Kinetic Approach to Fractional Exclusion Statistics

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Abstract

Abstract: We show that the kinetic approach to statistical mechanics permits an elegant and efficient treatment of fractional exclusion statistics. By using the exclusion-inclusion principle recently proposed [Phys. Rev. E **49**, 5103 (1994)] as a generalization of the Pauli exclusion principle, which is based on a proper definition of the transition probability between two states, we derive a variety of different statistical distributions interpolating between bosons and fermions. The Haldane exclusion principle and the Haldane-Wu fractional exclusion statistics are obtained in a natural way as particular cases. The thermodynamic properties of the statistical systems obeying the generalized exclusion-inclusion principle are discussed.

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Following Haldane's formulation [1] of a generalized Pauli exclusion principle, many papers have been recently devoted to the study of fractional exclusion statistics by interpolation of bosonic and fermionic distributions [2]. There is an intrinsic connection between these fractional statistics and the interpretation of the fractional quantum Halleffect [3] and anyonic physics [4,5]. Murthy and Shankar [6] generalized the Haldane statistics to infinite dimensions and showed [7] that the one dimensional bosons interacting through the two-body inverse square potential $V_{ij} = g(g-1)(x_i - x_j)^{-2}$ of the Calogero-Sutherland model obeys fractional exclusion statistics in the sense of the Haldane interpretation. Using the Thomas-Fermi method, Sen and Bhaduri [8] considered the particle exclusion statistics for the Calogero-Sutherland model in the presence of an external, arbitrary one-body potential. Isakov [9] extended the Calogero-Sutherland model in the case of particles interacting through a generic two-body potential V_{ij} and derived the related exclusion statistics. Rajagopal [10] obtained the von Neumann entropy per state of the Haldane exclusion statistics. Nayak and Wilczek [11] studied the low-temperature properties, the fluctuations and the duality of exclusions statistics. Murthy and Shankar [6] also computed, for quasi-particles, in the Luttinger model, the exclusion statistics parameter g related to the exchange statistical parameter α of the quantum phase $e^{i\pi\alpha}$, and showed [7] that the parameter g can be completely determined by the second virial coefficient in the high temperature limit.

Haldane defined the statistics g of a particle by

$$g = -\frac{d_{N+\Delta N} - d_N}{\Delta N} \quad , \tag{1}$$

where d_N is the single-particle Hilbert-space dimension, when the coordinates of $N-1$ particles are kept fixed. Thus the dimension of the Hilbert space for the single particle states is a finite and extensive quantity that depends on the number of particles in the system. One can choose this dimension as

$$d_N = K - g(N-1) \quad , \tag{2}$$

and the statistical weight or degeneracy factor is

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$$W = \frac{(d_N + N - 1)!}{N! \, (d_N - 1)!} \quad . \tag{3}$$

This equation is a simple interpolation between the number of ways of placing N identical bosons or fermions in K single-particle independent states confined to a finite region of matter. The expression of the mean occupation number $n = N/K$ has been obtained in an implicit form by Wu [2].

$$n = \frac{1}{w(E) + g} \quad , \tag{4}$$

with w satisfying the relation

$$w(E)^g \, [1 + w(E)]^{1-g} = e^{\beta(E - \mu)} \quad . \tag{5}$$

In the special cases $g = 0$ and $g = 1$, Eq.(4) yields, respectively, the Bose-Einstein (BE) and Fermi-Dirac (FD) distributions.

Ilinski and Gunn [12] criticize the identity of relations (1) and (3) of Haldane and, using (2), derive a different W constructing a statistical mechanics which is not in complete agreement with Wu statistical mechanics [2].

Acharya and Swamy [13] and Polychronakos [14] have studied, in addition to the Haldane exclusion statistics, new fractional statistics with appealing features as positive probabilities and analytical expressions for all termodynamic quantities. We wish to recall the first work on intermediate statistics published by Gentile at the beginning of 1940 [15].

In a previous work [16] we have considered the kinetics of particles in a phase space of arbitrary dimensions D, obeying an exclusion-inclusion principle. We obtained the statistical distributions of the particle system as the stationary state of a non-linear kinetic equation. A crucial point of this formalism is the definition of the transition probability which can be written in various forms, containing the effects of the inclusion-exclusion principle through the distribution function $n = n(t, \mathbf{v})$, which is the mean occupation number of the state $\mathbf{v}$.

Setting $\pi(t, \mathbf{v} \to \mathbf{u})$ the transition probability from the state $\mathbf{v}$ to the state $\mathbf{u}$, the evolution equation of the distribution function $n(t, \mathbf{v})$ can be written as

$$\frac{\partial n(t, \mathbf{v})}{\partial t} = \int \left[\pi(t, \mathbf{u} \to \mathbf{v}) - \pi(t, \mathbf{v} \to \mathbf{u}) \right] d^D u \quad . \tag{6}$$

In ref. [16] we postulated the following expression of the transition probability :

$$\pi(t, \mathbf{v} \to \mathbf{u}) = r(t, \mathbf{v}, \mathbf{v} - \mathbf{u}) \, \varphi[n(t, \mathbf{v})] \, \psi[n(t, \mathbf{u})] \quad , \tag{7}$$

where $r(t, \mathbf{v}, \mathbf{v} - \mathbf{u})$ is the transition rate, $\varphi[n(t, \mathbf{v})]$ is a function depending on the occupational distribution at the initial state $\mathbf{v}$ and $\psi[n(t, \mathbf{u})]$ depends on the arrival state. The function $\varphi(n)$ must obey the condition $\varphi(0) = 0$ because the transition probability is equal to zero if the initial state is empty. Furthermore, the function $\psi(n)$ must obey the condition $\psi(0) = 1$ because, if the arrival state is empty, the transition probability is not modified. If we choose $\varphi(n) = n$ and $\psi(n) = 1$ we obtain the standard linear kinetics. The function $\psi(n)$ defines implicitely the inclusion-exclusion principle enhancing or inhibiting the transition probability.

In this letter we use the kinetic approach outlined above to generate new fractional exclusion statistics. The approach is appropriate both for non-interacting and interacting particles. In the following, we derive general expressions linking the main statistical and thermodynamic quantities to the transition probabilities of the kinetic theory. We obtain implicitely a fractional statistics which interpolates between BE and FD distributions in a single-particle Hilbert space whose dimension is an arbitrary function of n. We show that the Haldane-Wu statistics can be derived from this one by considering Brownian particles and by demanding that the Hilbert space dimension be a linear function of n. Another particular case within the family of statistics here introduced yields a statistics interpolating among BE, FD and Maxwell-Boltzmann (MB) distributions.

Let us consider Eqs. (6) and (7) and limit the discussion to the one-dimensional velocity space (the extension to a D-dimensional phase space is straightforward). We limit ourselves to first neighbors interactions, this being equivalent to truncate the Moyal expansion given by Eq.(7) of Ref. [16] at the second order and we obtain the following, generalized, non-linear Fokker-Planck equation

$$\frac{\partial n(t, v)}{\partial t} = \frac{\partial}{\partial v} \left[\left(J(t, v) + \frac{\partial D(t, v)}{\partial v} \right) \varphi(n) \psi(n) \right.$$
$$\left. + D(t, v) \left(\psi(n) \frac{\partial \varphi(n)}{\partial n} - \varphi(n) \frac{\partial \psi(n)}{\partial n} \right) \frac{\partial n(t, v)}{\partial v} \right] \quad . \tag{8}$$

$J(t, v)$ and $D(t, v)$ are the drift and diffusion coefficients, respectively, and are given by the first and the second order moments of the transition rate [16]. Eq.(8) is a continuity equation for the distribution function $n = n(t, v)$

$$\frac{\partial n(t, v)}{\partial t} + \frac{\partial j(t, v, n)}{\partial v} = 0 \quad , \tag{9}$$

where the particle current $j = j(t, v, n)$ is given by

$$j = -D \frac{\varphi(n) \psi(n)}{\overline{(\Delta n)^2}} \left[\frac{\partial \epsilon}{\partial v} \overline{(\Delta n)^2} + \frac{\partial n}{\partial v} \right] \quad . \tag{10}$$

We have defined

$$\overline{(\Delta n)^2} = \left\{ \frac{\partial}{\partial n} \log \left[\frac{\varphi(n)}{\psi(n)} \right] \right\}^{-1} \quad , \tag{11}$$

and

$$\frac{\partial \epsilon}{\partial v} = \frac{1}{D(t, v)} \left(J(t, v) + \frac{\partial D(t, v)}{\partial v} \right) \quad . \tag{12}$$

The function $\epsilon = \epsilon(v)$ is an adimensional single particle energy defined up to an additive, arbitrary constant. This energy is appropriate both for non-interacting and interacting particles. Nayak and Wilczek [11] have examined the particular case $\epsilon(v) \propto v^l$ with l integer. The case $l = 2$ corresponds to Brownian particles, the drift and diffusion coefficients being given by

$$J = \gamma v, \qquad D = \frac{\gamma}{\beta m} \quad . \tag{13}$$

The quantity γ is a dimensional constant and ϵ is given by: $\epsilon = \beta(E - \mu)$ with $E = \frac{1}{2} m v^2$ being the kinetic energy. $\beta = 1/T$ ($k_B = 1$) is the inverse of the temperature and μ is the chemical potential.

In stationary conditions, the particle current vanishes: $j(\infty, v, n) = 0$. Eq.(10) becomes a omogeneous first order differential equation, which can be integrated easily:

$$\frac{\psi(n)}{\varphi(n)} = \exp(\epsilon) \quad . \tag{14}$$

When the functions $\varphi(n)$ and $\psi(n)$ are fixed, Eq.(14) gives the statistical distribution $n = n(\epsilon)$ of the system.

We stress that, if we use as variable the single particle energy ϵ, it is easy to verify that the above-defined function $\overline{(\Delta n)^2} = <n^2> - <n>^2$, is equal to the second order density fluctuation [17]

$$\overline{(\Delta n)^2} = -\frac{\partial n}{\partial \epsilon} \quad . \tag{15}$$

This relation is very important because it reveals that the mean fluctuation of the occupation number is the crucial quantity that determines the equilibrium distribution.

The statistical distribution $n(\epsilon)$ can be obtained by the maximum entropy principle, fixing the total number of particles and energy of the system and using the standard Lagrange multiplier method. Setting $\mathcal{S}(n) = S(N)/K$ the thermodynamic limit of the entropy per state, we have

$$\frac{\partial}{\partial n}[\mathcal{S}(n) - \epsilon n] = 0 \quad , \tag{16}$$

From Eqs. (14) and (16) it is possible to obtain the entropy in terms of the functions $\varphi(n)$ and $\psi(n)$ as

$$\frac{\psi(n)}{\varphi(n)} = \exp\left[\frac{\partial \mathcal{S}(n)}{\partial n}\right] \quad , \tag{17}$$

or in terms of $\overline{(\Delta n)^2}$

$$\overline{(\Delta n)^2} = -\left[\frac{\partial^2 \mathcal{S}}{\partial n^2}\right]^{-1} \quad . \tag{18}$$

By using the Boltzmann principle with the identification of the entropy $\mathcal{S} = \log \mathcal{W}$, it is possible to find the relation between $\mathcal{W}$ and the functions $\varphi(n)$ and $\psi(n)$ in the following expression

$$\frac{\psi(n)}{\varphi(n)} = \exp\left[\frac{1}{\mathcal{W}(n)}\frac{\partial \mathcal{W}(n)}{\partial n}\right] \quad . \tag{19}$$

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It is well known that the partition function $\mathcal{Z}$ per state

$$\mathcal{Z} = e^{-\beta\Omega} \quad , \tag{20}$$

is related to the mean density n by means of

$$n = -\frac{\partial}{\partial\epsilon} \log \mathcal{Z} \quad . \tag{21}$$

Taking into account Eqs. (11), (15), (20) and (21), we may write the thermodynamic potential Ω as a function of $\varphi(n)$ and $\psi(n)$

$$\beta\Omega = \int n \frac{\partial}{\partial n} \log \frac{\psi(n)}{\varphi(n)} \, dn \quad , \tag{22}$$

and the density fluctuation as

$$\overline{(\Delta n)^2} = -n \left[\frac{\partial}{\partial n} \beta\Omega\right]^{-1} \quad . \tag{23}$$

Now let us consider the case $\varphi(n) = n$; for the function $\psi(n)$ we choose the particular form

$$\psi = d^{-\frac{\partial d}{\partial n}} (n+d)^{1+\frac{\partial d}{\partial n}} \quad . \tag{24}$$

where the function $d = d(n)$ must satisfy the condition $d(0) = 1$. In this case, Eq.(14) which defines the statistics, becomes:

$$d^{-\frac{\partial d}{\partial n}} (n+d)^{1+\frac{\partial d}{\partial n}} = n\,e^{\epsilon} \quad . \tag{25}$$

The function $\mathcal{W}$ becomes

$$\mathcal{W} = \frac{(n+d)^{n+d}}{n^n \, d^d} \quad , \tag{26}$$

and the entropy

$$\mathcal{S} = -n[w \log w - (1+w) \log(1+w)] \quad , \tag{27}$$

where $w = d/n$, so we find the general form of the von Neumann entropy per state [10]. The thermodynamic potential Ω becomes

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$$\beta\Omega = \left(n\frac{\partial d}{\partial n} - d\right)\log\frac{n+d}{d} \quad , \tag{28}$$

and the partition function $\mathcal{Z}$ can be rewritten as

$$\mathcal{Z} = \left(\frac{n+d}{d}\right)^{d-n\frac{\partial d}{\partial n}} \quad , \tag{29}$$

while the fluctuation becomes

$$\overline{(\Delta n)^2} = \frac{n\,d\,(n+d)}{\left(d - n\frac{\partial d}{\partial n}\right)^2 - n\,d\,(n+d)\frac{\partial^2 d}{\partial n^2}\log\frac{n+d}{d}} \quad . \tag{30}$$

The quantity d is an arbitrary function and we can use it to define a family of exclusion statistics. If we select a particular form of d, we observe that this one can depend not only on n but also on a parameter g: $d = d(g, n)$. In this way Eq.(25) defines, varying $g \in [0, 1]$, a fractional statistics. If we require that the statistics interpolates between BE and FD distributions we must set the two conditions: $d(0, n) = 1$ and $d(1, n) = 1 - n$. These two conditions imply that the single-particle Hilbert-space dimension be that of the Bose space when $g = 0$ and that of the Fermi one when $g = 1$. Let us call $S(N) = K\mathcal{S}(n)$ and $W(N)$ the entropy and statistical weight of the system of N particles lying in K states. From the Boltzmann principle $S = \log W$ we obtain $\mathcal{W} = W^{1/K}$. It is easy to see that $\mathcal{W}$, given by Eq.(26), is the thermodynamic limit of the statistical weight given by Eq. (3), where, this time, $\mathrm{d}_N = \mathrm{d}(g, N, K)$ is the single-particle Hilbert- space dimension, is an arbitrary function and admits as thermodynamic limit $d(g, n) = \mathrm{d}(g, N, K)/K$. If we consider a statistics interpolating between bosons and fermions, the conditions to which $d = d(g, n)$ must satisfy when $g = 0, 1$ can be derived in the thermodynamic limit from the two conditions $\mathrm{d}(0, N, K) = K$ and $\mathrm{d}(1, N, K) = K - N + 1$.

The Haldane-Wu choice of d given by Eq.(2) implies

$$d = 1 - g\,n \quad , \tag{31}$$

and requires that $\psi(n)$ be given by

$$\psi(n) = [1 - g\,n]^g\,[1 + (1 - g)\,n]^{1-g} \quad . \tag{32}$$

Equation (25) becomes

$$[1 - g\,n]^g\,[1 + (1 - g)\,n]^{1-g} = n\,e^\epsilon \tag{33}$$

and we obtain, in the case of Brownian particles, the statistics introduced by Haldane [1] and by Wu [2]. The partition function $\mathcal{Z}$ and the density fluctuation are given by

$$\mathcal{Z} = \frac{1 + (1 - g)n}{1 - gn} \quad . \tag{34}$$

$$\overline{(\Delta n)^2} = n\,(1 - gn)\,[1 + (1 - g)n] \quad . \tag{35}$$

As a second example of fractional statistics we consider the statistics defined by:

$$n = \frac{1}{\exp(\epsilon) - \kappa} \quad , \tag{36}$$

studied extensively in ref. [13,14,16]. For $\kappa = -1$, 0 and 1 one obtains the FD, MB and BE statistical distributions, respectively. In ref. [13] it is shown that the parameter κ is a function of the exchange statistical parameter α appearing in the quantum phase $e^{i\pi\alpha}$. This statistics is generated from

$$\psi(n) = 1 + \kappa n \quad . \tag{37}$$

In this case the partition function can be written as

$$\mathcal{Z} = (1 + \kappa n)^{1/\kappa} \quad , \tag{38}$$

and it is easy to verify that the density fluctuation $\overline{(\Delta n)^2}$ and the entropy $\mathcal{S}$ are given by

$$\overline{(\Delta n)^2} = n(1 + \kappa n) \quad , \tag{39}$$

$$\mathcal{S} = \frac{1}{\kappa}(1 + \kappa n)\log(1 + \kappa n) - n\log n \quad , \tag{40}$$

while the statistical weight $\mathcal{W}$ becomes

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$$\mathcal{W} = \frac{(1 + \kappa n)^{(1+\kappa n)/\kappa}}{n^n} \quad , \tag{41}$$

and can be obtained as the thermodynamic limit of the statistical weight:

$$W = |\kappa|^N \frac{\left(\frac{K}{|\kappa|}\right)!}{N! \left(\frac{K}{|\kappa|} - N\right)!} \quad , \quad \kappa < 0 \quad , \tag{42}$$

$$W = \frac{K^N}{N!} \quad , \quad \kappa = 0 \quad , \tag{43}$$

$$W = \kappa^N \frac{\left(\frac{K}{\kappa} + N - 1\right)!}{N! \left(\frac{K}{\kappa} - 1\right)!} \quad , \quad \kappa > 0 \quad . \tag{44}$$

In conclusion, we have shown that the effects of the exclusion-inclusion principle can be taken into account in the particle kinetics by means of a proper definition of the transition probability. We have derived a non-linear evolution equation which admits as steady solutions the exclusion statistical distributions. We have linked the main thermodynamic properties of the system with the function $\psi(n)$ which generates the exclusion statistics. As first application of the theory we have considered a family of exclusion statistics in which the single-particle Hilbert-space dimension is an arbitrary function of the mean occupational number and contains, as particular case, the Haldane-Wu statistics. As second application we have considered an exclusion statistics interpolating among FD, MB and BE distributions.

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