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Gauge independence of the bubble nucleation rate in theories with radiative symmetry breaking

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Abstract

In field theories where a metastable false vacuum state arises as a result of radiative corrections, the calculation of the rate of false vacuum decay by bubble nucleation depends on the effective potential and the other functions that appear in the derivative expansion of the effective action. Beginning with the Nielsen identity, we derive a series of identities that govern the gauge dependence of these functions. Using these, we show, to leading nontrivial order, that even though these functions are individually gauge-dependent, one obtains a gauge-independent result for the bubble nucleation rate. Our formal arguments are complemented by explicit calculations for scalar electrodynamics in a class of R_ξ gauges.

1 Introduction

In addition to the minimum energy true vacuum state, many quantum field theories have one or more metastable "false vacua" that can decay to the true vacuum by the nucleation of bubbles of the stable vacuum. Methods have been developed for calculating the rate of this process either at zero temperature [1] or at high temperature [2]. However, these must be modified in the case of theories in which symmetry breaking arises as a result of radiative corrections [3]. While a scheme for dealing with such cases (at zero temperature) has been developed [4], it leads to an expression for the bubble nucleation rate that is not manifestly gauge-independent. In this paper we address the issue of this gauge dependence.

The standard approach [1] to the calculation of the bubble nucleation rate at zero temperature is based on finding a "bounce" solution of the classical Euclidean field equations. The nucleation rate per unit volume Γ may be written as

$$\Gamma = Ae^{-B} \tag{1.1}$$

where B is the Euclidean action of the bounce solution and A is an expression involving functional determinants that is generally equal to a numerical factor of order unity times a dimensionful quantity determined by the characteristic mass scales of the theory.

A problem arises if radiative corrections modify the vacuum structure of the theory. Theories in which this happens generally have no bounce solution; even if a bounce does exist, the nucleation rate calculation based on the bounce is not reliable. However, by integrating out certain fields at the outset, one can derive a modified algorithm [4] that can be applied to this situation. The results of this method are conveniently expressed in terms of the functions that appear in the derivative expansion

$$S^{\rm eff} = \int d^4x \left[V^{\rm eff}(\phi) + \frac{1}{2} Z(\phi)(\partial_\mu \phi)^2 + \cdots \right] \tag{1.2}$$

of the Euclidean effective action. (The dots represent terms containing four or more derivatives; these do not enter the calculation at the order to which we work.) These functions can in turn be expanded in power series in the couplings. For example, in a gauge theory with weak (i.e., $O(e^4)$) scalar self-couplings, the effective potential is of order e^4 and may be written, using an obvious notation, as

$$V^{\rm eff} = V^{\rm eff}_{e^4} + V^{\rm eff}_{e^6} + \cdots \, , \tag{1.3}$$

while

$$Z = 1 + Z_{e^2} + \cdots \, . \tag{1.4}$$

The first step in this approach is to use the leading approximation to the effective action to determine a bounce solution $\phi_b(x)$ through the equation

$$\Box\phi_b = \frac{\partial V_{e^4}^{\rm eff}}{\partial\phi} \,. \tag{1.5}$$

The desired nucleation rate is then given by

$$\Gamma = A' e^{-(B_0+B_1)} \tag{1.6}$$

where

$$B_0 = \int d^4x \left[V_{e^4}^{\rm eff}(\phi_b) + \frac{1}{2}(\partial_\mu\phi_b)^2 \right] \tag{1.7}$$

turns out to be of order e^{-4} while

$$B_1 = \int d^4x \left[V_{e^6}^{\rm eff}(\phi_b) + \frac{1}{2} Z_{e^2}(\phi_b)(\partial_\mu\phi_b)^2 \right] \tag{1.8}$$

is of order e^{-2}. The calculation of the pre-exponential factor is much more complicated than in the standard case; in particular A' cannot be expressed solely in terms of the functions appearing in Eq. (1.2). Nevertheless, one finds that, just as in the standard case, A' is equal to a numerical factor of order unity times a dimensionful factor determined by the mass scales of the theory.

Like any physically measurable quantity, the nucleation rate should be gauge independent. Since the leading terms in the effective potential are gauge independent, there is no difficulty in this regard with respect to either B_0 or the bounce solution itself. However, both of the functions that enter in B_1 are known [5] to depend on gauge. Our goal is to show that, nevertheless, these combine to give a gauge-independent contribution to the nucleation rate. Although we do not explicitly examine the prefactor A', we expect that our methods could be extended — albeit with considerably more technical complication — to show that it too is independent of gauge.

Our approach is based on the Nielsen identity [6], which describes the gauge dependence of the effective action, and which has been used to show that gauge-independent physical quantities can be obtained from a gauge-dependent effective potential [7]. In Sec. 2 we present a compact derivation of this identity, following the method of Kobes, Kunstatter, and Rebhan [8]. However, the usual form of the identity is not quite sufficient for our purposes. Instead, what we need is a series of identities, each of which gives the gauge dependence of one of the functions appearing in the derivative expansion (1.2). Although the identity for the effective potential is well-known, the remaining identities are, to our knowledge, new. In Sec. 3 we derive these from the master identity and then use them to give a general proof of the gauge-independence of B_1. To complement this formal proof, we have verified the relevant identities by explicit calculations for the case of scalar quantum electrodynamics in R_ξ gauges. These calculations, which expand upon the work of Aitchison and Fraser [9], are described in Sec. 4. Section 5 contains some concluding comments. Some two-loop effective potential calculations are presented in an Appendix.

2 The Nielsen identity

In this section we use the method of Ref. [8] to derive the Nielsen identity. We consider a gauge theory with fields denoted by ϕ_i. The classical action S is invariant under a set of infinitesimal gauge transformations of the form

$$\delta_g \phi_i = \Delta_i^\beta \theta_\beta \tag{2.1}$$

where the Δ_i^β are linear operators. (We will henceforth suppress the index β; for scalar electrodynamics, which we examine in greatest detail, there is only a single gauge parameter θ in any case.) By choosing a gauge-fixing function $F(\phi_i)$ and introducing Fadeev-Popov ghosts η and $\bar\eta$, we can write the generating functional of connected Green's functions as

$$\exp\left[iW(J,F)\right] = \int [\mathcal{D}\phi_i][\mathcal{D}\eta][\mathcal{D}\bar\eta] \exp\left[iI(F) + i\int d^4y J^i(y)\,\phi_i(y)\right] \tag{2.2}$$

where

$$I(F) = S - \int d^4x \left\{ \frac{\left[F(\phi)\right]^2}{2\xi} + \bar\eta \frac{\delta F}{\delta \phi_i}\Delta_i \eta \right\} \tag{2.3}$$

is invariant under the BRST transformations

$$\delta_{\rm BRST}\,\phi_i = \zeta\,\Delta_i\eta, \qquad \delta_{\rm BRST}\,\bar\eta = -\zeta\frac{1}{\xi}F, \qquad \delta_{\rm BRST}\,\eta = 0 \tag{2.4}$$

with ζ an arbitrary Grassman number.

In presenting the derivation, it is convenient to adopt a compact notation where

$$\langle \mathcal{O}(\phi)\rangle \equiv e^{-iW} \int [\mathcal{D}\phi][\mathcal{D}\eta][\mathcal{D}\bar\eta]\mathcal{O}(\phi,\eta,\bar\eta) \exp\left[iI + i\int d^4y J^i(y)\phi_i(y)\right] \tag{2.5}$$

for any operator $\mathcal{O}$. Now note that if $\mathcal{O}$ is linear in the ghost fields, its odd Grassman character leads to the vanishing of this quantity. In particular,

$$\langle \bar\eta\, G\rangle = 0 \tag{2.6}$$

for any functional $G[\phi(x)]$. Applying the BRST transformation (2.4) to this equation results in the identity

$$\left\langle \delta_{\rm BRST}[\bar\eta(x)\, G(x)] + i\bar\eta(x) G(x) \int d^4y J^i(y)\, \delta_{\rm BRST}\phi_i(y) \right\rangle = 0 \tag{2.7}$$

which may be rewritten, using the anticommutivity of η and $\bar\eta$, as

$$-\left\langle \frac{1}{\xi} F(x)\, G(x) + \bar\eta(x) \frac{\delta G(x)}{\delta\phi_i(x)}\, \Delta_i\eta(x) \right\rangle = -i \int d^4y J^i(y)\, \langle \Delta_i\eta(y)\, \bar\eta(x)\, G(x)\rangle\ . \tag{2.8}$$

Now consider the effect of an infinitesimal change $F \to F + \Delta F$ in the gauge-fixing function. Recalling Eqs. (2.2) and (2.3), we see that the change in the generating functional $W[J]$ is simply the integral over x of the left hand side of Eq. (2.8), with G set equal to ΔF. Hence,

$$\Delta W = -i \int d^4x\, d^4y\; J^i(y) \langle \Delta_i\eta(y)\, \bar\eta(x)\, \Delta F(x)\rangle\, . \tag{2.9}$$

Recalling that the effective action is related to $W[J]$ by the Legendre transformation

$$S_{\rm eff} = W - \int d^4x J^i(x)\, \frac{\delta W}{\delta J_i(x)} = W - \int d^4x J^i(x)\, \phi_i(x) \tag{2.10}$$

we find that

$$\Delta S_{\rm eff} = i \int d^4x\, d^4y\; \frac{\delta S_{\rm eff}}{\delta\phi_i(y)} \langle \Delta_i\eta(y)\, \bar\eta(x)\, \Delta F(x)\rangle_{\rm 1PI} \tag{2.11}$$

where the subscript 1PI indicates that only the contributions from one-particle irreducible graphs are to be included. In particular, an infinitesimal change $d\xi$ in the gauge parameter is equivalent to the choice $\Delta F = -(F/2\xi) d\xi$. Hence,

$$\xi \frac{\partial S_{\rm eff}}{\partial \xi} = -\frac{i}{2} \int d^4x\, d^4y \frac{\delta S_{\rm eff}}{\delta\phi_i(y)} \langle \Delta_i\eta(y)\, \bar\eta(x)\, F(x)\rangle_{\rm 1PI} \tag{2.12}$$

$$= \int d^4y \frac{\delta S_{\rm eff}}{\delta\phi_j(y)} H_j[\phi(z), y] \tag{2.13}$$

where

$$H_j[\phi(z), y] = -\frac{i}{2} \int d^4x \langle \Delta_i\eta(y)\, \bar\eta(x)\, F(x)\rangle_{\rm 1PI}\ . \tag{2.14}$$

Eq. (2.13) is the Nielsen identity.

3 Derivative expansion of the Nielsen identity and formal proof of gauge independence of the nucleation rate

To study the gauge-dependence of the bubble nucleation rate, we will need a set of identities that are obtained by making derivative expansions of both sides of Eq. (2.13). For simplicity, consider the case where the effective action depends on only a single field $\phi(x)$. There is then only a single functional $H[\phi(x), y]$, which can be expanded as

$$H[\phi(x), y] = C(\phi) + D(\phi)(\partial_\mu \phi)^2 + \cdots \tag{3.1}$$

where all terms on the right are understood to be evaluated at point y and the dots represent terms with more than two derivatives. Inserting this, together with the expansion (1.2) of the effective action, into Eq. (2.13), gives

$$\begin{aligned} &\xi \frac{\partial}{\partial \xi} \int d^4x \left[V^{\text{eff}}(\phi) + \frac{1}{2} Z(\phi)(\partial_\mu \phi)^2 + \cdots \right] \\ &\qquad = \int d^4x \left[C(\phi) + D(\phi)(\partial_\mu \phi)^2 + \cdots \right] \left[\frac{\partial V^{\text{eff}}}{\partial \phi} + \frac{1}{2} \frac{\partial Z}{\partial \phi} (\partial_\mu \phi)^2 - \partial_\mu \left[Z(\phi)\, \partial_\mu \phi \right] + \cdots \right] . \end{aligned} \tag{3.2}$$

If this identity is to hold for arbitrary $\phi(x)$, then not only must the integrands on the two sides be equal point by point, but the terms with equal number of derivatives must be separately equal. Thus, the terms with no derivatives obey

$$\xi \frac{\partial V^{\text{eff}}}{\partial \xi} = C \frac{\partial V^{\text{eff}}}{\partial \phi} \tag{3.3}$$

while from the terms with two derivatives we obtain

$$\xi \frac{\partial Z}{\partial \xi} = C \frac{\partial Z}{\partial \phi} + 2D \frac{\partial V^{\text{eff}}}{\partial \phi} + 2Z \frac{\partial C}{\partial \phi} \, . \tag{3.4}$$

(Eq. (3.3), which can be obtained immediately from Eq. (2.13) by choosing $\phi(x)$ to be a constant, appears in Ref. [6].)

We now specialize to the case of a gauge theory with gauge coupling e and scalar self-couplings of order e^4. As indicated in Eq. (1.3), the effective potential begins with terms of order e^4, while $Z(\phi) = 1 + O(e^2)$. Analysis of the relevant graphs shows that $C(\phi)$ starts at order e^2 and $D(\phi)$ is of order unity. The terms of order e^4 in Eq. (3.3) yield

$$\xi \frac{\partial V_{e^4}^{\rm eff}}{\partial \xi} = 0\,. \tag{3.5}$$

Now recall that the bounce solution $\phi_b(x)$ is determined, through Eq. (1.5), by $V_{e^4}^{\rm eff}$. Since Eq. (3.5) shows that the latter is gauge independent, both $\phi_b(x)$ and B_0, the leading contribution to the exponent of the nucleation, are independent of ξ.

To study the gauge dependence of B_1, we need the order e^6 terms of Eq. (3.3),

$$\xi \frac{\partial V_{e^6}^{\rm eff}}{\partial \xi} = C_{e^2} \frac{\partial V_{e^4}^{\rm eff}}{\partial \phi}\,, \tag{3.6}$$

as well as the terms of order e^2 in Eq. (3.4),

$$\frac{1}{2}\xi \frac{\partial Z_{e^2}}{\partial \xi} = \frac{\partial C_{e^2}}{\partial \phi}\,. \tag{3.7}$$

These equations, together with Eq. (1.8), imply that

$$\begin{aligned}
\xi \frac{\partial B_1}{\partial \xi} &= \xi \frac{\partial}{\partial \xi} \int d^4x \left[V_{e^6}^{\rm eff} + \frac{1}{2} Z_{e^2} (\partial_\mu \phi_b)^2 \right] \\
&= \int d^4x \left[C_{e^2} \frac{\partial V_{e^4}^{\rm eff}}{\partial \phi} + \frac{\partial C_{e^2}}{\partial \phi} (\partial_\mu \phi_b)^2 \right] \\
&= \int d^4x \left[C_{e^2} \frac{\partial V_{e^4}^{\rm eff}}{\partial \phi} + (\partial_\mu C_{e^2})(\partial_\mu \phi_b) \right] \\
&= \int d^4x \, C_{e^2} \left[\frac{\partial V_{e^4}^{\rm eff}}{\partial \phi} - \Box \phi_b \right]
\end{aligned} \tag{3.8}$$

where all quantities are to be evaluated with $\phi(x)$ set equal to the bounce solution $\phi_b(x)$. Eq. (1.5), which determines the bounce, shows that the last expression on the right hand side must vanish, and hence that

$$\xi \frac{\partial B_1}{\partial \xi} = 0\,. \tag{3.9}$$

This verifies that, at least up to pre-exponential terms of order unity, the bubble nucleation rate is gauge independent.

4 Scalar Electrodynamics

4.1 Basics

We now illustrate these formal arguments by explicit calculations for the case of scalar electrodynamics. The Lagrangian, which we write in the form

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}(\partial_\mu\Phi_1 - eA_\mu\Phi_2)^2 + \frac{1}{2}(\partial_\mu\Phi_2 + eA_\mu\Phi_1)^2 - V(\Phi) \tag{4.1}$$

with

$$V(\Phi) = \frac{1}{2}m^2\Phi^2 + \frac{\lambda}{4!}\Phi^4 \tag{4.2}$$

and $\Phi \equiv (\Phi_1^2 + \Phi_2^2)^{1/2}$, is invariant under the gauge transformation

$$\delta_g A_\mu = \partial_\mu\theta, \qquad \delta_g\Phi_1 = e\Phi_2\theta, \qquad \delta_g\Phi_2 = -e\Phi_1\theta\,. \tag{4.3}$$

If $m^2 > 0$, the tree-level potential has a minimum at $\Phi = 0$. In order that one-loop effects be able to change the vacuum structure and give a symmetry-breaking minimum at $\Phi = \langle\Phi\rangle \neq 0$, we must require that both λ and m^2 be anomalously small, of order e^4 and $e^2\langle\Phi\rangle^2$, respectively.

For calculating the bubble nucleation rate it is sufficient to evaluate the terms in the derivative expansion of the effective action for $\Phi_2 = A_\mu = 0$. With this in mind, we will consider the class of gauges determined by the gauge-fixing function

$$F = (\partial_\mu A^\mu + ev\Phi_2)\ . \tag{4.4}$$

(The gauge-dependence of the effective potential in these gauges was studied in detail by Aitchison and Fraser [9]; in the following discussion we will make use of a number of their results.) The Nielsen identity (2.13) then involves only the single functional

$$H_{\Phi_1}[\phi(z), y] = -\frac{ie}{2}\int d^4x\langle(\Phi_2(y)\,\eta(y)\,\bar{\eta}(x)\ (\partial_\mu A^\mu(x) + ev\Phi_2(x))\rangle\,. \tag{4.5}$$

The effective action can be obtained as the sum of one-particle irreducible vacuum graphs in the theory obtained from the Lagrangian (4.1) by making the shift $\Phi_1 \to \Phi_1 + \phi$ and then dropping all terms linear in the quantum fields. The vertex factors for these graphs can be simply read off from the resulting Lagrangian in the standard fashion (see, e.g., Ref. [9]). The propagators require a bit more work. Following the usual approach, one would obtain from the Lagrangian (together with the gauge-fixing and ghost terms) the effective Φ_1, Φ_2, A_μ, and ghost propagators

$$G_1(k) = \frac{i}{k^2 - \tilde{m}_1(\phi)^2} \tag{4.6}$$

$$G_2(k) = \frac{i(k^2 - \xi e^2\phi^2)}{D(k)} \tag{4.7}$$

$$\begin{aligned} G_{\mu\nu}(k) &= G^T_{\mu\nu}(k) + G^L_{\mu\nu}(k) \\ &= i\frac{-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}}{k^2 - e^2\phi^2} - \frac{i[\xi(k^2 - \tilde{m}_2^2) - e^2v^2]}{D(k)}\frac{k_\mu k_\nu}{k^2} \end{aligned} \tag{4.8}$$

$$G_g = \frac{i}{k^2 + e^2 v\phi} \tag{4.9}$$

as well as the mixed Φ_2–A_μ propagator

$$G_{2\mu}(k) = \frac{e(\xi\phi + v)\,k_\mu}{D(k)} \tag{4.10}$$

where the momentum flow is understood to flow from the Φ_2 end to the A_μ end. In these expressions

$$D(k) = k^4 - k^2(\tilde{m}_2^2 - 2e^2v\phi) + e^2\phi^2(e^2v^2 + \xi\tilde{m}_2^2) \tag{4.11}$$

and

$$\tilde{m}_1^2(\phi) = m^2 + \frac{\lambda}{2}\phi^2 = V''(\phi) \tag{4.12}$$

$$\tilde{m}_2^2(\phi) = m^2 + \frac{\lambda}{6}\phi^2 = \frac{V'(\phi)}{\phi}\,. \tag{4.13}$$

These propagators are not quite what we need. Our assumption that λ is of order e^4 not only makes some of the one-loop terms comparable to the tree-level terms, but also implies that some multi-loop graphs are not suppressed relative to graphs with fewer loops; specifically, the insertion of transverse photon loops along a scalar propagator does not increase the order of the graph. To restore the validity of our expansion, these insertions must be summed. This can be done simply by replacing the propagators given in Eqs. (4.6-4.8) by "dressed" propagators in which the $\tilde{m}_a^2$ are replaced by

$$m_1^2(\phi) = V''_{e^4}(\phi) \tag{4.14}$$

$$m_2^2(\phi) = \frac{V'_{e^4}(\phi)}{\phi}\,. \tag{4.15}$$

(To avoid double-counting, subtractions are needed for certain graphs with two or more loops; these corrections only affect contributions of higher order than those we will be considering.)

Before we proceed to verify the identities, there is one more issue to be addressed. Many of the graphs contributing to the effective action have divergences that must be cancelled by appropriate counterterms. We will not display these explicitly, but all divergent integrals should be understood to be made finite by some gauge-invariant renormalization scheme (e.g., minimal subtraction in the context of dimensional regularization); when we refer to the magnitude of an integral, this should be understood as referring to the magnitude of its finite part.

4.2 The identity for the effective potential

To order e^2 the function $C(\phi)$ entering the identity (3.3) receives contributions only from the two graphs shown in Fig. 1. These combine to give [1]

$$\begin{aligned} C_{e^2} &= -\frac{ie}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+e^2v\phi)\,D(k)}\left[e(\xi\phi+v)\,k^2 - ev(k^2-\xi e^2\phi^2)\right] \\ &= -\frac{ie^2\phi\xi}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{D(k)}\,. \end{aligned} \tag{4.16}$$

The effective potential is obtained by summing the graphs with vanishing external momenta. The one-loop contributions may be split into three parts. First, the graphs with a transverse photon loop give a contribution

$$-\frac{3i}{2}\int \frac{d^4k}{(2\pi)^4}\ln(k^2-e^2\phi^2)\,. \tag{4.17}$$

From dimensional arguments, this integral is clearly of order $e^4\phi^4$. Hence, it combines with the tree-level potential to give

$$V^{\rm eff}_{e^4} = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 - \frac{3i}{2}\int \frac{d^4k}{(2\pi)^4}\ln(k^2-e^2\phi^2)\,. \tag{4.18}$$

This is manifestly gauge-independent, in accordance with Eq. (3.5). A second gauge-independent contribution, coming from the graphs with Φ_1-loop graphs, is

$$-\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\ln(k^2-m_1^2)\,. \tag{4.19}$$

[1] Apart from an overall sign arising from a difference in the definition in $C(\phi)$, these expressions are the same as those appearing in Ref. [9]

This is of order m_1^4, and hence contributes only to $V_{e^8}^{\rm eff}$. Finally, there is an order e^6 contribution

$$V_{e^6;1-{\rm loop}}^{\rm eff} = -\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\left[\ln D(k) - 2\ln(k^2+e^2v\phi)\right] \tag{4.20}$$

in which the first term arises from graphs with Φ_2, longitudinal photon, or mixed scalar-photon propagators while the second is due to those with a single ghost loop.

In addition to these one-loop contributions, there are a number of two-loop graphs that contribute to $V_{e^6}^{\rm eff}$. Some two-loop graphs have already been included in Eq. (4.20) as a result of the replacement $\tilde{m}_i^2 \to m_i^2$, including in particular the ξ-dependent "figure-eight" graph with one transverse photon loop and one Φ_2 loop. In the Appendix we show that although a number of the remaining graphs are ξ-dependent, they add together[2] to give a gauge-independent contribution to $V_{e^6}^{\rm eff}$. Hence, we can combine Eq. (4.20) with Eqs. (4.11), (4.15), and (4.16), to obtain

$$\xi\frac{\partial V_{e^6}^{\rm eff}}{\partial\xi} = \xi\frac{\partial V_{e^6;1-{\rm loop}}^{\rm eff}}{\partial\xi} = C_{e^2}\,\frac{\partial V_{e^4}^{\rm eff}}{\partial\phi}\,, \tag{4.21}$$

thus verifying Eq. (3.6).

4.3 The identity for $Z(\phi)$

We now turn to the identity (3.4), which we will verify to order e^2. We begin by recalling that $Z(\phi)$ can be calculated from the sum of one-particle irreducible graphs with one external line carrying momentum p, another carrying momentum $-p$, and all others with zero momentum. If the contribution of graph j is denoted by $I_j(p^2)$, then

$$Z = -i\,\frac{\partial}{\partial p^2}\sum I_j\bigg|_{p^2=0}\;. \tag{4.22}$$

Although there are many one-loop graphs contributing to the scalar self-energy, we will need to calculate only a few. Those graphs with quartic vertices are independent of the external momentum and hence do not contribute to $Z(\phi)$. Because of our assumption that

[2] The cancellation of the gauge-dependence among these graphs can be understood by considering the case $\lambda\sim e^2$, where the loop expansion is completely equivalent to an expansion in e^2. Apart from the appearance of $\tilde{m}_2^2$ rather than m_2^2, the one-loop approximation to $C(\phi)$ is precisely the same as C_{e^2} of Eq. (4.16). Hence, Eq. (3.3) can be satisfied both at the one-loop level in that case (as was shown in [9]) and in at $O(e^6)$ in our case only if this cancellation among the two-loop graphs occurs.

$\lambda = O(e^4)$, all graphs with a vertex arising from the scalar self-interaction are at least of order e^4 and can also be ignored here. Finally, the self-energy graph with a single ghost loop, although of order e^2, is ξ-independent. Thus, the entire ξ-dependence of $Z_{e^2}(\phi)$ comes from the four graphs shown in Fig. 2.

It is convenient to consider separately the terms containing the transverse and the longitudinal parts of the photon propagators. The only contribution with two transverse propagators, from graph b, is manifestly independent of ξ and so can be neglected here. Graphs a, b, and c each give contributions with a single transverse propagator; although separately these each contain ξ-dependent terms of order e^2, their sum is easily seen to be ξ-independent to this order.

This leaves the terms containing only longitudinal photons. These may be written as

$$I_j = e^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{D(k)\,D(p+k)}\, b_j(k,p) \tag{4.23}$$

where

$$\begin{aligned}
b_a &= -[(p+k)^2 - \xi e^2\phi^2]\,(\xi k^2 - e^2v^2)\,[k\cdot(2p+k)]^2\,k^{-2} \\
b_b &= 2e^2\phi^2\,(\xi k^2 - e^2v^2)\,[\xi(p+k)^2 - e^2v^2][k\cdot(p+k)]^2\,k^{-2}\,(p+k)^{-2} \\
b_c &= -4e^2\phi^2\,(\xi\phi + v)\,(\xi k^2 - e^2v^2)[k\cdot(p+k)]\,[k\cdot(2p+k)]\,k^{-2} \\
b_d &= e^2(\xi\phi+v)^2\,(k^2-p^2)\,k\cdot(k+2p)\,.
\end{aligned} \tag{4.24}$$

(In these expressions we have omitted terms proportional to m_2^2 since the contribution of these is at least $O(e^4)$. Note also that we have included a factor of two in the the contribution from graph c to take into account the fact that reversing the direction of p gives a second graph with the same value.) Summing these expressions, we obtain

$$\begin{aligned}
b_a + b_b + b_c + b_d = {} & \xi(p^2 - 2p\cdot k - k^2)D(p+k) - 2\xi p^2\left(e^2\phi v + e^4\phi^2 v^2\right) \\
& + \left\{ \frac{e^6\phi^4 v^2}{2}\left[3(k+p)^2 - 3k^2 + \frac{(p+k)^4}{k^2} - \frac{k^4}{(p+k)^2}\right] + \frac{e^4\phi^2\xi^2}{2}[k^4 - (p+k)^4]\right\} \\
& + \cdots
\end{aligned} \tag{4.25}$$

where the dots denote terms that are either ξ-independent, of order p^3, or else proportional to m_2^2 and thus of higher order in e. Because of their antisymmetry under the interchange

$k^2 \longleftrightarrow (p+k)^2$, the contributions of the terms in curly brackets to the integral in Eq. (4.23) cancel. Inserting the remaining terms into the integral and keeping terms proportional to p^2, we find that

$$\begin{aligned} Z_{e^2} &= -ie^2\xi \int \frac{d^4k}{(2\pi)^4} \left[\frac{1}{D(k)} - \frac{2\,(e^2\phi v + e^4\phi^2 v^2)}{[D(k)]^2} \right] + \xi\text{−independent terms} \\ &= -ie^2\xi \int \frac{d^4k}{(2\pi)^4} \left[\frac{1}{D(k)} - \frac{\phi}{[D(k)]^2}\frac{\partial D(k)}{\partial \phi} \right] + \xi\text{−independent terms} \\ &= 2\frac{\partial C_{e^2}}{\partial \phi} + \xi\text{−independent terms}\,. \end{aligned} \tag{4.26}$$

(In going from the first to the second line, contributions proportional to $\partial m_2^2/\partial\phi$ have been neglected as being of higher order.)

Differentiating this with respect to the gauge-parameter ξ gives

$$\begin{aligned} \xi\frac{\partial Z_{e^2}}{\partial \xi} &= 2\frac{\partial}{\partial \phi}\left[C_{e^2} + ie^2\phi\xi \int \frac{d^4k}{(2\pi)^4} \frac{e^2\phi^2\xi m_2^2}{[D(k)]^2} \right] \\ &= 2\frac{\partial C_{e^2}}{\partial \phi} + O(e^4)\,. \end{aligned} \tag{4.27}$$

This verifies the identity (3.7).

5 Concluding remarks

In this paper we have have shown how the Nielsen identity that describes the gauge dependence of the effective action can be converted into an infinite series of identities, one for each of the functions appearing in the derivative expansion of the effective action. Using these identities, we have shown, to leading nontrivial order, that one obtains a gauge-independent result for the bubble nucleation rate even in theories where the calculation of this rate involves the gauge-dependent higher order contributions to the effective action. This provides one more example to show that the gauge-dependence of the effective action does not prevent it from being a useful tool for obtaining gauge-independent physical results.

As an explicit example, we have verified the identities for the ξ-dependence of $V^{\rm eff}(\phi)$ and $Z(\phi)$ in the class of gauges defined by the gauge-fixing function (4.4). In fact, these gauges actually depend on a second parameter, v. (Note that nothing in our calculations requires that v be equal to the vacuum expectation value of ϕ.) Working from Eq. (2.11), we find that

$$v\frac{\partial V^{\text{eff}}}{\partial v} = C^v \frac{\partial V^{\text{eff}}}{\partial \phi} \tag{5.1}$$

and

$$v\frac{\partial Z}{\partial v} = C^v \frac{\partial Z}{\partial \phi} + 2D^v \frac{\partial V^{\text{eff}}}{\partial \phi} + 2Z\frac{\partial C^v}{\partial \phi}\,. \tag{5.2}$$

where $C^v(\phi)$ and $D^v(\phi)$ are obtained from the derivative expansion of

$$H^v_{\Phi_1}[\phi(z), y] = -ie^2 v \int d^4x \langle (\Phi_2(y)\,\eta(y)\,\bar{\eta}(x)\,\Phi_2(x)\rangle\,. \tag{5.3}$$

(Note the absence of the factor of 1/2 relative to Eq. (4.5).) In particular, the leading contributions to these identities comes from

$$C_{e^2} = -ie^2 v \int \frac{d^4k}{(2\pi)^4} \frac{(k^2 - \xi e^2 \phi^2)}{(k^2 + e^2 v\phi)\,D(k)} \tag{5.4}$$

The Fermi gauges, defined by $F = \partial_\mu A^\mu$, can be obtained from the R_ξ gauges we have considered by setting $v = 0$. However, because of the infrared divergences that afflict these gauges, the limit $v \to 0$ is somewhat nontrivial and the verification of the identities for these gauges must be done separately [10]. To see the cause of these difficulties, note that our assumptions about the magnitude of λ and m^2 imply that if $v = 0$ the zeros of $D(k)$ occur at values of k^2 of order $e^4\phi^2$. This has the effect of making some two-loop graphs (beyond those resummed by the conversion of $\tilde{m}_i^2$ to m_i^2) comparable to one-loop graphs. For example, in the calculation of the quantity $\partial C_{e^2}/\partial\phi$ on the left hand side of the Nielsen identity for Z_{e^2}, the terms involving $\partial m_2^2/\partial\phi$ are no longer higher order. The corresponding terms on the right hand side of the identity come from contributions to Z_{e^2} due to two-loop graphs and one-loop graphs with vertices proportional to λ; both types of contributions can be neglected for generic nonzero values of v.

A Appendix

In this appendix we show that, although individual two-loop graphs give gauge-dependent contributions to $V^{\text{eff}}_{e^6}$, their sum is ξ-independent. The first step is to identify the relevant graphs. All graphs with vertices proportional to λ give higher order contributions, and so can be omitted. Similarly, any graph with a loop containing only Φ_1 propagators is proportional to a power of m_1^2 and hence of higher order. Finally, there is a two-loop graph containing a ghost loop, but it is manifestly gauge-independent. The only nonzero graphs remaining are shown in Fig. 3.

It is convenient to decompose the photon propagators into transverse and longitudinal parts, and to examine separately the contributions from each. Consider first the contributions involving transverse photon propagators. The part of graph a involving two such propagators and the part of graph e involving one transverse photon have already been included in the one-loop calculation by the resummation that converted the $\tilde{m}_i^2$ to the m_i^2, and hence should be omitted. This leaves the portions of graphs a, b, and d that involve only a single transverse photon each. The contribution of these to the effective potential is

$$\begin{aligned}
-e^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} & \frac{g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}}{(k^2 - e^2\phi^2)\, D(p)\, [(k+p)^2 - m_1^2]} \Bigg[2e^2\phi^2 \left(\xi p^2 - e^2 v^2\right) \left(\frac{p_\mu p_\nu}{p^2}\right) \\
& - \frac{1}{2}(p^2 - \xi e^2\phi^2)\,(k+2p)_\mu (k+2p)_\nu - 2e^2\phi(\xi\phi + v)(k+2p)_\mu p_\nu \Bigg] \\
&= 2e^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \frac{1}{(k^2 - e^2\phi^2)\,[(k+p)^2 - m_1^2]} \left(1 - \frac{(p\cdot k)^2}{k^2p^2}\right) + \cdots
\end{aligned} \tag{A.1}$$

where the dots represent terms proportional to m_2^2. Not only is this result independent of ξ, but examination of the integrals shows it to be in fact of order e^8.

This leaves us with the terms involving only longitudinal photons. Let us denote the corresponding contribution from graph j by J_j. For the first four graphs this may be written in the form

$$J_j = -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \frac{1}{[(k+p)^2 - m_1^2]\, D(k)\, D(p)}\, a_j(k,p)\,. \tag{A.2}$$

Omitting terms proportional to m_2^2, whose effects are of higher order, one finds that

$$\begin{aligned}
a_a &= e^2\phi^2 \left(\xi k^2 - e^2v^2\right)\left(\xi p^2 - e^2v^2\right)(k\cdot p)^2\, k^{-2}\, p^{-2} \\
a_b &= -\frac{1}{2}\left(\xi k^2 - e^2v^2\right)\left(p^2 - \xi e^2\phi^2\right)\left(k^2 + 2k\cdot p\right)^2 k^{-2} \\
a_c &= \frac{e^2}{2}\left(\xi\phi + v\right)^2\left(k^2 + 2k\cdot p\right)\left(p^2 + 2k\cdot p\right) \\
a_d &= -2e^2\phi\left(\xi\phi + v\right)\left(\xi k^2 - e^2v^2\right)\left(k^2 + 2k\cdot p\right)(k\cdot p)\, k^{-2}\,.
\end{aligned} \tag{A.3}$$

The sum of these is

$$\begin{aligned}
a_a + a_b + a_c + a_d = &-\frac{1}{2}(p+k)^2 \left[(k^2 - \xi e^2\phi^2)\left(\xi p^2 - e^2v^2\right) + (k\cdot p)\left(2p^2\xi - e^2\phi^2\xi^2\right) \right] \\
&+ \frac{\xi}{2} D(p)\left[(p+k)^2 - p^2\right] + A(k,p) + \cdots
\end{aligned} \tag{A.4}$$

where $A(k,p)$ is an antisymmetric function of k and p and the dots represent terms that are either proportional to m_2^2, and thus of higher order, or else ξ-independent.

When this sum is inserted back into the integral, the term containing $A(k,p)$ vanishes because of its antisymmetry. The remaining terms give

$$\begin{aligned} J_a + J_b + J_c + J_d = & \frac{e^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \left[1 + \frac{m_1^2}{[(k+p)^2 - m_1^2]} \right] \frac{1}{D(k)\,D(p)} \\ & \times \Big[(k^2 - \xi e^2 \phi^2)(\xi p^2 - e^2 v^2) + (k \cdot p)(2p^2 \xi - e^2 \phi^2 \xi^2) \Big] \\ & - \frac{e^2 \xi}{2} \int \frac{d^4k}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \frac{[(p+k)^2 - p^2]}{[(k+p)^2 - m_1^2]\,D(k)} + \cdots . \end{aligned} \tag{A.5}$$

The terms in the first integral that are proportional to m_1^2 are at least $O(e^8)$ and can be omitted. In the second integral, let us make the change of variable $p \to p - k$. The resulting integral is then clearly the product of two integrals, one of which is proportional to m_1^2, and is thus also higher order. Hence,

$$J_a + J_b + J_c + J_d = \frac{e^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{(k^2 - \xi e^2 \phi^2)}{D(k)} \int \frac{d^4p}{(2\pi)^4} \frac{(\xi p^2 - e^2 v^2)}{D(p)} + \cdots . \tag{A.6}$$

This last expression is precisely equal to $-J_e$. Hence, the two-loop contribution to $V_{e^6}^{\rm eff}$ is ξ-independent, as was claimed.

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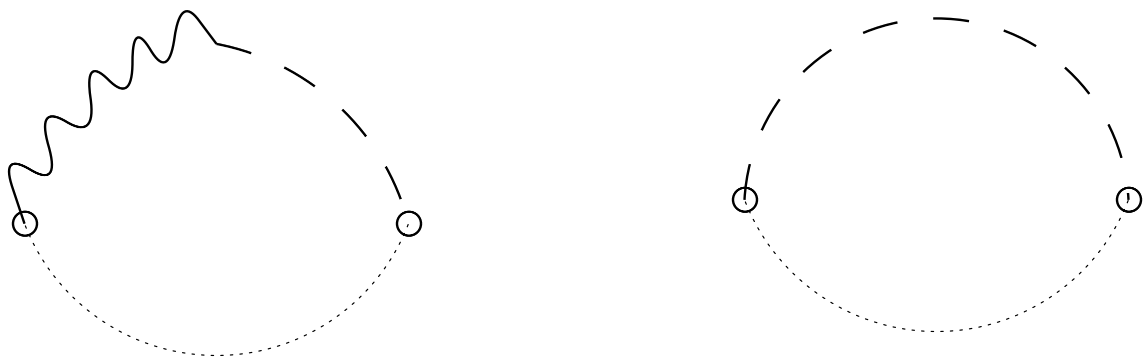

Figure 1: The two graphs that contribute to C_{e^2}. Photon, Φ_2, and ghost propagators are indicated by wiggly, long-dashed, and short-dashed lines, respectively.

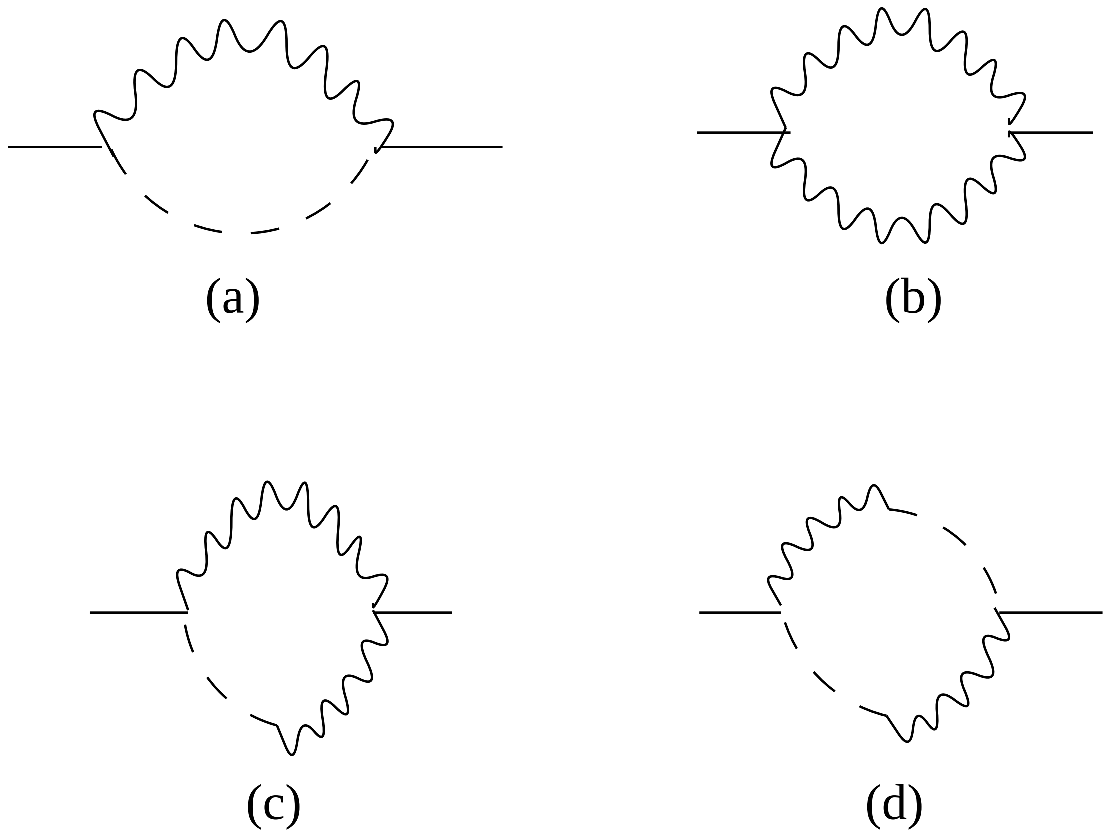

Figure 2: The graphs that contribute to Z_{e^2}. Solid lines represent Φ_1 propagators, with all other propagators shown as in Fig. 1.

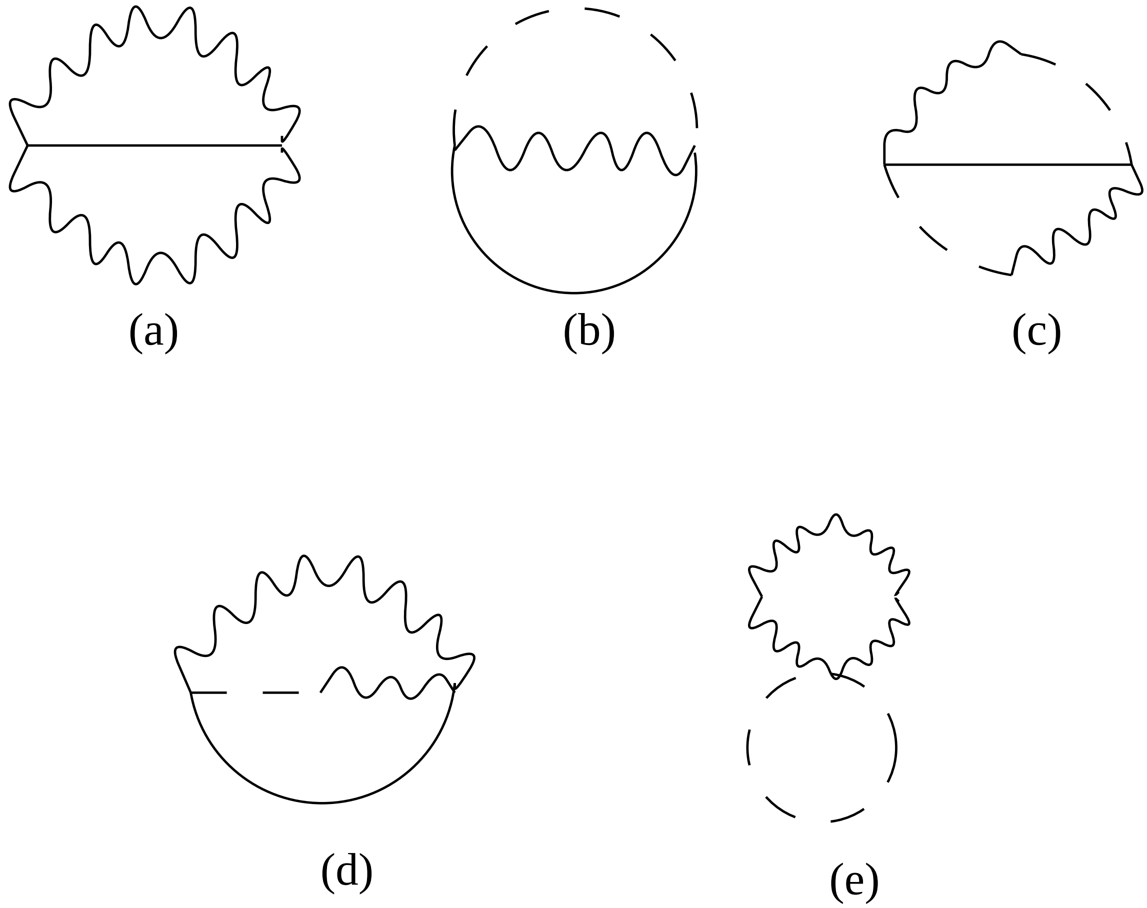

Figure 3: The two-loop graphs that contribute to V_{e^6}.