is determined entirely by the affine holonomy transformation associated with a closed curve γ going around the string, given by (1.2). But owing to the cylindrical symmetry of this geometry, this transformation should commute with ${M^2}_1$ which generates rotations about the axis of the string. The most general affine holonomy transformation which commutes with ${M^2}_1$ is of the form

$$\Phi_\gamma = \exp[-i(bP_o + cP_3 + a{M^2}_1 + d{M^3}_0)]. \tag{2.1}$$

Therefore, the most general external geometry should depend on the four parameters a, b, c and d. This geometry has been obtained, from the point of view of affine holonomy, by Tod [16] although the present argument which uses the gravitational phase operator (1.2) is somewhat simpler and more physical.

I shall consider here only the most general *stationary* exterior solution which depends only on three parameters ($d = 0$):

$$ds^2 = (dt + \beta d\phi)^2 - d\rho^2 - \alpha^2\rho^2 d\phi^2 - (dz + \gamma d\phi)^2, \tag{2.2}$$

where α, β and γ are constants related to a, b and c respectively. The special case of $\gamma = 0$ was considered by Deser et al [17] and Mazur [18]. It is worth noting that the usual linear holonomy around the cosmic string can only determine the parameter α. Whereas the translational part of the affine holonomy distinguishes metrics (2.2) with different values for (β, γ) [16], which shows the importance of affine holonomy. It follows from (1.4) and (1.5) that the rotational part of the affine holonomy, due to α, requires curvature inside the string. The translational part of the affine holonomy, due to β and γ, suggests (but does not require) the inclusion of torsion inside the string.

With a view towards this, rewrite (2.2) as $ds^2 = \eta_{ab}\theta^a\theta^b$, where the orthonormal co-frame field θ^a is

$$\theta^0 = dt + \beta d\phi, \theta^1 = d\rho, \theta^2 = \alpha\rho d\phi, \theta^3 = dz + \gamma d\phi. \tag{2.3}$$

Let e_a be the frame (vierbein) dual to θ^a: $\theta_\mu{}^b e^\mu{}_a = \delta^b_a$. The connection coefficients in this basis are $\omega_\mu{}^a{}_b \equiv \theta_\nu{}^a \nabla_\mu e^\nu{}_b = 0$, for all a, b, μ except for

$$\omega_{\hat{\phi}}{}^1{}_2 = -\omega_{\hat{\phi}}{}^2{}_1 = -\alpha, \tag{2.4}$$

assuming no torsion in the exterior. This external geometry is *affine flat*, i.e. $Q^a = 0, R^a{}_b = 0$ on using (1.5), and yet the *affine holonomy around the string is non trivial* [19].

Suppose γ is a closed curve around the string. Then from (1.2),

$$\begin{aligned}\Phi_\gamma = exp\left(-i\oint_\gamma \theta_\mu{}^0 P_0 dx^\mu\right) exp\left(-i\oint_\gamma \theta_\mu{}^3 P_3 dx^\mu\right) \\ \times Pexp\left[-i\oint_\gamma \left(\sum_{k=1}^{2} \theta_\mu{}^k P_k + \omega_\mu{}^1{}_2 M^2{}_1\right) dx^\mu\right].\end{aligned} \tag{2.5}$$

The three factors in (2.5) commute with one another. On comparing with (2.1) and using (2.3), $b = 2\pi\beta$ and $c = 2\pi\gamma$. The first factor in (2.5), which is a time translation, may be given a physical meaning as follows: Suppose an optical, neutron or superconducting interferometer encloses the string once and is at rest with respect to the above coordinate system. Then the above time translation gives rise to a "Sagnac" phase shift [6,20], which in the present case is $\Delta\phi_E = 2\pi\beta E$, where E is the frequency of the interfering particle (eigenvalue of P_0).

The second factor in (2.5), which is a spatial translation, may be given physical meaning by the following new effect: Suppose the beam at the beam splitter of the above interferometer has a $z-$ component of momentum p. I.e. p is the approximate eigenvalue of P_3. Then, this factor gives rise to the phase shift $\Delta\phi_p = 2\pi\gamma p$.

If in (1.6), coordinates and basis can be chosen such that $\Lambda(s,t) \simeq 1$ for all s, t, then Σ will be called infinitesimal. It follows from (1.6), (1.4) and (2.5) that when the

cross- section of the string is infinitesimal in this sense, it must necessarily contain torsion in order that the surface integral has the time translation contained in the line integral. Then $\Delta\phi$ may be regarded as a topological phase shift due to the enclosed torsion inside the string. It is possible for the string not to contain torsion, but only by violating the above infinitesimality assumption.

The simplest gravitational field equations in the presence of torsion are the Einstein-Cartan- Sciama- Kibble (ECSK) equations [21], which may be written in the form [22]

$$\frac{1}{2}\eta_{ijkl}\theta^l \wedge R^{jk} = -8\pi G t_i, \tag{2.6}$$

$$\eta_{ijkl}\theta^l \wedge Q^k = 8\pi G s_{ij}, \tag{2.7}$$

where t_i and s_{ij} are 3-form fields representing the energy-momentum and spin densities. I shall now obtain an exact solution of these equations for the interior of the cosmic string which matches the exterior solution (2.2). This will then give physical and geometrical meaning to the parameters α and β in (2.2). This solution will be different from earlier torsion string solutions [23] which have static interior metrics matched with exterior metrics which are different from (2.2).

The ρ and z coordinates in the interior will be chosen to be the distances measured by the metric in these directions. Since the exterior solution has symmetries in the t, ϕ, and z directions, it is reasonable to suppose the same for the interior solution. So, all functions in the interior will be functions of ρ only. So, I make the following ansatz in the interior:

$$\theta^0 = u(\rho)dt + v(\rho)d\phi, \theta^1 = d\rho, \theta^2 = f(\rho)d\phi, \theta^3 = dz + g(\rho)d\phi, \omega^2{}_1 = k(\rho)d\phi = -\omega^1{}_2, \tag{2.8}$$

all other components of $\omega^a{}_b$ being zero, and $ds^2 = \eta_{ab}\theta^a\theta^b \equiv g_{\mu\nu}dx^\mu dx^\nu$. Suppose also that there is a fluid in the interior whose energy density ϵ and spin density σ polarized in the z-direction are constant, and this spin has a constant current density τ in the z-direction. I. e.

$$t_0 = \epsilon\theta^1 \wedge \theta^2 \wedge \theta^3 = \epsilon f(\rho)d\rho \wedge d\phi \wedge dz,$$

$$s_{12} = -s_{21} = \sigma\theta^1 \wedge \theta^2 \wedge \theta^3 - \tau\theta^0 \wedge \theta^1 \wedge \theta^2$$
$$= \sigma f(\rho) d\rho \wedge d\phi \wedge dz - \tau u f(\rho) dt \wedge d\rho \wedge d\phi, \tag{2.9}$$

the other components of s_{ij} being zero. In terms of the components of the energy-momentum and spin tensors in the present basis, this means that $t^0{}_0 = \epsilon =$ constant and $s^0{}_{12} = \sigma =$ constant.

It is assumed that there is no surface energy-momentum or spin for the string. Then the metric must satisfy the junction conditions [24], which in the present case are

$$g_{\mu\nu}|_- = g_{\mu\nu}|_+, \partial_{\hat{\rho}} g_{\mu\nu}|_+ = \partial_{\hat{\rho}} g_{\mu\nu}|_- + 2K_{(\mu\nu)\hat{\rho}}, \tag{2.10}$$

where $K_{\alpha\beta\gamma} = \frac{1}{2}(-Q_{\alpha\beta\gamma} + Q_{\beta\gamma\alpha} - Q_{\gamma\alpha\beta})$ is the contorsion or the defect tensor, $|_+$ and $|_-$ refer to the limiting values as the boundary of the string is approached from outside and inside the string, respectively, and the hat denotes the corresponding coordinate component.

Substitute (2.8), (2.9) into the Cartan equations (2.7). The $(i,j) = (0,2), (0,3), (2,3)$ eqs. are automatically satisfied. The $(i,j) = (0,1), (1,3), (1,2)$ eqs. yield

$$f'(\rho) = k(\rho), u'(\rho) = 0, v'(\rho) = 8\pi G\sigma f(\rho), g'(\rho) = 8\pi G\tau f(\rho), \tag{2.11}$$

where the prime denotes differentiation with respect to ρ. Therefore, the continuity of the metric (eq. (2.10)) implies that since $u = 1$ at the boundary, $u(\rho) = 1$ everywhere. Now substitute (2.8), (2.9) into the Einstein equations (2.6). The $i = 0$ eq. yields

$$k'(\rho) = -8\pi G\epsilon f(\rho). \tag{2.12}$$

The $i = 1, 2, 3$ equations yield, respectively

$$t_1 = 0, t_2 = 0, t_3 = \frac{k'}{8\pi G} dt \wedge d\rho \wedge d\phi = -\epsilon\theta^0 \wedge \theta^1 \wedge \theta^2, \tag{2.13}$$

using (2.12). Hence, $t^3{}_3 = \epsilon = t^0{}_0$. From (2.11) and (2.12),

$$f''(\rho) + \frac{1}{\rho^{*2}} f(\rho) = 0, \tag{2.14}$$

where $\rho* = (8\pi G\epsilon)^{-1/2}$. In order for there not to be a metrical "cone" singularity at $\rho = 0$, it is necessary that $\theta^2 \sim \rho d\phi$ near $\rho = 0$. Hence, the solution of (2.14) is $f(\rho) = \rho * sin\frac{\rho}{\rho*}$. Then from (2.11), $k(\rho) = cos\frac{\rho}{\rho*}$, and requiring $v(0) = 0 = g(0)$ to avoid a conical singularity, $v(\rho) = 8\pi G\sigma\rho *^2 \left(1 - cos\frac{\rho}{\rho*}\right)$, and $g(\rho) = 8\pi G\tau\rho *^2 \left(1 - cos\frac{\rho}{\rho*}\right)$.

This gives the metric in the interior of the string to be

$$ds^2 = \left[dt + 8\pi G\sigma\rho *^2 \left(1 - cos\frac{\rho}{\rho*}\right) d\phi\right]^2 - d\rho^2 - \rho *^2 sin^2 \left(\frac{\rho}{\rho*}\right) d\phi^2$$
$$- \left[dz + 8\pi G\tau\rho *^2 \left(1 - cos\frac{\rho}{\rho*}\right) d\phi\right]^2, \tag{2.15}$$

and the connection is $\omega^2{}_1 = cos\frac{\rho}{\rho*} d\phi$. The only non vanishing components of torsion and curvature are

$$Q^0 = 8\pi G\sigma\rho * sin\left(\frac{\rho}{\rho*}\right) d\rho \wedge d\phi, Q^3 = 8\pi G\tau\rho * sin\left(\frac{\rho}{\rho*}\right) d\rho \wedge d\phi,$$
$$R^1{}_2 = \frac{1}{\rho*} sin\left(\frac{\rho}{\rho*}\right) d\rho \wedge d\phi = -R^2{}_1. \tag{2.16}$$

I apply now the junction conditions (2.10), which will show that ρ is discontinuous across the boundary. Denote the values of ρ for the boundary in the internal and external coordinate systems by ρ_- and ρ_+ respectively. From (2.1) and (2.15), $g_{\hat{t}\hat{\phi}}$,$g_{\hat{z}\hat{\phi}}$, and $g_{\hat{\phi}\hat{\phi}}$ are respectively continuous iff

$$\beta = 8\pi G\sigma\rho *^2 \left(1 - cos\frac{\rho_-}{\rho*}\right), \tag{2.17}$$

$$\gamma = 8\pi G\tau\rho *^2 \left(1 - cos\frac{\rho_-}{\rho*}\right), \tag{2.18}$$

$$\alpha\rho_+ = \rho * sin\frac{\rho_-}{\rho*}. \tag{2.19}$$

The remaining metric coefficients are clearly continuous. The only non zero contorsion terms which enter into (2.10) are obtained from (2.16) to be

$$K_{(\hat{\phi}\hat{t})\hat{\rho}} = -4\pi G\sigma\rho * sin\frac{\rho}{\rho*}, K_{(\hat{\phi}\hat{z})\hat{\rho}} = 4\pi G\tau\rho * sin\frac{\rho}{\rho*},$$
$$K_{\hat{\phi}\hat{\phi}\hat{\rho}} = (8\pi G)^2 \left(\tau^2 - \sigma^2\right) \rho *^3 \left(1 - cos\frac{\rho}{\rho*}\right) sin\frac{\rho}{\rho*}. \tag{2.20}$$

Using (2.19) and(2.20), it can now be verified that the remaining junction conditions (2.10) are satisfied provided $\alpha = cos\frac{\rho_-}{\rho*}$. The mass per unit length is

$$\mu \equiv \int_\Sigma \epsilon\theta^1 \wedge \theta^2 = \frac{1}{4G}\left(1 - cos\frac{\rho_-}{\rho*}\right) = \frac{1}{8\pi G}\int_\Sigma R^1{}_2, \tag{2.21}$$

where Σ is a cross-section of the string (constant t, z). Therefore, $\alpha = 1 - 4G\mu$. The angular momentum per unit length due to the spin density is

$$J \equiv \int_\Sigma \sigma\theta^1 \wedge \theta^2 = 2\pi\sigma\rho\,*^2\left(1 - cos\frac{\rho_-}{\rho*}\right) = \frac{1}{8\pi G}\int_\Sigma Q^0. \tag{2.22}$$

Hence, from (2.17), $\beta = 4GJ$. The angular momentum flux, which is along the $z-$axis, is

$$F \equiv \int_\Sigma \tau\theta^1 \wedge \theta^2 = 2\pi\tau\rho\,*^2\left(1 - cos\frac{\rho_-}{\rho*}\right) = \frac{1}{8\pi G}\int_\Sigma Q^3. \tag{2.23}$$

Hence, from (2.18), $\gamma = 4GF$.

The Sagnac phase shift and the new phase shift obtained earlier are therefore $\Delta\phi_E = ET^0$, and $\Delta\phi_p = pT^3$, where T^0 and T^3 are the fluxes of Q^0 and Q^3 through Σ. These are both topological phase shifts, analogous to the Aharonov-Bohm effect with the string playing the role of the solenoid, in that they are invariant as the curve γ is deformed so long as it is outside the string.

It was recently pointed out to me that the special case of the above solution corresponding to $\gamma = 0 = \tau$ was found by Soleng [25]. If torsion is absent, which in the ECSK theory means that spin density is zero, then $\beta = 0 = \gamma$, and the above solution reduces to the exact static solution of Einstein's theory found by Gott [15] and others [26], whose linearized limit was previously found by Vilenkin [14].

3. INTERACTION OF A QUANTUM COSMIC STRING WITH A QUANTUM PARTICLE

Suppose now that the cosmic string is treated quantum mechanically. Then its gravitational field also should be treated quantum mechanically. It is then possible to form

a quantum superposition of the gravitational fields corresponding to different values of (α, β, γ) of the solution obtained above.

It was shown [27] that the following new physical effect is obtained when the cosmic string is in a superposition of quantum states corresponding to different values of β: A measurement on a quantum cosmic string that puts it in this superposition of geometries would change the intensity of the wave function of a particle in a *simply connected* region near the cosmic string, even though each of the superposed flat geometries in this region has no effect on the wave function. This is unlike the Aharonov-Bohm effect in which the wave function needs to go all the way around the multiply connected region surrounding the solenoid in order to be affected by the solenoid.

I shall now treat this effect using the variables θ^a and $\omega^a{}_b$, and generalize this effect further. Owing to the translational symmetry along the direction of the string, its gravitational field is equivalent to that of a point particle in $2+1$ dimensional gravity. Using the latter variables, Witten [28] has constructed a quantum theory of $2+1$ dimensional gravity which is finite. The effect which will be treated now therefore will also provide physical meaning to $2+1$ dimensional quantum gravity.

In the gravitational phase operator (1.2), θ^a and $\omega^a{}_b$ are now operators owing to the fact that the gravitational field they represent is quantized. Using (2.3) and (2.4), (1.2) may be written in terms of α, β and γ, which are also operators, as the product of two commuting exponentials:

$$\begin{aligned}\Phi_\gamma = \exp\left[-i\int_\gamma (dtP_0 + \beta d\phi P_0 + dzP_3 + \gamma d\phi P_3)\right] \\ \times P\exp\left[-i\int_\gamma \left(d\rho P_1 + \alpha\rho d\phi P_2 - \alpha d\phi M^2{}_1\right)\right].\end{aligned} \tag{3.1}$$

From the end of section 2 it follows that

$$\alpha = 1 - 4G\hat{\mu}, \beta = 4G\hat{J}, \gamma = 4G\hat{F}, \tag{3.2}$$

where $\hat{\mu}$, $\hat{J}$ and $\hat{F}$ are the quantum mechanical operators corresponding to the mass, angular

momentum and angular momentum flux per unit length of the string. The latter operators are assumed to commute with one another.

Suppose that the quantum state of the cosmic string is initially in the superposition

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle), \tag{3.3}$$

where $|\psi_1\rangle$ and $|\psi_2\rangle$ are normalized eigenstates of α, β and γ with the same eigenvalue for α and the other eigenvalues being (β_1, γ_1) and (β_2, γ_2) respectively. According to (3.2), these different values of (β, γ) correspond to different eigenstates of $\hat{J}$ and $\hat{F}$, respectively, of the fluid that the string is made of. So, the superposition (3.3) may be obtained by putting the quantum mechanical particles which constitute this fluid in the corresponding superposition by, say, letting them interact with another quantum mechanical system. Suppose also that a test particle outside the string is approximately an eigenstate of its energy P_0 and momentum in the z-direction P_3, with eigenvalues E and p respectively. This is possible because the last two operators commute with each other due to the symmetry of the gravitational field of the string in the z-direction.

The test particle is initially far away from the string in the normalized state $|\zeta_0\rangle$ and is slowly brought towards the string without changing E or p. Suppose the interaction of $|\zeta_0\rangle$ with $|\psi_1\rangle$ and $|\psi_2\rangle$ changes the state of the combined system to $|\psi_1\rangle|\zeta_1\rangle$ and $|\psi_2\rangle|\zeta_2\rangle$ respectively. Then by the linearity of quantum mechanics, the interaction of $|\zeta_0\rangle$ with $|\psi_0\rangle$ gives rise to the entangled state for the combined system

$$|\chi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle|\zeta_1\rangle + |\psi_2\rangle|\zeta_2\rangle). \tag{3.4}$$

Now a measurement is made on the string and it is found to be in the superposition

$$|\psi\rangle = a|\psi_1\rangle + b\psi_2\rangle,$$

where $|a|^2 + |b|^2 = 1$. The corresponding state of the test particle is

$$|\zeta\rangle = \langle\psi|\chi\rangle = \frac{1}{\sqrt{2}}(a^*|\zeta_1\rangle + b^*|\zeta_2\rangle) \tag{3.5}$$

The wave function corresponding to this state is to a good approximation

$$\zeta(\mathbf{x},t) = \frac{1}{\sqrt{2}}\left[a^* \exp\{-i(\beta_1 E + \gamma_1 p)(\phi-\phi_0)\} + b^* \exp\{-i(\beta_2 E + \gamma_2 p)(\phi-\phi_0)\}\right]\zeta_0'(\mathbf{x},t). \tag{3.6}$$

To obtain (3.6), one may solve the wave equation for the interaction of the test particle with the cosmic string in each of the states $|\psi_1\rangle$ and $|\psi_2\rangle$ and superpose the two solutions, or one may act on the state of the combined system by (3.1). Then $\zeta_0'(\mathbf{x},t)$ is seen to be the result of the action on ζ_0 of the part of (3.1) that does not depend on β and γ and is therefore the same for both of the superposed states. The constant ϕ_0 depends on the phase difference between these states, which in turn depends on the details of the interaction.

The intensity is

$$\zeta^*\zeta(\mathbf{x},t) = \left(\frac{1}{2} + |ab|\cos[\{(\beta_1-\beta_2)E + (\gamma_1-\gamma_2)p\}(\phi-\phi_0)+\delta]\right)|\zeta_0'(\mathbf{x},t)|^2. \tag{3.7}$$

It follows that the intensity would oscillate as a function of ϕ. The number of oscillations per unit angular distance ϕ is

$$\nu = \frac{1}{2\pi}\{(\beta_1-\beta_2)E + (\gamma_1-\gamma_2)p\} = \frac{2G}{\pi}\{(J_1-J_2)E + (F_1-F_2)p\}, \tag{3.8}$$

on using (3.2). This effect may be regarded geometrically as being due to the difference between two affine connections, which is a tensor field. This explains why this effect may occur for a wave function that is in a simply connected region outside the string. Because unlike each affine connection which has zero curvature, and can therefore have physical influence only through its non trivial holonomy around the string, the above tensor field may have local influences.

4. QUANTUM GENERAL COVARIANCE AND SPACE-TIME POINTS

In general, if there is a quantum superposition of gravitational fields, by a quantum diffeomorphism, or simply a q- diffeomorphism, I mean performing different diffeomorphisms

on the superposed gravitational fields. Then the physical effect described in section 3 may be shown to be invariant under a particular q-diffeomorphism performed on the quantized gravitational field when $\gamma = 0$ [27,29]. I postulate that all physical effects are invariant under all q-diffeomorphisms. This suggests a generalization of the usual principle of general covariance for the classical gravitational field to the following *principle of quantum general covariance* in quantum gravity: The laws of physics should be covariant under q-diffeomorphisms.

On the other hand, the usual principle of general covariance requires covariance of the laws of physics under classical diffeomorphisms, or c-diffeomorphisms. A c-diffeomorphism is a diffeomorphism that is the same for all the superposed gravitational fields, and is thus a special case of a q-diffeomorphism. Therefore, the above principle of quantum general covariance generalizes the usual general covariance due to Einstein. Under a c-diffeomorphism, a given space-time point is mapped to the same space-time point for all of the geometries corresponding to the superposed gravitational fields. This is consistent with regarding the space-time manifold as real, i.e. a four dimensional ether.

It is instructive in this context to examine Einstein's resolution of the hole argument [30]: In 1913, Einstein and Grossmann [31] considered the determination of the gravitational field inside a hole in some known matter distribution by solving the gravitational field equations. If these field equations are generally covariant, then there are an infinite number of solutions inside the hole, which are isometrically related by diffeomorphisms. These geometries, which I shall call Einstein copies, may however be regarded as different representations of the same objective physical geometry. This follows if a space-time point inside the hole is defined operationally as the intersection of the world-lines of two material particles, or geometrically by the distances along geodesics joining this point to material points on the boundary of the hole. Under a c-diffeomorphism, such a point in one Einstein copy is mapped to a unique point in another Einstein copy. Both points may then be regarded as different representations of the same physical space-time point or an

event. So, if we restrict to just c-diffeomorphism freedom, space-time may be regarded as objective and real.

But the space-time points associated with each of the superposed gravitational fields, which are defined above in a c-diffeomorphism invariant manner, transform differently under a q-diffeomorphism. This means that in quantum gravity space-time points have no invariant meaning. However, protective observation suggests that quantum states are real [3]. Consequently, the space-time manifold, which appears to be redundant, may be discarded, and we may deal directly with the quantum states of the gravitational field. But then the curve γ in the gravitational phase operator (1.2) cannot be meaningfully defined as a curve in space-time. The resolution of this difficulty may be expected to lead us to a quantum theory of gravity that may be operational and geometrical.

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