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One-Loop Quadratic Divergences of the Dual Non-Linear Sigma Model in Four-Dimensional Spacetime

by

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Abstract

Following a review of the dual description of the non-linear sigma model we investigate the one-loop quadratic divergences. We use the covariant background field method for the general case and apply the results to the important example of $SU(2)$

1. Introduction

Following the recent dramatic work of Seiberg and Witten [1] in which a version of Olive-Montonen duality [2] was found within the context of $N=2$ supersymmetry, there has been strong motivation to investigate other possible dual models. One upshot of duality is the inversion of various coupling terms which may appear in a Lagrangian with the effect that in some sense the strong and weak coupling regimes of a theory become interchanged.

The non-linear sigma model has been known for some time to adequately describe the dynamics of pions [3] (and mesons in general) at energies which are small compared with the inverse confinement radius of QCD. The non-linear sigma model is inherently versatile and finds applications to a broad range of physics; in particular it has much to say about the symmetry breaking sector of the Standard Model and in this context has been much investigated by Dobado and Herrero [4] and Donoghue and Ramirez [5]. With the promise of interchanging high and low energy regimes, a study of the dual description of the non-linear sigma model could be particularly interesting phenomenologically.

Following the work of Buscher [6] and others [7] an algorithm exists for the construction of the dual of the non-linear sigma model in any number of dimensions. The algorithm consists of gauging a symmetry (an isometry) of the action by introducing non-propagating gauge fields whose field strength is constrained to vanish by means of a Lagrange multiplier. Integrating over the gauge fields instead yields the dual theory where now the Lagrange multiplier has been promoted to a full dynamical field.

In an earlier paper [8] we applied this algorithm to a general four-dimensional non-linear sigma model and found that the Lagrange multiplier was constrained to be a rank two antisymmetric tensor field, the dual model being composed of two real scalar fields (the pions $\pi^{\pm}$) and the rank two tensor field which was interpreted as the dual field associated with π^3. The interesting points of comparison between the original model and the dual model are expected to appear in the renormalisation behaviour, and it is this which we now begin to investigate by looking at the one-loop quadratic divergences. This should be regarded as nothing more than just a first step towards a more general study. In particular we shall have nothing to say about the logarithmic contributions for various reasons which shall become apparent as we proceed.

We begin by briefly reviewing the dual model.

2. A Dual Description of the Non-Linear Sigma Model in Four-Dimensional Spacetime

We take a target space whose coordinates, ρ^a, we split as $\rho^a = (\phi^i, \theta)$ and, without loss of generality, define a four-dimensional non-linear sigma model on this space as

$$S(\theta, \phi) = \int d^4x \left[\frac{1}{2}G(\phi)\,\partial^\mu\theta\partial_\mu\theta + G_i(\phi)\,\partial^\mu\phi^i\partial_\mu\theta + \frac{1}{2}g_{ij}(\phi)\,\partial^\mu\phi^i\partial_\mu\phi_j\right]. \tag{1}$$

This action is invariant under the global transformation $\theta \to \theta + \alpha$ and the duality emerges upon minimally gauging this global symmetry and adding a Lagrange multiplier term constraining the gauge field to be pure gauge [7]. We therefore consider

$$S(\theta, \phi) = \int d^4x \left[\frac{1}{2}G(\phi)\,D_\mu\theta D_\mu\theta + G_i(\phi)\,\partial_\mu\phi^i D_\mu\theta + \frac{1}{2}g_{ij}(\phi)\,\partial_\mu\phi^i\partial_\mu\phi^j - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\lambda_{\rho\sigma}F_{\mu\nu}\right] \tag{2}$$

where $D_\mu\theta = \partial_\mu\theta + A_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. $\lambda_{\rho\sigma}$ is the rank two antisymmetric tensor field referred to in the introduction and appears here as a Lagrange multiplier enforcing the pure gauge condition. As hinted earlier, the dual model appears upon integrating out the gauge field and promoting the Lagrange multiplier to be a fully dynamical gauge field. After performing this manipulation we obtain [8]

$$S(\lambda, \phi) = \int d^4x \left[\frac{1}{2}G_{ij}\partial_\mu\phi^i\partial^\mu\phi^j - \frac{1}{2G}\epsilon_{\mu\nu\rho\sigma}\partial_\nu\lambda_{\rho\sigma}\epsilon_{\mu\nu'\rho'\sigma'}\partial_{\nu'}\lambda_{\rho'\sigma'} + \frac{1}{G}\epsilon_{\mu\nu\rho\sigma}\partial_\nu\lambda_{\rho\sigma}G_i\partial_\mu\phi^i\right] \tag{3}$$

where G_{ij} is given from the tensors appearing in (1) by

$$G_{ij} = g_{ij} - \frac{1}{G}G_iG_j. \tag{4}$$

The dual action is therefore seen to be related to (1) via the interchange

$$\epsilon_{\mu\nu\rho\sigma}\partial_\nu\lambda_{\rho\sigma} \to G_i\partial_\mu\phi^i \tag{5}$$

which is in the spirit of the duality encountered in electromagnetism between the electric and magnetic fields. The apparent problem associated with the available degrees of freedom (six) of the rank two antisymmetric field is discussed in detail in [8] and [9] where the gauge freedom is noted to allow just one physical degree of freedom - consistent with its interpretation as the dual of a real scalar field.

In [8] we studied some phenomenology of this model and found that it reproduced known scattering amplitudes for the charged pions as derived from the original model. Here we consider the more general problem of the divergent behaviour of the model.

3. One-Loop Quadratic Divergences of the Dual Model

Having arrived at the result (3) for the dual model with the explicit inversion of various coupling terms, we now reparameterise the action such that all coupling terms are upstairs. This slightly eccentric decision is made purely for aesthetic reasons to make the following formulae easier to read.

We rewrite the Lagrangian (3) in the general form

$$\mathcal{L} = \frac{1}{2}G_{ij}\partial^\mu\phi^i\partial_\mu\phi^j + \epsilon_{\mu\nu\rho\sigma}G_i\partial^\mu\phi^i\partial_\nu A_{\rho\sigma} - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu\nu'\rho'\sigma'}G\partial_\nu A_{\rho\sigma}\partial_{\nu'}A_{\rho'\sigma'}. \tag{6}$$

To investigate the quadratic divergences we first expand the action using the background field method. This manifestly covariant [10] procedure is a well known computational tool in quantum field theories which allows us to compute radiative corrections whilst maintaining manifestly the symmetries of the theory under consideration [11]. The definition of a quantum field which translates as a vector, and the use of Riemann normal coordinates, leads to an expansion in which each term depends only on the tensors on the manifold. This geometric property of the expansion implies a corresponding geometric property of the counterterms obtained by calculating loop diagrams with the covariant background field vertices.

Using the background field method we expand the Lagrange density as

$$\mathcal{L}[\phi] = \mathcal{L}[\phi_{\rm cl}] + D\mathcal{L}|_{\phi_{\rm cl}} + \frac{1}{2}\, DD\mathcal{L}|_{\phi_{\rm cl}} + ... \tag{7}$$

with the action of the covariant derivatives being given by [12]

$$D\partial_\mu\phi^i = \nabla_\mu\xi^i \qquad D\nabla_\mu\xi^i = R^i_{jkl}\partial_\mu\phi^l\xi^j\xi^k$$
$$DG = \nabla_i G\xi^i \qquad DG_i = \nabla_j G_i\xi^j \qquad D\nabla_k\nabla_j G_i\xi^k. \tag{8}$$

Here ξ^i denote the quantum fluctuations of the ϕ^i fields whilst for the rank two antisymmetric tensor field, $A_{\rho\sigma}$, the quantum fluctuations are denoted $\lambda_{\rho\sigma}$ and the action of the covariant derivatives is

$$D\partial_\nu A_{\rho\sigma} = \partial_\nu\lambda_{\rho\sigma} \tag{9}$$

Upon expanding the Lagrange density (6) and inserting the expressions (8) where appropriate, we arrive at a propagator term for the ξ^i fields of the form

$$G_{ij}\nabla_\mu\xi^i\nabla_\mu\xi^j. \tag{10}$$

This can be manipulated into a useable form by moving to the tangent space via the following definitions [13]

$$\xi^a = e^a_i\xi^i \qquad \xi^i = E^i_a\xi^a, \tag{11}$$

allowing us to write

$$G_{ij}\nabla_\mu\xi^i\nabla_\mu\xi^j \to e^a_i e^b_j \eta_{ab}\nabla_\mu\xi^i\nabla_\mu\xi^j = \eta_{ab}\nabla_\mu\xi^a\nabla_\mu\xi^b \tag{12}$$

where use has been made of the relations $G_{ij} = e^a_i e^b_j \eta_{ab}$ and $\nabla_\mu e^a_i = 0$. Having made the shift to the tangent space our covariant derivative is now defined via the spin connection, ω^a_{ib},

$$\nabla_\mu\xi^a = \partial_\mu\xi^a + \omega^a_{ib}\xi^b\partial_\mu\phi^i. \tag{13}$$

Applying this transformation to all ξ^i terms we parameterise the Lagrangian for the quantum fields as

$$\begin{aligned}\mathcal{L} = \frac{1}{2}DDS &= \frac{1}{2}\eta_{ab}\partial^\mu\xi^a\partial_\mu\xi^b - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu\nu'\rho'\sigma'}\partial_\nu\widetilde{\lambda}_{\rho\sigma}\partial_{\nu'}\widetilde{\lambda}_{\rho'\sigma'} \\ &+ A_{ab}\xi^a\xi^b + B^\mu_{ab}\xi^a\partial_\mu\xi^b + C^{\mu\nu\rho}_a\xi^a\partial_\mu\widetilde{\lambda}_{\nu\rho} + D^{\mu\nu}_a\xi^a\widetilde{\lambda}_{\mu\nu} \\ &+ E^{\mu\rho\sigma}_a\partial_\mu\xi^a\widetilde{\lambda}_{\rho\sigma} + F^{\mu\nu\rho\sigma}\widetilde{\lambda}_{\mu\nu}\widetilde{\lambda}_{\rho\sigma} + G^{\mu\nu\rho\sigma\tau}\widetilde{\lambda}_{\mu\nu}\partial_\rho\widetilde{\lambda}_{\sigma\tau}\end{aligned} \tag{14}$$

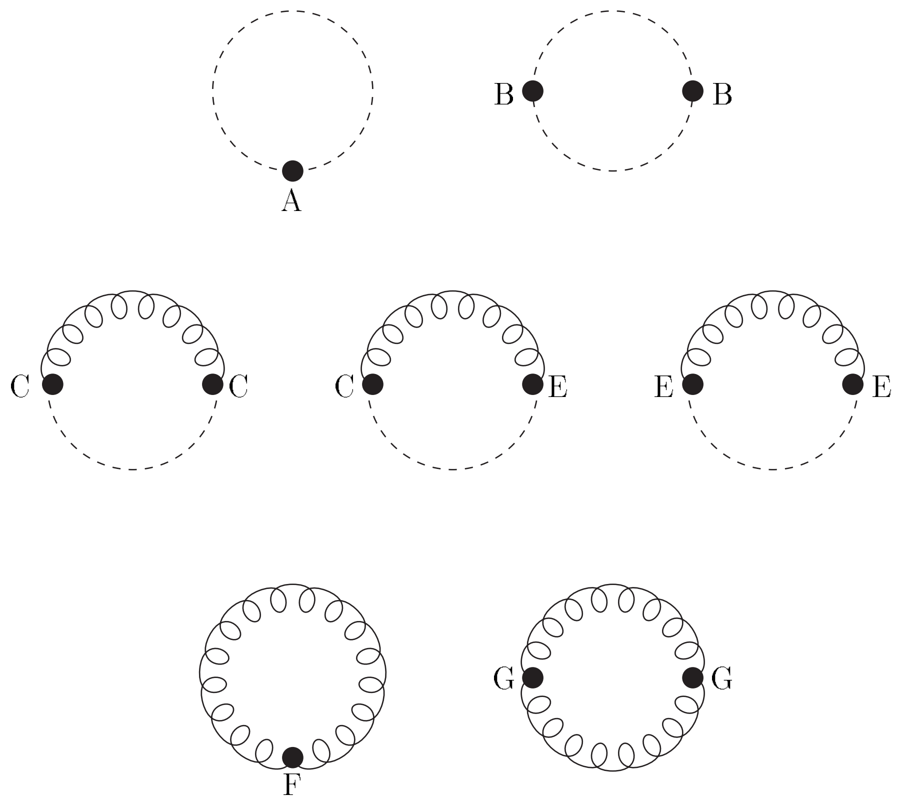
where, to get the kinetic term for the $\lambda_{\rho\sigma}$ fields into a convenient form, we have introduced $\widetilde{\lambda}_{\rho\sigma} = \sqrt{G}\lambda_{\rho\sigma}$ with the suitably covariantised derivative

$$\partial_\nu\lambda_{\rho\sigma} \to \frac{1}{\sqrt{G}}\partial_\nu\widetilde{\lambda}_{\rho\sigma} - \frac{1}{2}\frac{1}{G\sqrt{G}}\nabla_m G\partial_\nu\phi^m\widetilde{\lambda}_{\rho\sigma}. \tag{15}$$

Using this parameterisation we obtain

$$\begin{aligned}2A_{ab} &= \partial^\mu\phi^i\partial_\mu\phi^j\left(\eta_{cd}\omega^c_{ia}\omega^d_{jb} + R_{jmki}E^m_a E^k_b\right) \\ &+ \epsilon_{\mu\nu\rho\sigma}\partial_\nu A_{\rho\sigma}\left(\nabla_k\nabla_j G_i E^j_a E^k_b + G_m R^m_{jki}E^j_a E^k_b + 2\nabla_j G_m\omega^c_{ib}E^m_c E^j_a\right) \\ &- \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu\nu'\rho'\sigma'}\partial_\nu A_{\rho\sigma}\partial_{\nu'}A_{\rho'\sigma'}\nabla_j\nabla_i G E^i_a E^j_b \\ \\ B^\mu_{ac} &= \left(\eta_{cb}\omega^b_{ia}\partial_\mu\phi^i + \epsilon_{\mu\nu\rho\sigma}\partial_\nu A_{\rho\sigma}\nabla_j G_i E^i_c E^j_a\right) \\ \\ C^{\nu\rho\sigma}_a &= \frac{1}{\sqrt{G}}\epsilon_{\mu\nu\rho\sigma}\left[\left(2\nabla_{[j}G_{i]} + \frac{1}{2G}G_j\nabla_i G\right)E^j_a\partial_\mu\phi^i - \epsilon_{\mu\nu'\rho'\sigma'}\partial_{\nu'}A_{\rho'\sigma'}\nabla_i G E^i_a\right] \\ \\ D^{\rho\sigma}_a &= -\frac{\nabla_j G\partial_\nu\phi^j}{2G\sqrt{G}}\epsilon_{\mu\nu\rho\sigma}\left[\left(\nabla_m G_i E^m_a + G_m\omega^d_{ia}E^m_d\right)\partial_\mu\phi^i - \epsilon_{\mu\nu'\rho'\sigma'}\partial_{\nu'}A_{\rho'\sigma'}\nabla_i G E^i_a\right] \\ \\ E^{\mu\rho\sigma}_a &= -\frac{1}{2}\frac{1}{G\sqrt{G}}\epsilon_{\mu\nu\rho\sigma}\partial_\nu\phi^i G_j\nabla_i G E^j_a \\ \\ F^{\rho\sigma\rho'\sigma'} &= -\frac{1}{8G^2}\epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu\nu'\rho'\sigma'}\partial_\nu\phi^i\partial_{\nu'}\phi^j\nabla_i G\nabla_j G \\ \\ G^{\rho'\sigma'\nu\rho\sigma} &= \frac{1}{2G}\epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu\nu'\rho'\sigma'}\partial_{\nu'}\phi^i\nabla_i G\ .\end{aligned} \tag{16}$$

Using cut-off regularisation at the scale Λ, the Feynman diagrams which contribute $\sim \Lambda^2$ divergences are (dashed lines = ξ, coiled lines = λ):



Evaluating these diagrams we finally obtain the one-loop effective Lagrangian

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\partial^\mu \phi^i \partial_\mu \phi^j G^R_{ij} + \epsilon_{\mu\nu\rho\sigma}\partial_\nu A_{\rho\sigma}\partial_\mu \phi^i G^R_i - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu\nu'\rho'\sigma'}\partial_\nu A_{\rho\sigma}\partial_{\nu'} A_{\rho'\sigma'} G^R \tag{17}$$

with the renormalised tensors

$$\begin{aligned}
G^R_{ij} &= G_{ij} + J\left[R_{ij} - \frac{6}{G}G^{mn}\nabla_{[n}G_{i]}\nabla_{[m}G_{j]}\right] \\
G^R_i &= G_i - \frac{J}{2}\left[G^{mn}\nabla_n\nabla_m G_i - G^j R_{ji} - \frac{3}{G}G^{mn}\nabla_m G \nabla_{[n}G_{i]}\right] \\
G^R &= G + \frac{J}{2}\left[G^{mn}\left(\frac{3}{G}\nabla_m G - \nabla_m\right)\nabla_n G - \nabla_j G_i \nabla_m G_n \left(G^{mj}G^{in} - G^{jn}G^{im}\right)\right]
\end{aligned} \tag{18}$$

and

$$J = \int_0^\Lambda d^4k \frac{1}{k^2} = -\frac{1}{16\pi^2}\Lambda^2 \tag{19}$$

where we have Wick rotated and $d^4k = \frac{1}{2}k^2 \sin^2\theta \sin\phi dk^2 d\theta d\psi$ is the integration measure in Euclidean space.

This is our main result and represents the general one-loop corrected effective Lagrangian for the Abelian dualed sigma model. [1] Having derived the dual version of the non-linear sigma model and the very general renormalisation to one-loop, we now specialise to the important case of $SU(2)$.

4. Application to $SU(2)$

We take as our starting point the non-linear sigma model parameterised by the matrix field $U(x)$ belonging to the quotient space $SU(2)_L \otimes SU(2)_R / SU(2)_{L+R}$,

$$U(x) = \exp\left(i\tau.\phi/\Lambda\right), \tag{20}$$

where ϕ^a, $a = 1,2,3$, are the Goldstone bosons associated with the symmetry breaking $SU(2)_L \otimes SU(2)_R \to SU(2)_{L+R}$, τ^a are the 2×2 Pauli matrices and Λ some energy scale. (When applied to the symmetry breaking sector of the Standard Model Λ is fixed at the scale of the Higgs VEV = 246GeV).

Under an $SU(2)_L \otimes SU(2)_R$ transformation the matrix U transforms as $U \to LUR^\dagger$ and the $SU(2)$ invariant sigma model can be written

$$\mathcal{L} = \frac{\Lambda^2}{4}\mathrm{Tr}\partial^\mu U \partial_\mu U^{-1}. \tag{21}$$

To obtain the dual version of this model we must first massage (21) into the form (1) which we achieve by separating out one field - the field which shall carry the Abelian symmetry. When this is done (see [8] for definitions) we recover the form of (1) with

$$\begin{aligned} g_{ij} &= \delta^{ij}\frac{\sin^2\Omega}{\Omega^2} + \frac{1}{\Omega^2\Lambda^2}\left[1 - \frac{\sin^2\Omega}{\Omega^2}\right]\pi^i\pi^j \\ G_i &= \epsilon_{ij}\frac{1}{\Lambda}\pi^j\frac{\sin^2\Omega}{\Omega^2} \qquad G = 1. \end{aligned} \tag{22}$$

Once in this form we use the rules given in (3) and (4) to write down the dual verison of the $SU(2)$ non-linear sigma model to $O(p^2)$

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial_\nu\lambda_{\rho\sigma}\epsilon_{\mu\nu'\rho'\sigma'}\partial_{\nu'}\lambda_{\rho'\sigma'} + \frac{1}{\Lambda}\epsilon_{\mu\nu\rho\sigma}\partial_\nu\lambda_{\rho\sigma}\epsilon_{ij}\pi^j\partial_\mu\pi^i\frac{\sin^2\Omega}{\Omega^2} \\ &+ \frac{1}{2}\partial^\mu\pi^i\partial_\mu\pi^i\frac{\sin^2\Omega}{\Omega^2}\left(1-\sin^2\Omega\right) + \frac{1}{2}\frac{1}{\Lambda^2\Omega^2}\left[1 - \frac{\sin^2\Omega}{\Omega^2} + \frac{\sin^4\Omega}{\Omega^2}\right](\pi.\partial_\mu\pi)^2, \end{aligned} \tag{23}$$

[1] We are only interested here in divergences $\sim \Lambda^2$ and ignore the logarithmic contributions which can only be sensibly interpreted in the full momentum expansion of which the sigma model just described is the $O(p^2)$ leading term.

where in the language of (6) we have

$$G_{ij} = \delta^{ij}\frac{\sin^2\Omega}{\Omega^2}\left[1-\sin^2\Omega\right] + \frac{\pi^i\pi^j}{\Lambda^2\Omega^2}\left[1-\frac{\sin^2\Omega}{\Omega^2}+\frac{\sin^4\Omega}{\Omega^2}\right]$$

$$G_i = \epsilon_{ij}\frac{1}{\Lambda}\pi^j\frac{\sin^2\Omega}{\Omega^2} \qquad G = 1. \tag{24}$$

We can now immediately use (17) to investigate the $\sim\Lambda^2$ divergences appearing at the one-loop level in the dual $SU(2)$ non-linear sigma model. Given the relations in (24) and inserting into (18) we arrive at the surprisingly compact expression

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial^\mu\pi^i\partial_\mu\pi^j G_{ij}(1-2J) - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu\nu'\rho'\sigma'}\partial_\nu\lambda_{\rho\sigma}\partial_{\nu'}\lambda_{\rho'\sigma'}(1-2J)\\ &+ \epsilon_{\mu\nu\rho\sigma}\partial_\nu\lambda_{\rho\sigma}\partial_\mu\pi^i G_i.\end{aligned} \tag{25}$$

This is the one-loop corrected effective $SU(2)$ Lagrangian for the Abelian dualed sigma model and can be compared with the results presented in [14] for the conventional non-linear sigma model.

5. Conclusions

We have presented a completely general discussion of the quadratic renormalisation behaviour of the dual non-linear sigma model at the one loop level and applied it to the important phenomenological case of $SU(2)$. This is the first step in a general study of the divergent behaviour of the dual non-linear sigma model where we hope to find a formalism which will facilitate a study of the infrared behaviour (fixed points) which in itself will enable a detailed comparison between this model and the standard non-linear sigma model.

The next stage in the programme will be to extend the investigations to accommodate the logarithmic divergences. Here there is a considerable increase in the number of Feynman diagrams which we need to consider to one loop (≈ 20) and in addition the infrared singularities which emerge as the rank two antisymmetric tensor fields go soft in certain diagrams demand careful attention. It is hoped to pursue this line of investigation in a subsequent paper.

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