# Aspects of Classical and Quantum Nambu Mechanics

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July, 1995

**Abstract**

We present recent developments in the theory of Nambu mechanics, which include new examples of Nambu-Poisson manifolds with linear Nambu brackets and new representations of Nambu-Heisenberg commutation relations.

**Mathematics Subject Classification (1991)** 70H99, 58F07

# 1 Introduction

Nambu mechanics is a generalization of classical Hamiltonian mechanics, introduced by Yoichiro Nambu [1]. Recently [2], its basic principles have been formulated in an invariant geometrical form similar to that of Hamiltonian mechanics. In [3] new examples of classical dynamical systems were given, which can be described by this formalism. Despite the elegance and beauty of Nambu mechanics, it turns out be somewhat restrictive [2] with many basic problems waiting to be solved.

This letter begins with a few new examples of Nambu-Poisson manifolds. The main result of this section is the generalization of the linear Poisson bracket to the Nambu bracket. In the next section we consider the quantization of Nambu mechanics via the canonical formalism of Nambu-Heisenberg commutation relations and present their new representations.

# 2 Nambu-Poisson Manifolds

Let $M$ denote a smooth finite dimensional manifold and $C^\infty(M)$ the algebra of infinitely differentiable real valued functions on $M$. Recall that [2] $M$ is a called a Nambu-Poisson manifold if there exits a $\mathbb{R}$-multi-linear map

$$\{\ ,\ldots,\ \}\ :\ [C^\infty(M)]^{\otimes n} \to C^\infty(M) \tag{1}$$

called a Nambu bracket of order $n$ such that $\forall f_1, f_2, \ldots, f_{2n-1} \in C^\infty(M)$,

$$\{f_1, \ldots, f_n\} = (-1)^{\epsilon(\sigma)}\{f_{\sigma(1)}, \ldots, f_{\sigma(n)}\}, \tag{2}$$

$$\{f_1 f_2, f_3, \ldots, f_{n+1}\} = f_1\{f_2, f_3, \ldots, f_{n+1}\} + \{f_1, f_3, \ldots, f_{n+1}\}f_2, \tag{3}$$

and

$$\begin{aligned}\{\{f_1, \ldots, f_{n-1}, f_n\}, f_{n+1}, \ldots, f_{2n-1}\} + \{f_n, \{f_1, \ldots, f_{n-1}, f_{n+1}\}, f_{n+2}, \ldots, f_{2n-1}\} \\ + \cdots + \{f_n, \ldots, f_{2n-2}, \{f_1, \ldots, f_{n-1}, f_{2n-1}\}\} = \{f_1, \ldots, f_{n-1}, \{f_n, \ldots, f_{2n-1}\}\},\end{aligned} \tag{4}$$

where $\sigma \in S_n$—the symmetric group of $n$ elements—and $\epsilon(\sigma)$ is its parity. Equations (2) and (3) are the standard skew-symmetry and derivation properties found for the ordinary ($n = 2$) Poisson bracket, whereas (4) is a generalization of the Jacobi identity and was called in [2] the fundamental identity.

The dynamics on a Nambu-Poisson manifold $M$ (i.e. a phase space) is determined by $n-1$ so-called Nambu-Hamiltonians $H_1, \ldots, H_{n-1} \in C^\infty(M)$ and is governed by the following equations of motion

$$\frac{df}{dt} = \{f, H_1, \ldots, H_{n-1}\},\ \forall f \in C^\infty(M). \tag{5}$$

A solution to the Nambu-Hamilton equations of motion produces an evolution operator $U_t$ which by virtue of the fundamental identity preserves the Nambu bracket structure on $C^\infty(M)$.

The Nambu bracket is geometrically realized by a Nambu tensor field $\eta \in \wedge^n TM$, a section of the $n$-fold exterior power $\wedge^n TM$ of the tangent bundle $TM$, such that

$$\{f_1, \dots, f_n\} = \eta(df_1, \dots, df_n), \tag{6}$$

which in local coordinates $(x_1, \dots, x_n)$ is given by

$$\eta = \eta_{i_1 \dots i_n}(x) \frac{\partial}{\partial x_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{i_n}}, \tag{7}$$

where repeated indices are assumed to be summed.

It was stated in [2] that the fundamental identity (4) is equivalent to the following algebraic and differential constraints on the Nambu tensor $\eta_{i_1 \dots i_n}(x)$:

$$\Sigma_{ij} + P(\Sigma)_{ij} = 0, \tag{8}$$

for all multi-indices $i = \{i_1, \dots, i_n\}$ and $j = \{j_1, \dots, j_n\}$ from the set $\{1, \dots, N\}$, where

$$\Sigma_{ij} = \eta_{i_1 \dots i_n}\eta_{j_1 \dots j_n} + \eta_{j_n i_1 i_3 \dots i_n}\eta_{j_1 \dots j_{n-1} i_2} + \dots + \eta_{j_n i_2 \dots i_{n-1} i_1}\eta_{j_1 \dots j_{n-1} i_n} - \eta_{j_n i_2 \dots i_n}\eta_{j_1 \dots j_{n-1} i_1}, \tag{9}$$

and $P$ is the permutation operator which interchanges the indices $i_1$ and $j_1$ of $2n$-tensor $\Sigma$, and

$$\begin{aligned} &\sum_{l=1}^{N} \left( \eta_{l i_2 \dots i_n} \frac{\partial \eta_{j_1 \dots j_n}}{\partial x_l} + \eta_{j_n l i_3 \dots i_n} \frac{\partial \eta_{j_1 \dots j_{n-1} i_2}}{\partial x_l} + \dots + \eta_{j_n i_2 \dots i_{n-1} l} \frac{\partial \eta_{j_1 \dots j_{n-1} i_n}}{\partial x_l} \right) \\ = &\sum_{l=1}^{N} \eta_{j_1 j_2 \dots j_{n-1} l} \frac{\partial \eta_{j_n i_2 \dots i_n}}{\partial x_l}, \end{aligned} \tag{10}$$

for all $i_2, \dots, i_n, \ j_1, \dots, j_n = 1, \dots, N$.

It was noted in [2] that the equation $\Sigma_{ij} = 0$ is equivalent to the condition that $n$-tensor $\eta$ is decomposable so that any decomposable element in $\wedge^n V$, where $V$ is an $N$-dimensional vector space over $\mathbb{R}$, endows $V$ with the structure of a Nambu-Poisson manifold. In particular, the totally antisymmetric $n$-tensor in $\mathbb{R}^n$ defines a Nambu bracket. In addition to that we have the following result.

**Lemma 1** *The completely antisymmetric constant $(n-1)$-tensor $\eta_{i_1 \dots i_{n-1}}$, where $i_1, \dots, i_{n-1} = 1, \dots, n$ is a Nambu tensor.*

**Proof**. Note that $\eta$ is a $n-1$-tensor in $n$-dimensional space and not in $n-1$-dimensions. It is sufficient to prove that $\eta$ is decomposable. For the case $n = 4$ one has

$$\begin{aligned} &\eta_{ijk} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k} = \left( \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \right) + \left( \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_4} \right) \\ &+ \left( \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4} \right) + \left( \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4} \right) \\ &= \frac{1}{4} \left[ \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} \right) \wedge \left( \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} \right) \right. \\ &\left. \wedge \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} \right) \right]. \end{aligned} \tag{11}$$

For general $n$ a similar elementary exterior algebra proof can be given. One can also show that completely antisymmetric constant $n$-tensor $\eta_{i_1 \dots i_{n-1} a}$ in $\mathbb{R}^N$, where $i_1, \dots, i_{n-1} = 1, \dots, n$ and $a = n+1, n+2, \dots, N$ is decomposable and, therefore, is a Nambu tensor. This gives a triple Nambu bracket on the vector space $\mathbb{R}^N$ for any $N \geq 3$.

As an example, one can use this bracket to describe the integrable system of two vortices in an incompressible ideal fluid with the Hamiltonian given by

$$H = \ln\left[(q_1 - q_2)^2 + (p_1 - p_2)^2\right]. \tag{12}$$

This system has the integrals of motion

$$\begin{aligned} I_1 &= q_1 + q_2 \\ I_2 &= p_1 + p_2 \\ I_3 &= p_1^2 + p_2^2 + q_1^2 + q_2^2 \end{aligned} \tag{13}$$

and was realized as a Nambu four bracket in [3]. It can be also represented via a Nambu triple bracket as follows:

$$\{f, I_1, H\} = (1/2)\eta_{ijk}\frac{\partial f}{\partial x_i}\frac{\partial I_1}{\partial x_j}\frac{\partial H}{\partial x_k}, \tag{14}$$

where $i, j, k = 1, 2, 3, 4$ and $x_1 = p_1,\ x_2 = p_2,\ x_3 = q_1,\ x_4 = q_2$. One can also represent it as

$$\{f, I_2, H\} = (-1/2)\eta_{ijk}\frac{\partial f}{\partial x_i}\frac{\partial I_2}{\partial x_j}\frac{\partial H}{\partial x_k}. \tag{15}$$

So far we have been dealing with constant Nambu tensors. Next we shall consider Nambu tensors linear in their coordinates. Namely, let $V$ be a $n+1$-dimensional vector space over $\mathbb{R}$ with coordinates $x_1, \dots, x_{n+1}$ and define the following $n$-tensor

$$\eta_{i_1 \dots i_n}(x) = \epsilon^{i_{n+1}}_{i_1 \dots i_n} x_{i_{n+1}}, \tag{16}$$

where $\epsilon^{i_{n+1}}_{i_1 \dots i_n} = \epsilon_{i_1 \dots i_{n+1}}$ is a completely antisymmetric $n+1$-tensor.

**Lemma 2** *The tensor $\eta_{i_1 \dots i_n}(x)$ is a Nambu tensor.*

**Proof**. First consider the $n$=3 case:

$$\eta_{ijk}(x) = \epsilon^{l}{}_{ijk} x_l, \tag{17}$$

where $i, j, k, l = 1, 2, 3, 4$. We must show that the tensor $\eta_{ijk}(x)$ satisfies both the algebraic and differential conditions (8)-(10). First consider the differential constraint (10), which reads

$$(\epsilon_{ijkl}\epsilon_{mnoi} + \epsilon_{oikl}\epsilon_{mnji} + \epsilon_{ojil}\epsilon_{mnki} - \epsilon_{mnil}\epsilon_{ojki})x_l = 0. \tag{18}$$

In order to prove it, write down the decomposability condition $\Sigma_{ij} = 0,\ i, j = 1, 2, 3, 4$ for the 4-tensor $\epsilon_{ijkl}$

$$\epsilon_{ijkl}\epsilon_{mnop} + \epsilon_{oikl}\epsilon_{mnjp} - \epsilon_{oijl}\epsilon_{mnkp} + \epsilon_{ojki}\epsilon_{mnlp} - \epsilon_{ojkl}\epsilon_{mnip} = 0. \tag{19}$$

Setting $p = i$ in the above expression gives

$$\epsilon_{ijkl}\epsilon_{mnoi} + \epsilon_{oikl}\epsilon_{mnji} + \epsilon_{ojil}\epsilon_{mnki} - \epsilon_{ojki}\epsilon_{mnil} = 0, \tag{20}$$

which is equivalent to the equation (18). Now, the algebraic condition (8) for $\eta_{ijk}(x)$ reads:

$$\begin{aligned}(\epsilon_{i_1i_2i_3l}\epsilon_{j_1j_2j_3k} + \epsilon_{ki_1i_3l}\epsilon_{j_1j_2j_3i_2} + \epsilon_{ki_2i_1l}\epsilon_{j_1j_2j_3i_3} + \\ \epsilon_{j_1i_2i_3l}\epsilon_{i_1j_2j_3k} + \epsilon_{kj_1i_3l}\epsilon_{i_1j_2j_3i_2} + \epsilon_{ki_2j_1l}\epsilon_{i_1j_2j_3i_3})x_l x_{j_3} = 0.\end{aligned} \tag{21}$$

To prove it, write down the equation $\Sigma_{ij} + P(\Sigma_{ij}) = 0$ for $\epsilon_{ijkl}$ and contract it with two $x$'s as follows:

$$\begin{aligned}(\epsilon_{i_1i_2i_3l}\epsilon_{j_1j_2j_3k} + \epsilon_{ki_1i_3l}\epsilon_{j_1j_2j_3i_2} + \epsilon_{ki_2i_1l}\epsilon_{j_1j_2j_3i_3} + \epsilon_{ki_2i_3i_1}\epsilon_{j_1j_2j_3l} + \\ \epsilon_{j_1i_2i_3l}\epsilon_{i_1j_2j_3k} + \epsilon_{kj_1i_3l}\epsilon_{i_1j_2j_3i_2} + \epsilon_{ki_2j_1l}\epsilon_{i_1j_2j_3i_3} + \epsilon_{ki_2i_3j_1}\epsilon_{i_1j_2j_3l})x_l x_{j_3} = 0.\end{aligned} \tag{22}$$

The fourth and eighth terms in the above equation vanish since $\epsilon_{j_1j_2j_3l}x_l x_{j_3} = 0$ and thus equation (22) is equivalent to equation (21). This completes the proof for $n = 3$; the proof for $n > 3$ is similar.

**Remark 1** This construction generalizes the standard linear Poisson bracket on a vector space $V$, given by the Poisson tensor $\eta_{ij}(x) = c_{ij}^k x_k$. The Jacobi identity for this bracket is equivalent to the Jacobi identity for the structure constants $c_{ij}^k$ so that the dual space $g = V^*$ has a Lie algebra structure. This classical construction goes back to Sophus Lie and plays a fundamental role in representation theory. In [2] and [4] a generalization of Lie algebras to the case of higher order operations, called Nambu-Lie gebras, were introduced. The example of a linear Nambu bracket (16) provides the dual space $V^*$ with the Nambu-Lie structure.

**Remark 2** Note that algebraic constraint on a Nambu tensor $\eta_{ijk} = c_{ijk}$ according to [2, Remark 1] contains the Jacobi identity for $c_{ijk}$ interpreted as structure constants. Namely, writing down the algebraic constraint for Nambu tensor $c_{ijk}$:

$$\begin{aligned}c_{i_1i_2i_3}c_{j_1j_2j_3} + c_{j_3i_1i_3}c_{j_1j_2i_2} + c_{j_3i_2i_1}c_{j_1j_2i_3} + \\ c_{j_1i_2i_3}c_{i_1j_2j_3} + c_{j_3j_1i_3}c_{i_1j_2i_2} + c_{j_3i_2j_1}c_{i_1j_2i_3} = 0,\end{aligned} \tag{23}$$

and setting $j_2 = i_1$ gives

$$c_{i_1i_2i_3}c_{j_1i_1j_3} + c_{j_3i_1i_3}c_{j_1i_1i_2} + c_{j_3i_2i_1}c_{j_1i_1i_3} = 0, \tag{24}$$

which is the Jacobi identity for the totally skew-symmetric structure constants $c_{ijk}$. This opens a possibility of using the structure constants of simple Lie algebras as Nambu tensors. Unfortunately, except for the structure constants of $sl(2, \mathbb{C})$, they do not satisfy the fundamental identity and, therefore, can not serve as Nambu tensors. Indeed, it follows from the Cartan classification that either $sl(3, \mathbb{C})$ or $sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C})$ are subalgebras of all simple Lie algebras. However, it can be shown directly that the structure constants for these Lie algebras do not satisfy the fundamental identity.

So far, all examples of Nambu tensors turn out to be decomposable. This leads us to a conjecture that all Nambu tensors are decomposable. In other words, equation $\Sigma_{ij}+P(\Sigma_{ij}) = 0$ for all multi-indices $i$ and $j$ should imply that $\Sigma_{ij} = 0$.

# 3 Representations of Nambu-Heisenberg Commutation Relations

There exist different approaches towards a quantization of Nambu mechanics, such as deformation quantization in the spirit of [5] and Feynman path integral approach, based on the action principle for Nambu mechanics [2]. Here we will comment only on the method of canonical quantization, which is based on the Heisenberg commutation relations

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = I, \tag{25}$$

and its generalization to higher order algebraic structures proposed by Nambu [1]:

$$[A_1, \ldots, A_n] =^{def} \sum_{\sigma \in S_n} (-1)^{\epsilon(\sigma)} A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(n)} = cI, \tag{26}$$

where $I$ is the unit operator and $c$ is a constant. In [2] it was called the Nambu-Heisenberg commutation relation. In particular, for $n = 3$ we have

$$[A_1, A_2, A_3] = A_1A_2A_3 - A_1A_3A_2 + A_3A_1A_2 - A_3A_2A_1 + A_2A_3A_1 - A_2A_1A_3. \tag{27}$$

Denote by $\zeta$ the primitive $n^{th}$ root of unity, $\zeta^n = 1$, and by $\mathbb{Q}[\zeta]$ the corresponding cyclotomic field: an algebraic extension of the field of rational numbers $\mathbb{Q}$ by $\zeta$. When $n$ is a prime number, the minimal polynomial for $\zeta$ has the form $1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1} = 0$. Let $\mathbb{Z}[\zeta]$ be a ring of algebraic integers in $\mathbb{Q}[q]$, i.e.

$$\mathbb{Z}[\zeta] = \{\omega = m_1 + m_2\zeta + m_3\zeta^2 + \cdots + m_{n-2}\zeta^{n-2} \mid m_1, \ldots, m_{n-2} \in \mathbb{Z}\},$$

and let $\mathcal{H}_n$ will be a vector space over $\mathbb{C}$ with a basis $\{|\omega >\}$, parametrized by the elements $\omega \in \mathbb{Z}[\zeta]$.

It was shown in [2] that the $n = 3$ Nambu-Heisenberg commutation relation admits the following representation:

$$\begin{aligned} A_1|\omega > &= (\omega + \zeta + 1)|\omega + 1 >, \\ A_2|\omega > &= (\omega + \zeta)|\omega + \zeta >, \\ A_3|\omega > &= \omega|\omega + \zeta^2 >, \end{aligned} \tag{28}$$

where $|\omega >\in \mathcal{H}_3$ and $c = \zeta^2(\zeta + 1)$.

In [2] it was also mentioned that the Nambu-Heisenberg commutation relations for general $n$ admit a natural representation in the vector space $\mathcal{H}_n$. Here we present the following result, which can be proved directly (using a symbolic calculations package).

**Lemma 3** *The Nambu-Heisenberg commutation relations for $n = 5$ and $n = 7$ admit the following representations.*

*1)* $[A_1,\ldots,A_5]=(-2\zeta^4+3\zeta^2+3\zeta-2-2\zeta^{-2})I$

$$\begin{aligned} A_1|\omega> &= (\omega+\zeta^3+\zeta^2+\zeta+1)|\omega+1>, \\ A_2|\omega> &= (\omega+\zeta^3+\zeta^2+\zeta)|\omega+\zeta>, \\ A_3|\omega> &= (\omega+\zeta^3+\zeta^2)|\omega+\zeta^2>, \\ A_4|\omega> &= (\omega+\zeta^3)|\omega+\zeta^3> \\ A_5|\omega> &= \omega^2|\omega+\zeta^4>, \quad |\omega>\in\mathcal{H}_5. \end{aligned} \tag{29}$$

*2)* $[A_1,\ldots,A_7]=7(3+3\zeta^{-1}+\zeta^{-2}-3\zeta^2+\zeta^{-3}-3\zeta^3)I$

$$\begin{aligned} A_1|\omega> &= (\omega+\zeta^5+\zeta^4+\zeta^3+\zeta^2+\zeta+1)|\omega+1> \\ A_2|\omega> &= (\omega+\zeta^5+\zeta^2+\zeta^3+\zeta^2+\zeta)|\omega+\zeta> \\ A_3|\omega> &= (\omega+\zeta^5+\zeta^4+\zeta^3+\zeta^2)|\omega+\zeta^2> \\ A_4|\omega> &= (\omega+\zeta^5+\zeta^4+\zeta^3)|\omega+\zeta^3> \\ A_5|\omega> &= (\omega^2+\zeta^5+\zeta^4)|\omega+\zeta^4> \\ A_6|\omega> &= (\omega+\zeta^5)|\omega+\zeta^5> \\ A_7|\omega> &= \omega^3|\omega+\zeta^6>, \quad |\omega>\in\mathcal{H}_7. \end{aligned} \tag{30}$$

**Remark 3** Although we are absolutely certain that there exists a natural representation for any $n$, we were unable to construct it explicitly. We only mention that somehow composite and prime $n$'s differ dramatically in this respect (which shows in representations for $n=4$ and $n=6$, which we do not present here) and it is rather difficult to present explicit constructions for prime $n$.

One possibility of realizing these representations was suggested to us by Igor Frenkel, who proposed that operators $A_i$ may be interpreted as special difference operators acting on a linear span of a root lattice for simple Lie algebras. Namely, for $n=3$ consider the root system $\Phi$ of the Lie algebra ($sl(3,\mathbb{C})$). It consists of six roots $\pm\alpha=\pm(1,0)$, $\pm\beta=\pm(-1,\sqrt{3})/2$, and $\pm\theta=\pm(\alpha+\beta)$. The simple roots can be chosen to be $\alpha$ and $\beta$ and the Weyl group of $\Phi$, which is isomorphic to $S_3$, is given by $W(\Phi)=\{1,\ r_\alpha,\ r_\beta,\ r_\alpha r_\beta,\ r_\beta r_\alpha,\ r_\alpha r_\beta\ r_\alpha\}$, where $r_\alpha$ and $r_\beta$ stand for corresponding reflections. The root lattice $\Lambda_r$ of $sl(3,\mathbb{C})$ is isomorphic to $\mathbb{Z}[\zeta]$ with $\zeta^3=1$ via $\alpha\to 1,\ \beta\to(-1+\sqrt{-3})/2=\zeta$, so that the vector space $\mathcal{H}_3$ can be parametrized by the simple roots of $sl(3,\mathbb{C})$

$$|\omega>=|m_1\alpha+m_2\beta> \quad m_1,m_2\in\mathbb{Z}.$$

For any $\sigma\in\Lambda_r$ define a shift operator $A_\sigma$ in $\mathcal{H}_3$ by

$$A_\sigma|\omega>=[r_\alpha r_\beta(\sigma)+\omega]|\omega+\sigma>. \tag{31}$$

For the case $n=3$ we can write down the representation of the Nambu-Heisenberg commutation relations in terms of these operators as

$$[A_\alpha,\ A_\beta,\ A_{-\theta}]=9I,$$

or, in the previous notation,

$$\begin{aligned} A_1|\omega> &= (\omega+\zeta)|\omega+1>, \\ A_2|\omega> &= (\omega-\zeta-1)|\omega+\zeta>, \\ A_3|\omega> &= (\omega+1)|\omega+\zeta^2>, \end{aligned}$$

since $r_\alpha r_\beta(\alpha) = \beta \to \zeta,\ r_\alpha r_\beta(\beta) = -\theta \to -\zeta-1,\ r_\alpha r_\beta\ (-\theta) = \alpha \to 1.$ We wonder if similar representations exist for difference operators related with other root systems, namely with that of $sl(n, \mathbb{C})$.

**Acknowledgements**
We are grateful to Professor I. Frenkel for helpful discussions. This work was supported in part by the U.S. Department of Energy grant DE-F602-888ER40388 (R.C.) and by the NSF grant DMS-92-04092 (L.T.).

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