Representation-theoretic derivation of the Temperley-Lieb-Martin algebras

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Abstract

Explicit expressions for the Temperley-Lieb-Martin algebras, *i.e.*, the quotients of the Hecke algebra that admit only representations corresponding to Young diagrams with a given maximum number of columns (or rows), are obtained, making explicit use of the Hecke algebra representation theory. Similar techniques are used to construct the algebras whose representations do not contain rectangular subdiagrams of a given size.

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1 Introduction

The Temperley-Lieb algebra plays an important role in a wide range of areas of mathematics and physics. It grew out of a study of relations between the percolation and colouring problem [1], and has since been used in studies of integrable models in statistical mechanics [2], von Neumann algebras [3], representations of braid groups, and knot and link invariants [4, 5, 6].

The Temperley-Lieb algebra $TL_n(q)$ is known to be a quotient of the Hecke algebra $H_n(q)$ to an algebra, whose irreducible representations are classified by Young diagrams with at most two columns or, equivalently, two rows [4, 5]. $H_n(q)$ is defined as a free unital associative algebra generated by $g_1, \ldots, g_{n-1}$ subject to the relations

$$\begin{array}{ll} g_i^2 = (q-1)g_i + q & i = 1,\, 2,\, \cdots,\, n-1 \\ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} & i = 1,\, 2,\, \cdots,\, n-2 \\ g_i g_j = g_j g_i & \text{if } |i-j| \geq 2 \end{array} \tag{1}$$

where q is a complex number. It may be considered as a deformation of the group algebra $\mathbf{C}S_n$ of the symmetric group S_n. In the limit $q = 1$ the g_i are identified with transpositions $(i, i{+}1)$. For generic values of q the representation theory of $H_n(q)$ closely resembles the representation theory of S_n. In particular, irreducible representations of $H_n(q)$ can be labelled by Young diagrams with n boxes [7, 8].

To obtain the defining relations of $TL_n(q)$ we first define

$$e_j = \frac{g_j + 1}{q^{\frac{1}{2}}}, \tag{2}$$

rewrite the first and third of relations (1) in terms of the generators e_j

$$e_j^2 = (q^{1/2} + q^{-1/2})e_j \ , \tag{3}$$

$$e_i e_j = e_j e_i \qquad \text{if } |i-j| \geq 2, \tag{4}$$

and, instead of the second of relation (1) impose the condition

$$e_j e_{j\pm 1} e_j - e_j = 0. \tag{5}$$

Thus, the Temperley-Lieb algebra $TL_n(q)$ is the free associative algebra generated by $e_1, \ldots e_{n-1}$, subject to the relations (3), (4), and (5).

The Temperley-Lieb algebra has been applied to the analysis of integrable models with the quantum group $U_q(su(2))$ symmetry [9]. In order to analyse integrable models such as spin chains or diffusion-reaction processes [10], which have $U_q(su(N))$ symmetry, one has to consider multi-column (or multi-row) generalisations of the Temperley-Lieb algebra, that have recently been introduced by Martin [2]. These Temperley-Lieb-Martin algebras are defined as the quotients of the Hecke algebra $H_n(q)$ that admit only irreducible representations described by Young diagrams with at most N columns. Such an algebra for $N = 3$ was also considered by Sochen [11]. Another generalisation of the Temperley-Lieb algebra, that excludes Young diagrams containing a rectangular Young subdiagram with specified numbers of both rows and columns, was considered by Martin and Rittenberg [12]. In this paper we give a simple derivation of these algebras, making explicit use of the representation theory of the Hecke algebra $H_n(q)$, and, in particular, of the properties of the Murphy operators [13, 14]. We find that with a certain choice of generators the defining relations of the multi-column Temperley-Lieb algebra are obtained as higher order iterates of the Temperley-Lieb braiding relation, eq. (5).

Our paper is organised as follows. In Section 2 we describe some elements of the representation theory of Hecke algebras, focusing on the construction of projection operators. In Section 3 we state and prove the main result concerning the structure of the Temperley-Lieb-Martin algebras. In Section 4 we discuss the structure of a double quotient of the Hecke algebra which leads to an algebra whose Young diagrams do not contain a rectangle of a given shape. In Section 5 we make some concluding remarks.

We assume that q is a real positive number.

2 Projection operators in the Hecke algebra $H_n(q)$

In this section we define and discuss the basic properties of certain projection operators in the Hecke algebra $H_n(q)$. We start, however, recalling some elementary facts about the Young diagrams. We denote by $\Gamma_n \equiv [\lambda_1, \lambda_2, \cdots, \lambda_k]$ the Young diagram of n boxes arranged into k rows of respective lengths $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$.

Definition 2.1 *The Young diagram* $\Gamma_n \equiv [\lambda_1, \lambda_2, \cdots, \lambda_k]$ *contains the Young diagram* $\Gamma'_m \equiv [\lambda'_1, \lambda'_2, \cdots, \lambda'_\ell]$ *if* $k \geq \ell$ *and* $\lambda_i \geq \lambda'_i, \quad i = 1, 2, \cdots, \ell$.

The following lemma and corollary are immediate consequences of the Littlewood-Richardson rule for the outer product [15].

Lemma 2.2 *Any Young diagram obtained as the outer product of two Young diagrams* Γ_n *and* Γ_m *contains both.*

Corollary 2.3 *A Young diagram with no more than* ℓ *columns (rows) could only be obtained as a direct product of Young diagrams with no more than* ℓ *columns (rows).*

Both Lemma 2.2 and Corollary 2.3 will be used in the proof of the main result in Section 3.

Definition 2.4 *The* q*-content of the* j*'th box in the* i*'th row of a Young diagram is* $q[j-i]_q$*, where* $[k]_q \equiv \frac{q^k-1}{q-1} = 1 + q + \cdots + q^{k-1}$.

In the analysis of the structure of Hecke algebras an important role is played by the Murphy operators [13, 14] L_p, given by

$$\begin{aligned} L_p &= g_{p-1} + \frac{1}{q} g_{p-2} g_{p-1} g_{p-2} + \frac{1}{q^2} g_{p-3} g_{p-2} g_{p-1} g_{p-2} g_{p-3} \\ &\qquad + \cdots + \frac{1}{q^{p-2}} g_1 g_2 \cdots g_{p-2} g_{p-1} g_{p-2} \cdots g_2 g_1 \; ; \qquad p = 2,\, 3,\, \cdots,\, n. \end{aligned}$$

Any two Murphy operators commute with one another. The symmetric polynomials in the Murphy operators span the center of the Hecke algebra. A state labelled by the sequence of Young diagrams $\Gamma_2 \subset \Gamma_3 \subset \cdots \subset \Gamma_n$ is an eigenstate of all the Murphy operators $L_2,\, L_3,\, \cdots,\, L_n$. The eigenvalue of L_i corresponding to this state is the q-content of the box that has been added to Γ_{i-1} to obtain Γ_i.

The fundamental invariant of $H_n(q)$, $\sum_{i=2}^n L_i$, has been shown to fully characterise its irreducible representations [16], and could therefore be used to construct projection operators onto subspaces consisting of irreducible representations with any desired specification. However, using the properties of the Murphy operators such projection operators can be written down in an even simpler form. This is particularly simple for the one-dimensional single row $[n]$ or single column $[1^n]$ irreducible representations. Take $[n]$ for example and note that it is the only irreducible representation of $H_n(q)$ for which the box at position $(2,1)$ does not exist. The q-content of this box is -1. Thus, for all irreducible representations but $[n]$ one of the Murphy operators must assume the eigenvalue -1. Therefore, $\tilde{C}_n \equiv \prod_{i=2}^n (L_i + 1)$ vanishes on all irreducible representations except $[n]$. The eigenvalues of the Murphy operators corresponding to the various boxes of $[n]$ are q, $q+q^2$, $q+q^2+q^3$, $\cdots$, $q+q^2+\cdots+q^{n-1}$, respectively, so that the operator $\tilde{C}_n$ assumes the value $[n]_q!$ where $[i+1]_q! = [i]_q![i+1]_q$. Thus, the normalised projection operator on the single row irreducible representation of $H_n(q)$ is

$$C_n = \prod_{j=2}^{n} \frac{L_j + 1}{[j]_q} \; .$$

This projection operator can be written as a sum over the $n!$ reduced words that furnish a basis of $H_n(q)$. Explicitly the first two operators come out as

$$\begin{aligned} C_2 &= \frac{1}{[2]_q}(1+g_1)\,, \\ C_3 &= \frac{1}{[3]_q!}(1+g_1+g_2+g_1g_2+g_2g_1+g_1g_2g_1)\,. \end{aligned}$$

Since C_n is manifestly symmetric in the Murphy operators, it is central in $H_n(q)$. Being a projection operator, it is idempotent. One important consequence is that for an arbitrary polynomial $F(g_1, g_2, \cdots, g_{n-1})$ in the generators of $H_n(q)$ the identity

$$F(g_1, g_2, \cdots, g_{n-1})C_n = F(q, q, \cdots, q)C_n \tag{6}$$

is satisfied in any irreducible representation of $H_n(q)$. This follows from the fact that on the single-row irreducible representation all g_i are represented by q, and on all other irreducible representations both sides of eq. (6) vanish.

For our construction of quotients of the Hecke algebra $H_n(q)$ it is necessary to consider the Hecke subalgebras generated by sets of consecutive generators g_i, g_{i+1}, $\cdots$, $g_{i+\ell-2}$, which are isomorphic with $H_\ell(q)$. We shall denote such algebras by $H_\ell^{(i)}(q)$, where the superscript specifies the generator with the lowest index. Within those subalgebras the Murphy operators are

$$L_j^{(i)} = g_{j+i-2} + \frac{1}{q} g_{j+i-3}g_{j+i-2}g_{j+i-3} + \cdots + \frac{1}{q^{j-2}} g_i g_{i+1} \cdots g_{j+i-2} \cdots g_{i+1} g_i \;;\; j = 2,\, 3,\, \cdots,\, \ell$$

and the projection operator onto the one-row irreducible representation of $H_\ell^{(i)}(q)$ is

$$C_\ell^{(i)} = \prod_{j=2}^{\ell} \frac{L_j^{(i)} + 1}{[j]_q} \; .$$

We often suppress the superscript when the lowest generator is g_1.

We now derive a recurrence relation for the single-row projection operators. First we use the idempotency of C_{i+1} and the fact that it commutes with L_{i+2} to note that

$$C_{i+2} = C_{i+1} \frac{L_{i+2} + 1}{[i+2]_q} = C_{i+1} \frac{L_{i+2} + 1}{[i+2]_q} C_{i+1}. \tag{7}$$

Next, using eq. (6) we obtain

$$C_{i+1}\frac{L_j^{(2)}+1}{[j]_q}C_{i+1}=C_{i+1}\ ;\quad j=2,\,3,\,\cdots,\,i, \tag{8}$$

$$C_{i+1}L_{i+2}C_{i+1}=C_{i+1}g_{i+1}C_{i+1}[i+1]_q, \tag{9}$$

and

$$C_{i+1}L_{i+1}^{(2)}C_{i+1}=C_{i+1}g_{i+1}C_{i+1}[i]_q\ . \tag{10}$$

From eqs. (9) and (10) it follows that

$$C_{i+1}\frac{L_{i+2}+1}{[i+2]_q}C_{i+1}=\frac{[i+1]_q^2}{[i]_q[i+2]_q}C_{i+1}\frac{L_{i+1}^{(2)}+1}{[i+1]_q}C_{i+1}-\frac{q^i}{[i]_q[i+2]_q}C_{i+1}. \tag{11}$$

Substitution of eq. (11) in eq. (7) and use of eq. (8) yields

$$C_{i+2}=\frac{[i+1]_q^2}{[i]_q[i+2]_q}C_{i+1}^{(1)}C_{i+1}^{(2)}C_{i+1}^{(1)}-\frac{q^i}{[i]_q[i+2]_q}C_{i+1}^{(1)}.$$

We can now renormalise the projection operators $C_{i+1}^{(j)}$ and define

$$e_j^{(i)}\equiv[[i+1]]_qC_{i+1}^{(j)}\ , \tag{12}$$

where $[[k]]_q\equiv\frac{q^{k/2}-q^{-k/2}}{q^{1/2}-q^{-1/2}}=q^{-(k-1)/2}[k]_q$. The elements $e_j^{(i)}$ of $H_n(q)$ have the following properties

$$\left(e_j^{(i)}\right)^2=[[i+1]]_qe_j^{(i)} \tag{13}$$

and

$$e_j^{(i+1)}=\frac{1}{[[i]]_q[[i+1]]_q}\left(e_j^{(i)}e_{j+1}^{(i)}e_j^{(i)}-e_j^{(i)}\right)\ . \tag{14}$$

Finally, we can use the automorphism $g_{j+k}\mapsto g_{j+i-k}$, $\quad k=0,\,1,\,\cdots,\,i$, in $H_{i+2}^{(j)}(q)$, under which $C_{i+2}^{(j)}$ is invariant, to show that

$$e_j^{(i)}e_{j+1}^{(i)}e_j^{(i)}-e_j^{(i)}=e_{j+1}^{(i)}e_j^{(i)}e_{j+1}^{(i)}-e_{j+1}^{(i)}\ . \tag{15}$$

Equations (15) ensure that the $e_j^{(i+1)}$ can be equivalently defined by interchanging indices j with $j+1$ on the right hand side of (14). We also notice that equation (15) is of the same form as the second equation in (1) written in terms of the generators e_i (2). Indeed, for $i=1$, $e_j^{(1)}=e_j$ and thus (15) together with (13) and (4) are the defining relations of the Hecke algebra $H_n(q)$. One should notice, however, that for general i, $e_j^{(i)}e_{j+k+1}^{(i)}\neq e_{j+k+1}^{(i)}e_j^{(i)}$ if $k=1,\ldots,i-1$. Hence, the $e_j^{(i)}$ do not generate any Hecke algebra for $i>1$. But for $i=n-2$ there are only two elements $e_1^{(n-2)}$, $e_2^{(n-2)}$ and thus they generate the Hecke algebra $H_3(\tilde q)$ where $\tilde q$ is determined by the equation $[[2]]_{\tilde q}=[[n-1]]_q$.

3 The Temperley-Lieb-Martin algebras

In this section we derive the algebraic relations defining the quotient of the Hecke algebra corresponding to irreducible representations with at most ℓ columns. To do so we start from $H_{\ell+1}(q)$, the lowest order Hecke algebra for which such restriction is meaningful. Since the only irreducible representation of $H_{\ell+1}(q)$ on which $e_1^{(\ell)}$ does not vanish is $[\ell+1]$, the desired quotient corresponds to $e_1^{(\ell)} = 0$.

We can now state the main result of this article.

Theorem 3.1 *In the Hecke algebra $H_n(q)$, let $e_i^{(\ell)}$, where $\ell = 1, 2, \cdots, n-1$ and $i = 1, 2, \cdots, n-\ell$, be given by eq. (12). Then the following are equivalent:*

1. $e_i^{(\ell)} = 0\ ;\ i = 1, 2, \cdots, n-\ell.$

2. *$H_n(q)$ is restricted to have irreducible representations labelled by Young diagrams with at most ℓ columns.*

Proof. The Hecke algebra $H_n(q)$, $n \geq \ell + 1$, can be written as a direct sum of the three subalgebras $H_i^{(1)}(q)$, $H_{\ell+1}^{(i)}(q)$ and $H_{n-i-\ell+1}^{(i+\ell)}(q)$, where the first and/or the last could be the trivial algebra $H_1(q)$. Therefore, the irreducible representations of $H_n(q)$ are direct products of the irreducible representations of the three Hecke subalgebras specified. To show that 1 follows from 2 we note that if only irreducible representations with at most ℓ columns are allowed for $H_n(q)$, then by Corrolary 2.3 only such irreducible representations are allowed for each of the subalgebras. In particular, for the $H_{\ell+1}^{(i)}(q)$ algebra the irreducible representation $[\ell+1]$ is excluded. Consequently, $e_i^{(\ell)}$ inevitably vanishes (*cf.* eq. (12)).

To show that 2 follows from 1 we note that from 1 $\mathrm{tr}(e_1^{(\ell)}) = 0$. Recall that in $H_{\ell+1}(q)$ $\mathrm{tr}(e_1^{(\ell)})$ vanishes on all irreducible representations with not more than ℓ columns and is positive on the irreducible representation $[\ell+1]$. Given any irreducible representation Γ_n of $H_n(q)$, $n > \ell+1$, the trace of $e_1^{(\ell)}$ can be evaluated recursively via [17]

$$\mathrm{tr}(e_1^{(\ell)})_{\Gamma_n} = \sum_{\Gamma_{n-1} \subset \Gamma_n} \mathrm{tr}(e_1^{(\ell)})_{\Gamma_{n-1}} \ ,$$

where $\Gamma_{n-1} \subset \Gamma_n$ means that Γ_{n-1} is one of the Young diagrams obtained by eliminating a box in Γ_n. Now, if Γ_n consists of more than ℓ columns it means that the iterative process carries the positive contribution initially due to $\mathrm{tr}(e_1^{(\ell)})_{[\ell+1]}$ and that cannot be annuled since there are no negative contributions. Hence, for all but irreducible representations with at most ℓ columns $\mathrm{tr}(e_1^{(\ell)}) > 0$. $\square$

Noting that for $\ell = 2$ the first statement of Theorem 3.1 is simply the relation (5) that defines the Temperley-Lieb algebra we have the following well known

Corollary 3.2 *([4, 5]) The Temperley-Lieb algebra $TL_n(q)$ is a quotient of the Hecke algebra $H_n(q)$ admitting only irreducible representations corresponding to Young diagrams with not more than two columns.*

Using eq. (3) it is a simple matter to reduce the condition obtained for $\ell = 3$, *i.e.*, $e_i^{(3)} = 0$, to the form derived by Sochen in [11].

Remark 3.3 From the proof of Theorem 3.1 it follows that the requirement $e_1^{(\ell)} = 0$ is sufficient to eliminate all Young diagrams with more than ℓ columns. Once this is established it is a simple matter to show that $e_i^{(\ell)} = 0$ for all $i \leq n - \ell$. This can be done either by invoking the other half of Theorem 3.1 (the fact that $e_i^{(\ell)} = 0$ is necessary for the irreducible representations to contain at most ℓ columns), or by noting that $\mathrm{tr}(e_i^{(\ell)}) = \mathrm{tr}(e_1^{(\ell)})$ [17], and that $e_i^{(\ell)}$ is central in $H_\ell^{(i)}(q)$, i.e. within any irreducible representation it is a multiple of the unit matrix.

Trivial modifications yield the multi-row Temperley-Lieb algebra. Precisely, we introduce the projection operator

$$R_i = \prod_{j=2}^{i} \frac{L_j - q}{(-q)[j]_{q^{-1}}} ,$$

that annihilates all Young diagrams with $i+1$ boxes in which the box $(2,1)$, whose q-content is q, is present. Thus, the sole surviving Young diagram is the single-column (i.e. $(i+1)$-row) diagram $[1^{i+1}]$. Instead of eq. (12) we obtain

$$f_j^{(i)} = [[i+1]]_{\sqrt{q}} R_{i+1}^{(j)} .$$

In particular, for $i = 1$ we obtain

$$f_i = q^{\frac{1}{2}}(q - g_i) .$$

Eqs. (13) and (14) remain unchanged in form, except that $f_j^{(i)}$ replaces $e_j^{(i)}$. In Theorem 3.1 "columns" should be replaced by "rows".

4 Elimination of rectangular subdiagrams

In this section we define the quotient of the Hecke algebra that corresponds to Young diagrams which do not contain a rectangular subdiagram consisting of ℓ_v rows each of length ℓ_h. We start by considering the lowest order Hecke algebra for which such a restriction is meaningful, $H_{\ell_h \ell_v}(q)$. Let

$$C_{\ell_h} = \frac{\prod_{i=2}^{\ell_h \ell_v} \left(L_i - q[\ell_h]_q \right)}{\prod_{i=1}^{\ell_h} \prod_{j=1}^{\ell_v}{}' (q[j-i]_q - q[\ell_h]_q)}$$

be the projection operator that eliminates diagrams in which the top box in the column $\ell_h + 1$ is occupied, and

$$R_{\ell_v} = \frac{\prod_{i=2}^{\ell_h \ell_v} \left(L_i - q[-\ell_v]_q \right)}{\prod_{i=1}^{\ell_h} \prod_{j=1}^{\ell_v}{}' (q[j-i]_q - q[-\ell_v]_q)}$$

be the projection operator that eliminates diagrams in which the leftmost box in the row $\ell_v + 1$ is occupied. The prime in both expressions means that the product in the denominator excludes the factor $i = j = 1$. C_{ℓ_h} and R_{ℓ_v} are normalised to unity on the rectangular diagram $[\ell_h^{\ell_v}]$.

Since $R_{\ell_v} C_{\ell_h}$ vanishes on all irreducible representations except $[\ell_h^{\ell_v}]$, the quotient of interest is specified by setting $Q_{\ell_h, \ell_v} \equiv R_{\ell_v} C_{\ell_h} = 0$.

If an irreducible representation of $H_n(q)$, $n > \ell_h \ell_v$, does not contain the rectangular subdiagram $[\ell_h^{\ell_v}]$ we find that

$$Q^{(i)}_{\ell_h, \ell_v} \equiv R^{(i)}_{\ell_v} C^{(i)}_{\ell_h} = 0 \; ,$$

in all subalgebras $H^{(i)}_{\ell_h \ell_v}$, $i = 1,\ 2,\ \cdots,\ n+1-\ell_h \ell_v$. Here $R^{(i)}_{\ell_v}$ and $C^{(i)}_{\ell_h}$ are obtained from R_{ℓ_v} and C_{ℓ_h}, respectively, by replacing L_j by $L^{(i)}_j$.

On the other hand, if $Q^{(1)}_{\ell_h, \ell_v} = 0$ then in all irreducible representations of $H_n(q)$, $n > \ell_h \ell_v$, $\mathrm{tr}(Q^{(1)}_{\ell_h, \ell_v}) = 0$. In an irreducible representation Γ_n that contains the rectangle $[\ell_h^{\ell_v}]$ $\mathrm{tr}(Q^{(1)}_{\ell_h, \ell_v}) > 0$, since the recursive evaluation of this trace carries the positive contribution of $\mathrm{tr}(Q^{(1)}_{\ell_h, \ell_v})_{[\ell_h^{\ell_v}]}$. Recall that $[\ell_h^{\ell_v}]$ is the only irreducible representation of $H_{\ell_h \ell_v}(q)$ for which $\mathrm{tr}(Q^{(1)}_{\ell_h, \ell_v}) \neq 0$. Therefore if we require that $Q^{(1)}_{\ell_h, \ell_v} = 0$ no representation containing $[\ell_h^{\ell_v}]$ is allowed.

Thus, the vanishing of all $Q^{(i)}_{\ell_h, \ell_v}$, $i = 1,\ 2,\ \cdots,\ n+1-\ell_h \ell_v$, is a necessary and sufficient condition for the exclusion of irreducible representations that contain the rectangular subdiagram $[\ell_h^{\ell_v}]$.

As an example we consider the exclusion of $[2^2]$, i.e. $\ell_h = \ell_v = 2$, in $H_4(q)$. Here,

$$Q^{(1)}_{2,2} = \frac{(L_2-(q+q^2))(L_3-(q+q^2))(L_4-(q+q^2))(L_2+1+\frac{1}{q})(L_3+1+\frac{1}{q})(L_4+1+\frac{1}{q})}{(-q^2)(-1-q-q^2)(-q-q^2)(q+1+\frac{1}{q})(\frac{1}{q})(1+\frac{1}{q})} .$$

In fact, L_2 can only assume the eigenvalues q and -1, corresponding to the boxes $(1,2)$ and $(2,1)$, respectively. Therefore, the factors containing L_2 in the numerator do not annihilate any Young diagram and appear to be superfluous. To write a normalised projector in a simpler form we define the projectors Q_s and Q_a, onto the two vectors spanning the $[2,2]$ irreducible representation, i.e. $[2][2,1][2,2]$ and $[1,1][2,1][2,2]$, respectively. Thus,

$$Q_s = \frac{(L_2+1)(L_3-q-q^2)(L_4-q-q^2)(L_4+1+\frac{1}{q})}{(1+q)(-1-q-q^2)(-q-q^2)(1+\frac{1}{q})}$$

and

$$Q_a = \frac{(L_2-q)(L_3+1+\frac{1}{q})(L_4-q-q^2)(L_4+1+\frac{1}{q})}{(-1-q)(q+1+\frac{1}{q})(-q-q^2)(1+\frac{1}{q})} .$$

Clearly, Q_s+Q_a is equal to unity within the irreducible representation $[2,2]$, and vanishes otherwise.

In [12] a different condition was proposed to exclude the $[2,2]$ irreducible representation of $H_4(q)$. Namely, it was required that the operator

$$Q_{mr} = e_1e_3e_2([[2]]_q - e_1)([[2]]_q - e_3) ,$$

should vanish. It can easily be checked, however, that Q_{mr} is not a projection operator and furthermore it is nilpotent, i.e., $(Q_{mr})^2 = 0$.

5 Conclusions

In this paper we have presented a simple method for constructing quotients of the Hecke algebra $H_n(q)$, the irreducible representations of which are labelled by Young diagrams with suitably restricted shapes. Our construction is based on the use of the Murphy operators in $H_n(q)$. In particular, we have shown that the ℓ-column Temperley-Lieb-Martin algebra is defined by the relations

$$e^{(\ell)}_j e^{(\ell)}_{j\pm1} e^{(\ell)}_j = e^{(\ell)}_j, \quad j=1,\ldots,n-\ell \tag{16}$$

which have a form identical with the Temperley-Lieb relations (5).

The restriction of Hecke algebras to representations corresponding to Young diagrams of a given shape have been investigated in [12] in order to analyse the spectra of hamiltonians of integrable models with $U_q(su(N,M))$ symmetry such as the Perk-Schultz quantum chains, and also to analyse a certain class of diffusion-reaction processes [10]. Other restrictions were used in classifaction of conformal field theories [11]. We believe that our construction and, in particular, the simple form (16) of the relations defining the appropriate quotients can be used in further analysis of such models.

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