Mixing towards isotropization and gravitational-wave relic in string cosmology

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Abstract

The cosmological problem of *mixing*, with its various facets better known individually (as the horizon problem, the isotropy and entropy problems etc) is re-examined in the context of string cosmology and in terms of an open alternative to the mixmaster model. The mixing agent here is a gravitational wave with leading-order isotropization attained whe the first full wavelength is formed in the expanding young horizon. Standing and circularly polarized, this wave is inscribed within Bianchi-type VII_0 spatial sections of homogeneity. The finally emerging asymptotically flat FRW background, is intimately related to the axion scalar, in spite of the fact that the latter has virtually disappeared by that time. Surviving, however, is a relic gravitational wave with data and other observable imrints from the early string dynamics.

UOI-PH/95-319
hep-th/9507129

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1 Introduction

Following the extreme physics represented by the initial singularity and its immediate vicinity, there commence the somewhat better known aspects of the early dynamics. Ensuing thereof is the classical strong-field regime, currently illuminated in the context of string cosmology as exemplified by solutions from the effective string action (preferably exact to all orders in the string parameter α') [1],[2]. From this regime, we expect to draw resolutions for nearly all major cosmological problems as presently posed. Of particular importance and deserving re-examination in that context is the composite cosmological problem of *mixing*, with its various facets better known individually as the horizon problem, the isotropy and entropy problems, the sensitivity to initial conditions and so on [3],[4]. These last remarks also summarize the motivation and general objective of this paper. Actually, we will attempt a preliminary but nevertheless concrete study in terms of a specific 4D string background, realized (at least to leading order in α') by a Bianchi-type VII_0 model. We will present that model and see it evolve asymptotically towards a flat Friedmann-Robertson-Walker expansion. In the mixing mechanism and the subsequent isotropization attained practically within finite proper time, the main agent is a standing gravitational wave. A descendent of that wave seems to survive as a relic of the early dynamics. The hence made suggestion on its observability today could be of significance for string cosmology. Our main results will be presented in sections 2,3 and further discussed in 4.

To establish notation, we recall that the low-energy string effective action in 4D in the Einstein [conformal] frame may be expressed in terms of a non-linear $SL(2,R)/U(1)$ σ-model coupled to gravity as

$$S_{eff} = \int d^4x\sqrt{-g}\left(R + 2g^{\mu\nu}\frac{\partial_\mu S_+ \partial_\nu S_-}{(S_+ - S_-)^2}\right), \tag{1}$$

with the σ-model variables given in terms of the axion/dilaton pair by

$$S_\pm = b \pm i e^{\phi}. \tag{2}$$

The one-loop beta-function equations for conformal invariance follow from (1) as

$$\begin{aligned} R_{\mu\nu} + \frac{2}{(S_+ - S_-)^2}\partial_\mu S_+ \partial_\nu S_- &= 0, && (3)\\ \partial_\mu\left(g^{\mu\nu}\frac{\sqrt{-g}\partial_\nu S_\mp}{(S_+ - S_-)^2}\right) \pm 2\sqrt{-g}g^{\mu\nu}\frac{\partial_\mu S_+ \partial_\nu S_-}{(S_+ - S_-)^3} &= 0. && (4)\end{aligned}$$

The totally antisymmetric field strenght $H_{\lambda\mu\nu}$ (3-form H), generally expressible in terms of the potential $B_{\mu\nu}$ (2-form B) as

$$H = dB = -e^{-\phi}(db)^*, \tag{5}$$

has also been expressed here in terms of the axion scalar b and the dual * with respect to the 4D spacetime metric

$$ds^2 = -dt^2 + a_1(t)^2(\sigma^1)^2 + a_2(t)^2(\sigma^2)^2 + a_3(t)^2(\sigma^3)^2. \tag{6}$$

Such spacetimes possess homogeneous 3D spacelike hypersurfaces Σ^3 spanned by a basis of 1-forms $\{\sigma^i,\ i = 1, 2, 3\}$, which are invariant under the left action of some 3-parameter group of *transitive* Killing isometries [4],[5]. Presently, this group is of the mentioned type VII_0 in the Bianchi classification. Its structure constants can be read off the defining relations

$$d\sigma^1 = -\sigma^2 \wedge \sigma^3, \quad d\sigma^2 = -\sigma^3 \wedge \sigma^1, \quad d\sigma^3 = 0, \tag{7}$$

which are the (group-space) duals of the commutators of the group generators. The latter are realized as three independent killing vectors, taken along the principal directions of anisotropy. The metric coefficients a_i, as well as the b, ϕ fields are functions of the proper (co-moving) time t only.

2 The Bianchi-type VII_0 string background

Explicit holonomic-coordinate realizations for the σ^i can be found from the defining relations (7). One set of such coordinates x^i (with k an abitrary real constant) is

$$\begin{aligned} \sigma^1 &= \cos kx^3 dx^1 + \sin kx^3 dx^2, \\ \sigma^2 &= -\sin kx^3 dx^1 + \cos kx^3 dx^2, \\ \sigma^3 &= dx^3, \end{aligned} \tag{8}$$

subject to any transformation which preserves the structure of (7). We will also utilize the new coordinate time τ defined by

$$dt = a_1 a_2 a_3 d\tau = a^3 d\tau, \tag{9}$$

so that $a^{-1}da/dt$ is the mean Hubble constant of any comoving volume element in Σ^3. With a prime standing for $d/d\tau$ we may re-express (3), with its ii components giving

$$(\ln a_1^2)'' + a_1^4 - a_2^4 = 0, \quad (\ln a_2^2)'' + a_2^4 - a_1^4 = 0, \quad (\ln a_3^2)'' - (a_1^2 - a_2^2)^2 = 0, \tag{10}$$

while the $a_i(\tau)$ are also subject to the initial-value equation

$$(\ln a_1^2)'(\ln a_2^2)' + (\ln a_2^2)'(\ln a_3^2)' + (\ln a_3^2)'(\ln a_1^2)' + (a_1^2 - a_2^2)^2 = A^2, \tag{11}$$

essentialy the 00 equation in the set (3). In terms of the b, ϕ fields and the first-integral constant A, the pair of Eqs. (4) can be written as

$$b' + Ae^{2\phi} = 0, \qquad \phi'' + A^2 e^{2\phi} = 0. \tag{12}$$

The general solution thereof is

$$b = \left(\frac{N}{A}\right) \frac{\sinh(N\tau) + \sqrt{1 - \frac{A^2}{N^2}}\cosh(N\tau)}{\cosh(N\tau) + \sqrt{1 - \frac{A^2}{N^2}}\sinh(N\tau)}, \quad e^{-\phi} = \cosh(N\tau) + \sqrt{1 - \frac{A^2}{N^2}}\sinh(N\tau), \tag{13}$$

where N is another constant and where two more constants of integration have been absorbed to fix the origins of b, ϕ. The H 3-form follows from (5) as

$$H = A\sigma^1 \wedge \sigma^2 \wedge \sigma^3 = A dx^1 dx^2 dx^3. \tag{14}$$

which also identifies A as the magnitude of H per unit of invariant volume in Σ^3. We also note that the σ-model contributes in (1) as a source to the gravitational field with the energy-momentum tensor

$$\kappa^2 T_{\mu\nu} = -\frac{S'_+ S'_-}{(S_+ - S_-)^2} \,\mathrm{diag}\,(1, g_{ij}). \tag{15}$$

There remains the integration of Einstein's equations (10), for which the general solution may be expressed as

$$a_1^2 = e^{2P\tau + f}, \quad a_2^2 = e^{2P\tau - f}, \quad a_3^2 = e^{2P_3\tau + h}, \tag{16}$$

where P, P_3 are constants and f, h are solutions to the coupled system

$$f'' + 2e^{4P\tau}\sinh 2f = 0, \tag{17}$$

$$h'' + 4e^{4P\tau}(\sinh f)^2 = 0. \tag{18}$$

If f vanishes identically, we observe that the solution (16) attains $SO(2)$ or (if we also have $P_3 = P$) $SO(3)$ isotropy. In the latter case, (16) reduces to

$$a_1 = a_2 = a_3 = a = e^{P\tau}, \tag{19}$$

which describes flat FRW evolution. The importance (to be discussed shortly) of the special solution $f = 0$ of (17) is that it is an *attractor* in the sense that any other solution of (17) will eventually approach it at $\tau \to +\infty$ (for $P > 0$). We can also set $h = 0$ at that limit, in fact without loss of generality, due to presence of the general constant P_3 in (16). Further, one can show that the initial-value equation (11) may be re-expressed as

$$f'^2 - 4Ph' - 4e^{4P\tau}(\sinh f)^2 = 0, \tag{20}$$

to be considered together with the simultaneously valid content of (11) at $f = 0$, namely

$$4P^2 + 8PP_3 = A^2. \tag{21}$$

This is just a Kasner-like restriction on the Hubble constants and it obviously includes the $2P_3 = 2P = A/\sqrt{3}$ special case applicable at the $SO(3)$ (19). As other members in its class of Bianchi-type backgrounds, the present model has always an initial singularity, assumed to occur at $t = 0$ or $\tau = -\infty$, and no inflation [5].

3 Mixing and isotropization preliminaries

We will now examine the content of the type VII_0 spacetime, particularly as involved in the mixing mechanism and the induced isotropization within the Σ^3 spatial hypersurfaces. To fascilitate later discussion and better visualize what is going on, we will begin by concentrating for the moment on the mixing alone. We firstly need a clearer perspective on the actual descend of f towards the $f = 0$ solution of (17), already claimed as an attractor. As it turns out, after a brief initial adjustment to a generally lower value, f enters an oscillatory evolution (around the $f = 0$ limit) with monotonically decreasing amplitudes and increasing frequencies. This behavior can be shown and stated more rigorously if we restrict ourselves to sufficiently large times, namely values of t or τ such that the factor $\sinh 2f$ in the 'confining potential' in (17) is practically equal to $2f$. Then, under a re-definition of the independent variable as $x = e^{2P\tau}/P$, (17) is transformed to

$$x\frac{d^2 f}{dx^2} + \frac{df}{dx} + xf = 0. \tag{22}$$

This is the zeroth-order Bessel equation, whose general solution is a linear combination of the zeroth-order Bessel functions $J_0(x)$ and $Y_0(x)$. From their known behavior we immediately see that their approach to zero conforms with the generally claimed pattern, while for sufficiently large x in particular we find

$$f(\tau) = Fe^{-P\tau}\ \sin\left(\frac{e^{2P\tau}}{P}\right), \tag{23}$$

where F is a constant. The succession of zeroes and sign-changes of f corresponds to a succession of Kasner-like bounces, as can be seen in relation to (16) and (21). The resemblance to the mixing mechanism firstly observed in the mixmaster prototype [6] is rather striking, as we will further discuss later on.

To examine the nature and the dynamics of spatial anisotropy, we will firstly have to digress into its geometric foundation. To do that we re-express the metric (6) in holonomic coordinates so that, using (8) together with (16), we find

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j. \tag{24}$$

In this form, the part of the metric describing the geometry and evolution of the homogeneous Σ^3 sections (with $x^3 = z$ for brevity) is

$$g_{ij} = e^{2P\tau} \begin{pmatrix} e^f \cos^2 kz + e^{-f} \sin^2 kz & (e^f - e^{-f}) \cos kz \sin kz & 0 \\ (e^f - e^{-f}) \cos kz \sin kz & e^f \sin^2 kz + e^{-f} \cos^2 kz & 0 \\ 0 & 0 & e^h \end{pmatrix}. \tag{25}$$

This we may re-express as the sum of two contributions, a gravitational wave (as we will see in a moment) superimposed on a flat homogeneous background, namely as

$$g_{ij} = g_{ij}^{(0)} + g_{ij}^{(1)}. \tag{26}$$

The flat-bacground metric is

$$ds_0^2 = -dt^2 + g_{ij}^{(0)} dx^i dx^j, \tag{27}$$

with

$$g_{ij}^{(0)} = e^{2P\tau} \begin{pmatrix} \cosh f & 0 & 0 \\ 0 & \cosh f & 0 \\ 0 & 0 & e^h \end{pmatrix}. \tag{28}$$

On top of that, we have as a perturbation the remaining part of the metric

$$g_{ij}^{(1)} = e^{2P\tau} \sinh f \begin{pmatrix} \cos 2kz & \sin 2kz & 0 \\ \sin 2kz & -\cos 2kz & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{29}$$

This configuration is (i) traceless, (ii) covariantly constant and (iii) a properly defined eigenfunction of the wave operator, with all operations taken with repect to the background metric (27). While (i),(ii) can be easily verified, concerning (iii) we observe that $g_{ij}^{(1)}$ is an eigenfunction of ${}^{(3)}\nabla^2$, defined in Σ^3 with repect to $g_{ij}^{(0)}$. By a straightforward calculation we find that

$${}^{(3)}\nabla^2 g_{ij}^{(1)} = -\left(2ke^{-P\tau - h/2}\right)^2 g_{ij}^{(1)}, \tag{30}$$

which supplies the spatial part (namely in Σ^3) of the full wave operator acting on $g_{ij}^{(1)}$. The eigenvalue in (30) is minus the wavenumber $(2\pi/\lambda)$ squared, so that (29) can actually be viewed as a standing gravitational wave. Such configurations have already been studied to some extent in a general-relativistic context, and they are also realizable as the superposition of a pair of plain gravitational waves propagating in Σ^3 in opposite $(\pm z)$ directions [7]. Independently of these considerations, all quantities and, in particular, the wavelength of the standing wave can be calculated directly. Thus, from the invariance of (29) under increments of z by $\pm\pi$ and the resulting spacelike displacement ds as supplied by (24), we obtain

$$\lambda = \frac{\pi}{k} e^{P\tau} e^{h/2} = \frac{2\pi P}{k} e^{2P\tau} l_z, \tag{31}$$

a result which confirms our earlier identification of $-(2\pi/\lambda)^2$ as the eigenvalue in the rhs of (30). The second equality in(31) relates λ to l_z, namely the scale of the horizon in the z direction (along which the wave is developed), whose definition may be re-expressed as

$$l_z = a_3 \int dt/a_3 = \frac{1}{2P} e^{3P\tau} e^{h/2} = \frac{1}{2P} a^3. \tag{32}$$

These exact results also apply at the mentioned $\tau \to \infty$ limit where the horizon scale l_z becomes of the order of the cosmic time t as defined by (9). More explicitly, we find

$$l_z \to \frac{1}{2P} e^{3P\tau} \approx \frac{3}{2} t, \quad \lambda \to \frac{\pi}{k} e^{P\tau} \approx \frac{\pi}{k} a, \quad g_{ij}^{(0)} \to e^{2P\tau} \text{ diag } (1, 1, 1). \tag{33}$$

On the so attained asymptotic background, the wave (29) does not vanish (in spite of the presence of the vanishing f in its amplitude). Utilizing the asymptotic behavior of f in (23) we find

$$g_{ij}^{(1)} \to F e^{P\tau} \sin(e^{2P\tau}/P) \begin{pmatrix} \cos 2kz & \sin 2kz & 0 \\ \sin 2kz & -\cos 2kz & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{34}$$

which will be further discussed in the next section.

4 Discussion and conclusions

In the splitting (26) of the general metric down to the gravitational-wave perturbation (29) suprimposed on the flat background (27), *that* perturbation is neither small nor approximate. The wave, although it itself contributes to the anisotropy, it is at the same time the principal agent involved in the mixing and isotropization mechanism. Now, *some* approximation schemes will be needed for the closer examination of that mechanism, which is most effectively acting in-between two extreme epochs. I (the initial epoch), characterized by virtually arbitrary initial conditions and intractable dynamics, essentially involves the very complicated (possibly ergodic) part of the solutions which utilize (17) with $|f| \gg 1$. At the other extreme, involving $|f| \ll 1$, (the final epoch) F is dominated by the background metric (28), which by then has completely overgrown the perturbation (29), as it smoothly evolves towards the flat isotropic limit (19).

The I-epoch dynamics is characterized by arbitrarily large anisotropies, as seen from the general expression (16). The wavelengths λ are also very large relative to the young horizon l_z, as can be seen from (31). It follows that, during the I epoch, no full wavelength has yet been formed within the horizon. To the apparently finite number of zeroes of f (eventually zeroes of the Bessel functions from (22)) there corresponds a finite number of Kasner-like bounces in the early dynamics. The oscillations of f (and of h) also determine the time scale between the reversals of expansion/contraction at the bounces along the principal directions of anisotropy. These bounces induce mixing and isotropization, together with the generation of a considerable amount of entropy, essentially as in the Mixmaster prototype [6]. Although the latter has Σ^3 sections which are (deformed) S^3, the presently proposed open topology does not seem to make any difference in the applicability of the mixmaster approach [10].

In-between the I-F epochs, the wavelength λ continues to grow with the overall Hubble expansion while, simultaneously, it continues to shrink with respect to the (also expanding) horizon l_z, as seen from (31–33). Thus, the I-epoch $\lambda \gg l_z$ relationship will inevitably change to $\lambda \ll l_z$ in F so that $t_{\rm iso}$ can be formally defined as corresponding to the instant at which $\lambda = l_z$, occuring somewhere in the intermediate epoch. Around that time we also have $|f| \approx 1$, so that (22) and its solution could practically be used for a closer perspective on the mixing dynamics and on the degree of isotropy attained at $t_{\rm iso}$. Generally, from the $t =$ const. and $z =$ const. sections of (24) followed along t and z respectively, the wave is seen to be transverse circularly polarized and inscribed in the type-VII_0 geometry of Σ^3 (but not frozen therein) as they both evolve in time. The amplitudes of the wave are also translated within Σ^3 along z, while they are being differentially rotated in the x^1, x^2 plane by the local $SO(2)$ element contained in (29). This rotation is intimately related to (in fact it is a cosequense of) the helical structure of the type-VII_0 group of isometries in its transitive action (translations) defined by (7) etc., so that homogeneity in Σ^3 is respected at all times [7]. The emerging picture is that all points within a young horizon (at a sufficiently early Σ^3) are correlated as elements of a common dynamics involved in the formation of the above *single* gravitational wave. During the mixing, these elements evolve towards equivalence, namely indistinguishability of position and rate if expansion within Σ^3. This equivalence may be considered as having beeb attained *up to leading order* (in an approximation scheme having nothing to do with the α' expansion) as soon as the first full wavelength is created within the horizon, namely when $\lambda = l_z$ at $t = t_{\rm iso}$. The second-order approximation is realized with the formation of the second full wavelength within the horizon, and so on. The time required for the realization of each successive ordert is obviously decreasing, being inversely proportional to the monotonically increasing frequency of oscillations of f.

During the F (final) epoch, realized for $t \gg t_{\rm iso}$, λ (viewed as a time scale) is much smaller than t or the Hubble time, as follows from (31) etc. We conclude that the pertinent dynamics can be safely treated there in the adiabatic approximation. The dilaton dynamics, in principle fully recoverable from (13), has driven the model to lower couplings [8]. In fact e^{ϕ} diminishes as $e^{-|N|\tau}$ as $\tau \to \infty$, while the axion scalar b is trivialized towards a constant. Although H vanishes at that limit, its magnitude per unit of invariant volume remains constant, according to (5),(14) respectively. Had H been identically zero, meaning $A = 0$, there would be no $SO(3)$ isotropy limit (as follows from (21) etc). In that sense, the axion is intimately related to the asymptotic FRW structure, in spite of the fact that it has virtually disappeared by that time. All other quantities of immediate interest are given by (or may be derived from) the results (23) and (33, 34).

Of special interest is the relic gravitational wave (34), essentially as a surviving descendent of the mixing and isotropization process. Its amplitude, wavelength and frequency can be read off (34) and easily found to be constant multiplied by $e^{P\tau}$ factors, with the latter interpretable in terms of the overall cosmological (Hubble) expansion common to all scales and couplings. The maximum amplitude F in particular, although expected to be quite small, is of great importance because it determines the relative magnitude of (34) (considered as a perturbation) with respect to the background metric. (Rather than an arbitrary overall factor in the solution of the *homogeneous* equation (22), F should actually be viewed as descending from the non-linear (17) and therefore expressible in terms of $2P = A/\sqrt{3}$ and other numerical constants.) F will also determine the scale of the energy density stored in the relic wave, as well as its interrelation with the energy of the anisotropy in (15) wherefrom it has been originally drawn. This process continues well after any early energy exchange between interacting gravitational waves, namely well afte the time the universe became transparent to the gravitational radiation. The latter event (presumably very close to the initial singularity) is the gravitational analogue to the decoupling of the electromagnetic background (with *that* transparency established much later, at the surface of recombination). Relic waves thereoff may have survived to produce some *stochastic* (gravitational black body) background. However, although reasonably anticipated, it is not obvious how such a background would be observed [9]. In contrast to that, we have a single wave which survives in the present model. If indeed realized (even in an indirect context), such an observation would be of fundamental significance for string cosmology, in view of the plethora of data on the early dynamics encoded therein.

We have seen that our results really touch upon the composite problem of mixing, re-addressing its many-faceted aspects in the context of string cosmology as mentioned in the introduction. Certainly, they also touch upon the important question on their validity at the strong-field regime near the initial singularity. The straight answer is that they are not valid there – unless they can be extended to all orders in α' or, better yet, shown to descend from a 4D realization of a CFT. A closer examination could commence with the observation that our spacetime does not obviously belong to the broader class of null exact solutions [2], and with the issue of ergodicity of the solutions which involve (17) at $f \gg 1$ (also of relevance to the cosmological problem of sensitivity on initial conditions or the extremely early dynamics). Nevertheless, satisfactory resolutions seem to be already at hand for the isotropy (and entropy) problem and, in the context of the mixmaster approach [10], for the horizon problem as well.

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