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Interaction-Round-a-Face Models with Fixed Boundary Conditions: The ABF Fusion Hierarchy

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Abstract

We use boundary weights and reflection equations to obtain families of commuting double-row transfer matrices for interaction-round-a-face models with fixed boundary conditions. In particular, we consider the fusion hierarchy of the Andrews-Baxter-Forrester models, for which we find that the double-row transfer matrices satisfy functional equations with an $su(2)$ structure.

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1. Introduction

1.1 Overview

Two-dimensional lattice spin models in statistical mechanics have traditionally been solved by imposing periodic boundary conditions on the rows of the lattice. The Yang-Baxter equation, together with such boundary conditions, then leads to families of commuting row transfer matrices and hence solvability [1]. However, the work of Sklyanin [2] shows that, by using reflection equations, it is also possible to construct commuting double-row transfer matrices for vertex models with open boundary conditions. In this paper, we present a scheme, motivated by Sklyanin's formalism for open boundaries and Baxter's correspondence between vertex and interaction-round-a-face (IRF) models [3, 1], for obtaining solvable IRF models with fixed boundary conditions.

Although the usual bulk quantities of physical interest are independent of the boundary conditions in the thermodynamic limit, there are many surface properties, such as the interfacial tension, which are also important. Moreover, at criticality, the conformal spectra of lattice models do depend on the boundary conditions [4]. For these reasons it is of interest to study lattice models with non-periodic boundary conditions.

The layout of this paper is as follows. In the remainder of this section, we discuss related work on solvability with open boundaries and outline the formalism for vertex models. In

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Section 2, we present the general procedure for obtaining commuting double-row transfer matrices for IRF models with fixed boundary conditions and then specialise to the case of Andrews-Baxter-Forrester (ABF) [5] models. In Section 3, we consider fusion of IRF models with fixed boundary conditions and concentrate on the ABF fusion hierarchy. We conclude with a discussion of future work and three appendices, in which we prove some of the results used in the main text.

1.2 Background

Reflection equations were introduced by Cherednik [6] as a means of obtaining factorisable scattering matrices for particles on a semi-infinite line. Sklyanin [2] then considered these equations in the context of one-dimensional quantum spin chains and showed that they could be used to obtain integrable systems with non-periodic boundary conditions. When translated into the context of two-dimensional lattice models in statistical mechanics, Sklyanin's formalism provides a scheme for obtaining exactly solvable vertex models with open boundary conditions. More specifically, this procedure uses left and right boundary weights, represented by K matrices, in addition to the usual bulk vertex weights, represented by a spectral-parameter-dependent R matrix. From Sklyanin's work, it follows that if a particular R matrix satisfies the Yang-Baxter equation, the first and second inversion relations, and certain symmetries, and if corresponding K matrices can be found which satisfy the reflection equations, then families of commuting, open-boundary transfer matrices can be constructed. Each such transfer matrix involves vertex weights from two adjacent rows of the lattice as well as a left and right boundary weight.

Various modifications of the reflection equations have been considered since Sklyanin's original work. Mezincescu and Nepomechie [7, 8] and Yue and Chen [9] have generalised the formalism to allow for R matrices satisfying less restrictive symmetries, while recently, Kulish [10] has, independently from us, obtained reflection equations for IRF models. Another variation is spectral-parameter-independent reflection equations, which have been studied by Kulish and Sklyanin [11] and Kulish, Sasaki and Schweibert [12].

The reflection equations have been solved to give K matrices corresponding to the R matrices of a number of models. Cherednik [6] and Yue and Chen [9] found diagonal, elliptic solutions for the $Z_n \times Z_n$ Belavin model, which is related to the Lie algebra $A_{n-1}^{(1)}$. Sklyanin used Cherednik's solution for $n = 2$ to obtain trigonometric, diagonal K matrices for the six-vertex model, or XXZ chain, while non-diagonal K matrices for the six-vertex model have been obtained by de Vega and González-Ruiz [13] and Ghoshal and Zamolodchikov [14]. For the eight-vertex model, elliptic, diagonal K matrices have been found by Cuerno and González-Ruiz [15], while non-diagonal K matrices have been found by de Vega and González-Ruiz [16], Hou and Yue [17] and Inami and Konno [18].

The original six-vertex model corresponds to the spin-$\frac{1}{2}$ representation of $A_1^{(1)}$. Mezincescu, Nepomechie and Rittenberg [19] have also found diagonal, trigonometric K matrices for the Zamolodchikov-Fateev 19-vertex model, which corresponds to the spin-1 representation of $A_1^{(1)}$. Furthermore, diagonal, trigonometric K matrices for the $n(2n-1)$-vertex models which correspond to the fundamental representation of $A_{n-1}^{(1)}$ have been found by de Vega and González-Ruiz [13], while non-diagonal K matrices for these models have been found by Abad and Rios [20].

The models mentioned so far have all been based on non-exceptional, untwisted affine Lie algebras and we now outline the models based on other Lie algebras, for which K matrices are known. For the 19-vertex Izergin-Korepin model, which corresponds to the fundamental representation of the twisted affine Lie algebra $A_2^{(2)}$, diagonal, trigonometric K matrices have been obtained by Mezincescu and Nepomechie [21] and non-diagonal K matrices have been obtained by Kim [22]. For the 175-vertex model based on the exceptional Lie algebra $G_2^{(1)}$, diagonal K matrices have been obtained by Yung and Batchelor [23], and for the $spl(2,1)$-related 15-vertex t-J model, diagonal K matrices have been obtained by González-Ruiz [24]. Finally, it should be noted that Mezincescu and Nepomechie [21, 8] have shown that the identity matrix satisfies the right reflection equation for vertex models corresponding to the fundamental representations of the algebras $A_n^{(1)}$, $A_n^{(2)}$, $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$.

Having found K matrices for a particular model, it remains to obtain the eigenspectra of the corresponding double-row transfer matrices. This has been done, using various forms of the Bethe ansatz, by Kulish and Sklyanin [2, 25], Artz, Mezincescu, Nepomechie and Rittenberg [19, 26, 27, 28], Destri, De Vega and González-Ruiz [29, 30, 31, 32, 24], Yung and Batchelor [33, 23], Foerster and Karowski [34], and Zhou [35]. In all of this work, constant or trigonometric, diagonal K-matrices were used. In many cases, the important property of quantum algebra invariance—that is, commutation of the transfer matrix or Hamiltonian with elements of the associated quantum algebra—was obtained, by choosing specific, usually constant, K matrices. Indeed, the fact that quantum algebra invariance can not be achieved using standard periodic boundary conditions has been a strong motivation for much of the work on open boundary conditions. We also note that in [29, 30, 33], diagonal-to-diagonal open-boundary transfer matrices were considered. These were obtained by setting to zero the spectral parameters of alternate vertex weights in the double-row transfer matrices.

Other directions in which recent work with open boundaries has proceeded include the consideration of loop models by Yung and Batchelor [33], and the use of vertex operators by Jimbo, Kedem, Kojima, Konno and Miwa [36]. There has also been a substantial return to the use of reflection equations in the theory of scattering on finite or semi-infinite lines, which has been initiated by the work of Ghoshal and Zamolodchikov [14, 37] and Fring and Köberle [38]. Of particular relevance here is the boundary crossing equation introduced in [14], since we use corresponding equations in our treatment of IRF models.

In this paper, we apply the fusion procedure to IRF models with fixed boundary conditions and obtain functional equations satisfied by the fixed-boundary double-row transfer matrices of the ABF fusion hierarchy. Fusion was introduced by Kulish, Reshetikhin and Sklyanin [39] as a means of obtaining new solutions to the Yang-Baxter equation, by combining R matrices from a known solution of the Yang-Baxter equation. In terms of the associated Lie algebra, if the original R matrices correspond to a particular representation, then the fused R matrices correspond to higher-dimensional representations.

Fusion was first applied to IRF models, and in particular to the ABF models, by Date, Jimbo, Kuniba, Miwa and Okado [40, 41, 42, 43]. Bazhanov and Reshetikhin [44] then obtained functional equations satisfied by the periodic-boundary row transfer matrices of the ABF fusion hierarchy and used these equations to derive Bethe ansatz solutions for the eigenspectra of the transfer matrices. These functional equations were also used by Klümper and Pearce [45] to obtain a generalised inversion identity. Fusion of other IRF models, leading to functional equations satisfied by periodic-boundary transfer matrices, has been implemented by Jimbo, Kuniba, Miwa and Okado [46] and Bazhanov and Reshetikhin [47] for the $A_n^{(1)}$ IRF models, by Zhou and Pearce [48] for the A–D–E models, which include the critical ABF models, and by Zhou, Pearce and Grimm [49] for the dilute A models.

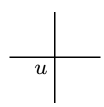
In the case of vertex models with open boundaries, the aim of fusion is to construct new solutions to the reflection equation, by combining K and R matrices from known solutions of the Yang-Baxter and reflection equations. A general formalism for this procedure has been presented by Mezincescu and Nepomechie [50] and has been applied to the eight-vertex model by Yue [51]. Furthermore, the boundary bootstrap equations, introduced in the context of scattering by Ghoshal and Zamolodchikov [14] and Fring and Köberle [38], correspond to the process of fusion. In each of these cases only level 2 fusion, in which two K matrices are combined with one R matrix, was considered explicitly, although it is clear that the process can be extended to higher fusion levels. In fact, Zhou [35] has recently applied fusion at arbitrary levels to the six-vertex model with open boundary conditions.

Finally, we note that the work of Saleur and Bauer [52] and of Destri and de Vega [30] has involved the consideration of ABF models with fixed boundary conditions, however in each of these cases the fixed boundary conditions were applied along diagonal rows of the lattice.

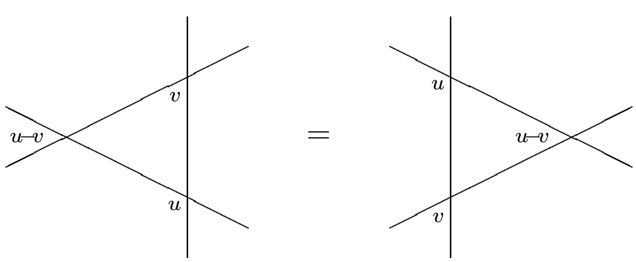
1.3 Vertex Models with Open Boundary Conditions

We now schematically outline the formalism for vertex models on which our treatment of IRF models is based. We note that the main differences between our formalism and that originally presented by Sklyanin [2] are that here only the first inversion relation is assumed, no R matrix symmetries are assumed, and the top row of the transfer matrix has the form $\boldsymbol{T}(\mu-u)^{t_0}$ rather than the form $\boldsymbol{T}(-u)^{-1}$, where $\boldsymbol{T}(u)$ is the form of the bottom row, μ is arbitrary and t_0 is transposition on the auxiliary space.

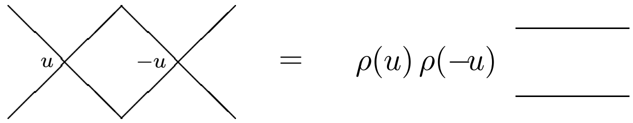
We are considering a vertex model with Boltzmann vertex weights



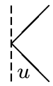
where u is a spectral parameter. We assume that these satisfy the Yang-Baxter equation



and the inversion relation



where ρ is a model-dependent scalar function. We now introduce left and right boundary weights



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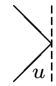
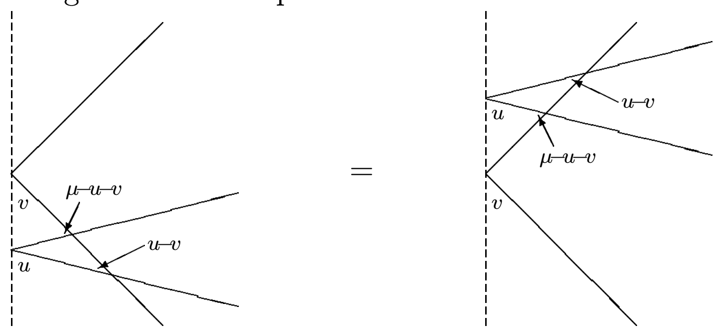


which satisfy left and right reflection equations



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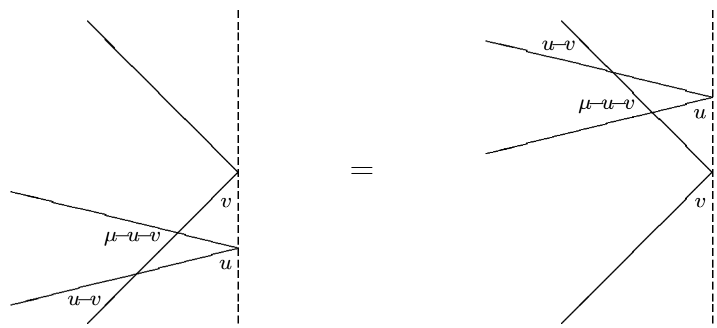
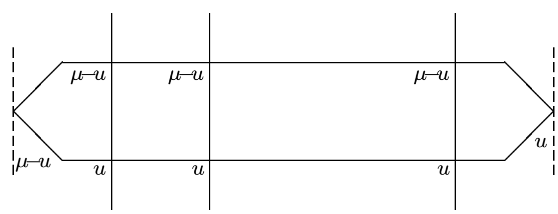


where μ is an arbitrary parameter. If we define a double-row transfer matrix as

$\boldsymbol{D}(u) \;=$



then it can be shown that, for any fixed value of μ, these matrices form a commuting family,

$$\boldsymbol{D}(u)\,\boldsymbol{D}(v) = \boldsymbol{D}(v)\,\boldsymbol{D}(u) \tag{1.1}$$

We also note that if we regard each value, u, of the spectral parameter as an effective angle

$$\theta(u) = \frac{\pi u}{\mu}$$

then the geometric angles in the diagrammatic Yang-Baxter and reflection equations correspond exactly with the effective angles given by values of the spectral parameter. This is due to the interpretation of these equations in terms of scattering.

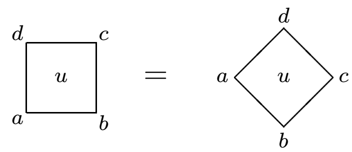
2. IRF Models with Fixed Boundary Conditions

2.1 Boltzmann Weights and Transfer Matrices

We now present our formalism for interaction-round-a-face (IRF) models, which was motivated by the preceding formalism for vertex models and Baxter's vertex-face correspondence [3, 1]. Our use of boundary crossing equations was also motivated by the work of Ghoshal and Zamolodchikov [14].

We are considering an IRF model with Boltzmann face weights

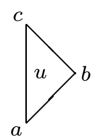
$$W\begin{pmatrix} d & c \\ a & b \end{pmatrix}\!\!\left|\, u\right) =$$



Here, the spins a, b, c, d take values from a discrete set and the spectral parameter u is a complex variable.

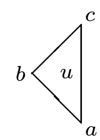
In order to accommodate fixed boundary conditions, we introduce left and right boundary weights

$$B_{\rm L}\!\left(\begin{matrix} c \\ a \end{matrix}\;\; b \,\middle|\, u\right) =$$



and

$$B_{\rm R}\!\left(b\;\;\begin{matrix} c \\ a \end{matrix} \,\middle|\, u\right) =$$

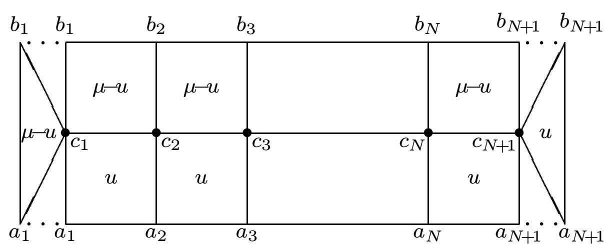


We now consider a lattice of width N and use these weights to construct a double-row transfer matrix. If $a_1, \ldots, a_{N+1}$ and $b_1, \ldots, b_{N+1}$ are two rows of spins, and μ is an arbitrary parameter, then the corresponding entry of the double-row transfer matrix is defined by

$$\langle a_1, \ldots, a_{N+1} | \boldsymbol{D}(u) | b_1, \ldots, b_{N+1} \rangle =$$

$$\sum_{c_1 \ldots c_{N+1}} B_{\rm L}\!\left(\begin{matrix} b_1 \\ a_1 \end{matrix}\;\; c_1 \,\middle|\, \mu - u\right) \left[\prod_{j=1}^{N} W\!\left(\begin{matrix} c_j & c_{j+1} \\ a_j & a_{j+1} \end{matrix} \,\middle|\, u\right) W\!\left(\begin{matrix} b_j & b_{j+1} \\ c_j & c_{j+1} \end{matrix} \,\middle|\, \mu - u\right)\right] B_{\rm R}\!\left(c_{N+1}\;\; \begin{matrix} b_{N+1} \\ a_{N+1} \end{matrix} \,\middle|\, u\right)$$

$$= \qquad (2.1)$$



In this and all subsequent diagrams, we use solid circles to indicate spins which are summed over and dotted lines to connect identical spins.

In general, there will be restrictions on the spins allowed on any neighbouring lattice sites, as specified by an adjacency matrix

$$A_{ab} = \begin{cases} 0\,, & a \text{ and } b \text{ may not be adjacent} \\ 1\,, & a \text{ and } b \text{ may be adjacent} \end{cases}$$

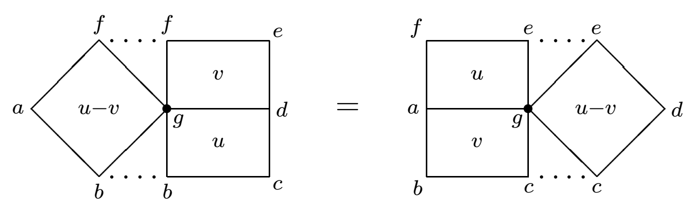
We assume that the face and boundary weights satisfy the adjacency condition as follows:

$$
\begin{aligned}
W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle|\, u\right) &= A_{ab}\, A_{bc}\, A_{cd}\, A_{da}\, W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle|\, u\right) \\
B_{\mathrm{L}}\left(\begin{matrix} c & \\ & b \\ a & \end{matrix}\middle|\, u\right) &= A_{ab}\, A_{bc}\, B_{\mathrm{L}}\left(\begin{matrix} c & \\ & b \\ a & \end{matrix}\middle|\, u\right) \\
B_{\mathrm{R}}\left(\begin{matrix} & c \\ b & \\ & a \end{matrix}\middle|\, u\right) &= A_{ab}\, A_{bc}\, B_{\mathrm{R}}\left(\begin{matrix} & c \\ b & \\ & a \end{matrix}\middle|\, u\right)
\end{aligned}
\tag{2.2}
$$

2.2 Local Relations

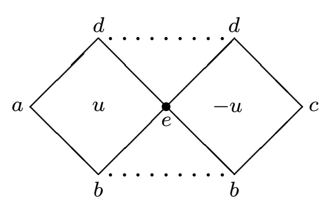
The face weights and boundary weights are assumed to satisfy the following local relations: the Yang-Baxter equation

$$
\begin{aligned}
&\sum_g W\left(\begin{matrix} f & g \\ a & b \end{matrix}\middle|\, u-v\right) W\left(\begin{matrix} g & d \\ b & c \end{matrix}\middle|\, u\right) W\left(\begin{matrix} f & e \\ g & d \end{matrix}\middle|\, v\right) = \\
&\qquad \sum_g W\left(\begin{matrix} a & g \\ b & c \end{matrix}\middle|\, v\right) W\left(\begin{matrix} f & e \\ a & g \end{matrix}\middle|\, u\right) W\left(\begin{matrix} e & d \\ g & c \end{matrix}\middle|\, u-v\right)
\end{aligned}
\tag{2.3}
$$



the inversion relation

$$
\sum_e W\left(\begin{matrix} d & e \\ a & b \end{matrix}\middle|\, u\right) W\left(\begin{matrix} d & c \\ e & b \end{matrix}\middle|\, -u\right) = \rho(u)\, \rho(-u)\, \delta_{ac}\, A_{ab}\, A_{ad} \tag{2.4}
$$

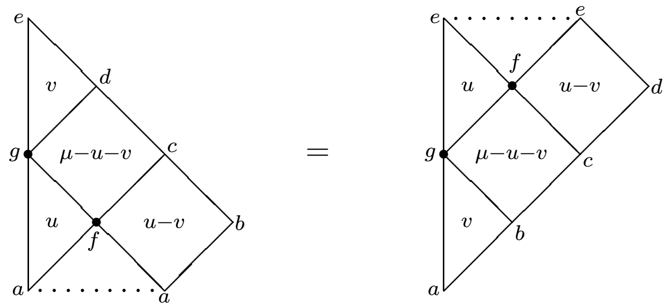


$$= \quad \rho(u)\, \rho(-u)\, \delta_{ac}\, A_{ab}\, A_{ad}$$

the left reflection equation

$$\sum_{fg} W\left(\begin{matrix} c & b \\ f & a \end{matrix}\middle| u-v\right) W\left(\begin{matrix} d & c \\ g & f \end{matrix}\middle| \mu-u-v\right) B_{\rm L}\left(\begin{matrix} g & \\ & f \\ a & \end{matrix}\middle| u\right) B_{\rm L}\left(\begin{matrix} e & \\ & d \\ g & \end{matrix}\middle| v\right) =$$

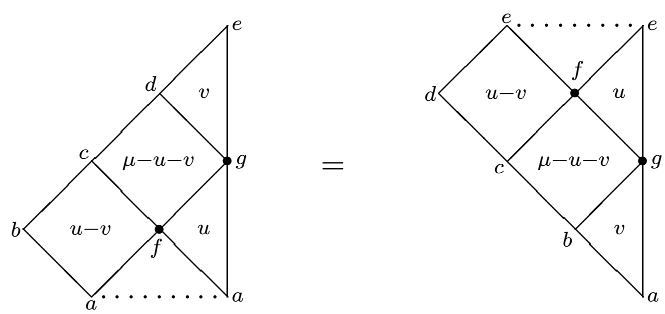
$$\sum_{fg} W\left(\begin{matrix} e & d \\ f & c \end{matrix}\middle| u-v\right) W\left(\begin{matrix} f & c \\ g & b \end{matrix}\middle| \mu-u-v\right) B_{\rm L}\left(\begin{matrix} e & \\ & f \\ g & \end{matrix}\middle| u\right) B_{\rm L}\left(\begin{matrix} g & \\ & b \\ a & \end{matrix}\middle| v\right) \qquad (2.5)$$



the right reflection equation

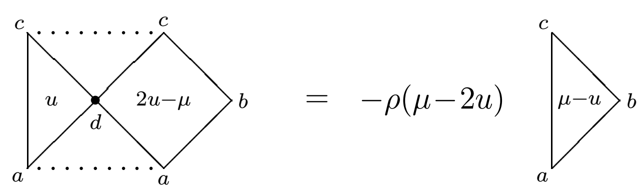
$$\sum_{fg} W\left(\begin{matrix} c & f \\ b & a \end{matrix}\middle| u-v\right) W\left(\begin{matrix} d & g \\ c & f \end{matrix}\middle| \mu-u-v\right) B_{\rm R}\left(\begin{matrix} & g \\ f & \\ & a \end{matrix}\middle| u\right) B_{\rm R}\left(\begin{matrix} & e \\ d & \\ & g \end{matrix}\middle| v\right) =$$

$$\sum_{fg} W\left(\begin{matrix} e & f \\ d & c \end{matrix}\middle| u-v\right) W\left(\begin{matrix} f & g \\ c & b \end{matrix}\middle| \mu-u-v\right) B_{\rm R}\left(\begin{matrix} & e \\ f & \\ & g \end{matrix}\middle| u\right) B_{\rm R}\left(\begin{matrix} & g \\ b & \\ & a \end{matrix}\middle| v\right) \qquad (2.6)$$



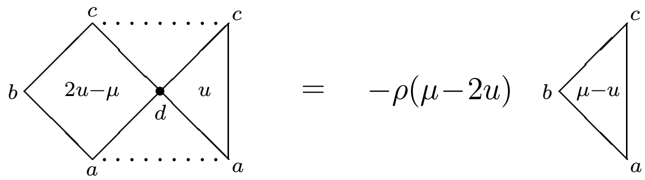
the left boundary crossing equation

$$\sum_{d} B_{\rm L}\left(\begin{matrix} c & \\ & d \\ a & \end{matrix}\middle| u\right) W\left(\begin{matrix} c & b \\ d & a \end{matrix}\middle| 2u-\mu\right) = -\rho(\mu-2u)\, B_{\rm L}\left(\begin{matrix} c & \\ & b \\ a & \end{matrix}\middle| \mu-u\right) \qquad (2.7)$$



and the right boundary crossing equation

$$\sum_d W\left(\begin{matrix} c & d \\ b & a \end{matrix}\middle| 2u-\mu\right)\, B_{\mathrm{R}}\left(d\ \begin{matrix} c \\ a \end{matrix}\middle| u\right) = -\rho(\mu-2u)\, B_{\mathrm{R}}\left(b\ \begin{matrix} c \\ a \end{matrix}\middle| \mu-u\right) \tag{2.8}$$



These equations are to be satisfied for all values of the external spins and all values of the spectral parameters. The function ρ is model-dependent and μ is the same fixed parameter as in (2.1).

We note that these local relations are consistent with the initial condition

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| 0\right) = -\rho(0)\,\delta_{ac}\,A_{ab}\,A_{ad} \tag{2.9}$$

(The minus sign is used here, and in the boundary crossing equations, in order to provide consistency with subsequent fusion equations.) More specifically, with this intitial condition we see that (2.3) holds for $u=v$ or $v=0$, that (2.4) holds for $u=0$, that (2.5) and (2.6) hold for $u=v$, and that (2.7) and (2.8) hold for $u=\mu/2$. Furthermore, we find that (2.3) holds for $u=0$, due to (2.4), while (2.5) and (2.6) hold for $v=\mu-u$ due to (2.7) and (2.8). Indeed, the inversion relation can be motivated by the Yang-Baxter equation together with the initial condition, while the boundary crossing equations can be motivated by the reflection equations together with the initial condition and the inversion relation.

It can also be seen that if the face weights satisfy certain reflection and rotation symmetries, then there are correspondences between solutions of the reflection equations. More specifically, if

$$B_{\mathrm{L}}\left(\begin{matrix} c \\ a \end{matrix}\ b\middle| u\right) = \tilde{B}_{\mathrm{L}}\left(\begin{matrix} c \\ a \end{matrix}\ b\middle| u\right) \quad\text{and}\quad B_{\mathrm{R}}\left(b\ \begin{matrix} c \\ a \end{matrix}\middle| u\right) = \tilde{B}_{\mathrm{R}}\left(b\ \begin{matrix} c \\ a \end{matrix}\middle| u\right)$$

satisfy (2.5) and (2.6), then the symmetry

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right) = W\left(\begin{matrix} b & c \\ a & d \end{matrix}\middle| u\right) \tag{2.10}$$

implies that so too do

$$B_{\mathrm{L}}\left(\begin{matrix} c \\ a \end{matrix}\ b\middle| u\right) = \tilde{B}_{\mathrm{L}}\left(\begin{matrix} a \\ c \end{matrix}\ b\middle| u\right) \quad\text{and}\quad B_{\mathrm{R}}\left(b\ \begin{matrix} c \\ a \end{matrix}\middle| u\right) = \tilde{B}_{\mathrm{R}}\left(b\ \begin{matrix} a \\ c \end{matrix}\middle| u\right)\ ,$$

the symmetry

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right) = W\left(\begin{matrix} d & a \\ c & b \end{matrix}\middle| u\right) \tag{2.11}$$

implies that so too do

$$B_{\mathrm{L}}\left(\begin{matrix} c & \\ & b \\ a & \end{matrix}\middle|\, u\right) = \tilde{B}_{\mathrm{R}}\left(\begin{matrix} & c \\ b & \\ & a \end{matrix}\middle|\, u\right) \quad \text{and} \quad B_{\mathrm{R}}\left(\begin{matrix} & c \\ b & \\ & a \end{matrix}\middle|\, u\right) = \tilde{B}_{\mathrm{L}}\left(\begin{matrix} c & \\ & b \\ a & \end{matrix}\middle|\, u\right) \;,$$

and the symmetry

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle|\, u\right) = W\left(\begin{matrix} b & a \\ c & d \end{matrix}\middle|\, u\right) \tag{2.12}$$

(which occurs if (2.10) and (2.11) both occur) implies that so too do

$$B_{\mathrm{L}}\left(\begin{matrix} c & \\ & b \\ a & \end{matrix}\middle|\, u\right) = \tilde{B}_{\mathrm{R}}\left(\begin{matrix} & a \\ b & \\ & c \end{matrix}\middle|\, u\right) \quad \text{and} \quad B_{\mathrm{R}}\left(\begin{matrix} & c \\ b & \\ & a \end{matrix}\middle|\, u\right) = \tilde{B}_{\mathrm{L}}\left(\begin{matrix} a & \\ & b \\ c & \end{matrix}\middle|\, u\right) \;.$$

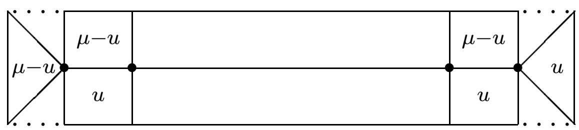
2.3 Crossing Symmetry of Double-Row Transfer Matrices

The double-row transfer matrices satisfy crossing symmetry,

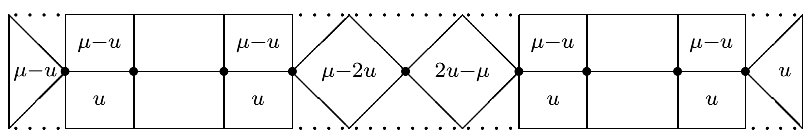
$$\boldsymbol{D}(u) = \boldsymbol{D}(\mu - u) \tag{2.13}$$

We prove this by considering an entry of $\boldsymbol{D}(u)$, applying the inversion relation at an arbitrary point, then using the Yang-Baxter equation N times, and finally applying both boundary crossing equations:

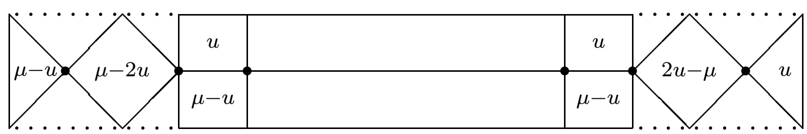
$$\boldsymbol{D}(u) \quad =$$



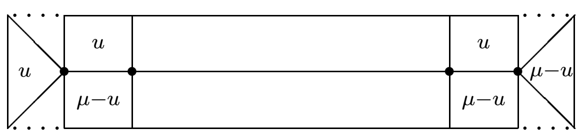
$$= \quad \frac{1}{\eta(u)}$$



$$= \quad \frac{1}{\eta(u)}$$



$$=$$



$$= \quad \boldsymbol{D}(\mu - u)$$

where $\eta(u) = \rho(\mu - 2u)\rho(2u - \mu)$.

2.4 Commutation of Double-Row Transfer Matrices

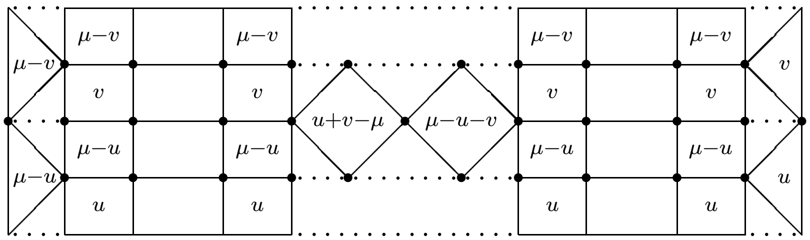
The double-row transfer matrices form a commuting family,

$$\boldsymbol{D}(u)\,\boldsymbol{D}(v) = \boldsymbol{D}(v)\,\boldsymbol{D}(u) \tag{2.14}$$

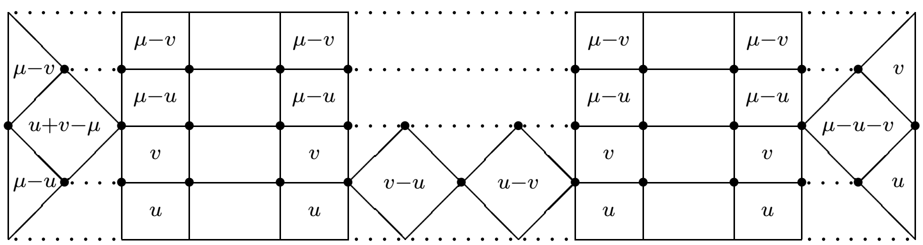
We prove this by the following steps, in each of which we use either the inversion relation, the Yang-Baxter equation N times, or the reflection equations:

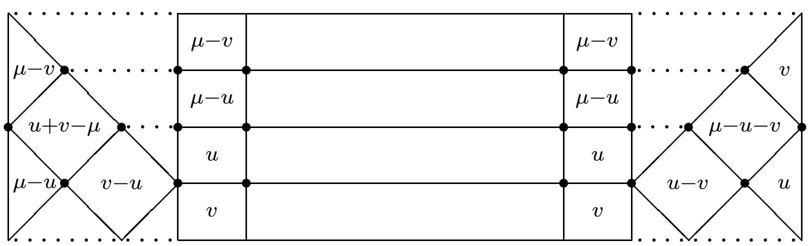
$\boldsymbol{D}(u)\,\boldsymbol{D}(v)$

$=$ 

$= \dfrac{1}{\eta(u,v)}$ 

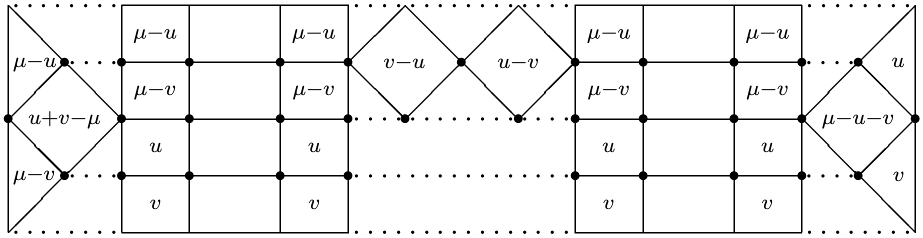
$= \dfrac{1}{\eta(u,v)}$ 

$= \dfrac{1}{\tilde{\eta}(u,v)}$ 

$= \dfrac{1}{\tilde{\eta}(u,v)}$ 

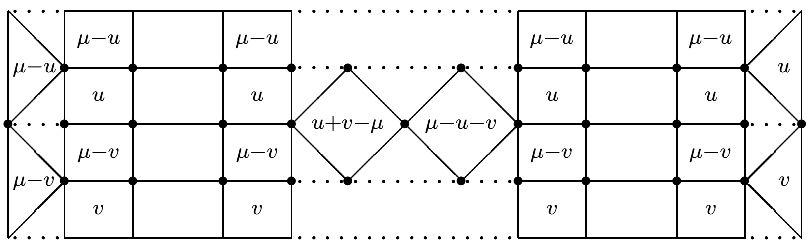
$= \dfrac{1}{\tilde{\eta}(u,v)}$ 

$$= \quad \frac{1}{\tilde{\eta}(u,v)}$$



$$= \quad \frac{1}{\eta(u,v)}$$

$$= \quad \frac{1}{\eta(u,v)}$$



$$=$$

$$= \quad \boldsymbol{D}(v)\,\boldsymbol{D}(u)$$

where $\eta(u,v) = \rho(u+v-\mu)\rho(\mu-u-v)$ and $\tilde{\eta}(u,v) = \rho(v-u)\rho(u-v)\rho(u+v-\mu)\rho(\mu-u-v)$.

2.5 ABF Models

We now consider the particular case of Andrews-Baxter-Forrester (ABF) restricted solid-on-solid models [5]. There is one such model for each integer $L \geq 3$. The spins—sometimes known also as heights—in the model labelled by L take the values

$$a \in \{1, 2, \ldots, L\}$$

and adjacent spins must differ by 1,

$$A_{ab} = \delta_{a,b-1} + \delta_{a,b+1} \tag{2.15}$$

For the model labelled by L, there is a fixed crossing parameter

$$\lambda = \frac{\pi}{L+1} \tag{2.16}$$

and the non-zero face weights are given by

$$
\begin{aligned}
W\left(\begin{matrix} a\pm1 & a \\ a & a\mp1 \end{matrix}\,\middle|\, u\right) &= \frac{\theta(\lambda-u)}{\theta(\lambda)} \\
W\left(\begin{matrix} a & a\pm1 \\ a\mp1 & a \end{matrix}\,\middle|\, u\right) &= \frac{\sqrt{\theta((a-1)\lambda)\,\theta((a+1)\lambda)}}{\theta(a\lambda)}\,\frac{\theta(u)}{\theta(\lambda)} \\
W\left(\begin{matrix} a & a\pm1 \\ a\pm1 & a \end{matrix}\,\middle|\, u\right) &= \frac{\theta(a\lambda\pm u)}{\theta(a\lambda)}
\end{aligned}
\tag{2.17}
$$

Here θ is a standard elliptic theta-1 function of fixed nome $\hat{q}$, with $-1<\hat{q}^2<1$,

$$
\theta(u)=\theta_1(u,\hat{q})=2\hat{q}^{1/4}\sin u\prod_{n=1}^{\infty}\left(1-2\hat{q}^{2n}\cos 2u+\hat{q}^{4n}\right)\left(1-\hat{q}^{2n}\right) \tag{2.18}
$$

At criticality, $\hat{q}=0$ and we can take $\theta(u)=\sin u$. The main properties of θ which we shall use are that it is odd

$$
\theta(u)=-\theta(-u) \tag{2.19}
$$

that it is periodic

$$
\theta(u)=-\theta(u+\pi) \tag{2.20}
$$

and that it satisfies the identity

$$
\begin{aligned}
&\theta(s+x)\,\theta(s-x)\,\theta(t+y)\,\theta(t-y)-\theta(s+y)\,\theta(s-y)\,\theta(t+x)\,\theta(t-x)\\
&\qquad=\theta(s+t)\,\theta(s-t)\,\theta(x+y)\,\theta(x-y)
\end{aligned}
\tag{2.21}
$$

It can be seen that the ABF face weights satisfy various simple relations: reflection and rotation symmetries, (2.10)–(2.12),

$$
W\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right)=W\left(\begin{matrix} d & a \\ c & b \end{matrix}\,\middle|\, u\right)=W\left(\begin{matrix} b & c \\ a & d \end{matrix}\,\middle|\, u\right)=W\left(\begin{matrix} b & a \\ c & d \end{matrix}\,\middle|\, u\right) \tag{2.22}
$$

crossing symmetry

$$
W\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right)=\sqrt{\frac{\theta(a\lambda)\,\theta(c\lambda)}{\theta(b\lambda)\,\theta(d\lambda)}}\;W\left(\begin{matrix} a & b \\ d & c \end{matrix}\,\middle|\, \lambda-u\right) \tag{2.23}
$$

full height reversal symmetry

$$
W\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right)=W\left(\begin{matrix} L+1-d & L+1-c \\ L+1-a & L+1-b \end{matrix}\,\middle|\, u\right) \tag{2.24}
$$

and the initial condition

$$
W\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, 0\right)=\delta_{ac}\,A_{ab}\,A_{ad} \tag{2.25}
$$

It is well-known that, essentially due to (2.21), the face weights also satisfy the Yang-Baxter equation, (2.3), and the inversion relation, (2.4), with the function ρ given by

$$\rho(u)=\frac{\theta(u-\lambda)}{\theta(\lambda)} \tag{2.26}$$

We now define, as the only non-zero ABF boundary weights,

$$\begin{aligned} B_{\rm L}\!\left(\begin{matrix} a \\ a \end{matrix}\; a\mp1 \,\middle|\, u\right) &= \sqrt{\frac{\theta((a\mp1)\lambda)}{\theta(a\lambda)}}\;\frac{\theta(u+\frac{\lambda-\mu}{2}\mp\xi_{\rm L}(a))\,\theta(u\pm a\lambda+\frac{\lambda-\mu}{2}\pm\xi_{\rm L}(a))}{\theta(\lambda)^2} \\ B_{\rm R}\!\left(a\mp1\; \begin{matrix} a \\ a \end{matrix} \,\middle|\, u\right) &= \sqrt{\frac{\theta((a\mp1)\lambda)}{\theta(a\lambda)}}\;\frac{\theta(u+\frac{\lambda-\mu}{2}\mp\xi_{\rm R}(a))\,\theta(u\pm a\lambda+\frac{\lambda-\mu}{2}\pm\xi_{\rm R}(a))}{\theta(\lambda)^2} \end{aligned} \tag{2.27}$$

where $\xi_{\rm L}(a)$ and $\xi_{\rm R}(a)$ are arbitrary parameters. In Appendix A, we show that the reflection equations, (2.5) and (2.6), are satisfied by the ABF face and boundary weights. We also show that the ABF weights, together with ρ given by (2.26), satisfy the boundary crossing equations, (2.7) and (2.8).

If $\xi_{\rm L}$ and $\xi_{\rm R}$ satisfy

$$\xi_{\rm L}(L+1-a)=-\xi_{\rm L}(a)\,,\qquad \xi_{\rm R}(L+1-a)=-\xi_{\rm R}(a) \tag{2.28}$$

then the ABF boundary weights satisfy full height reversal symmetry

$$\begin{aligned} B_{\rm L}\!\left(\begin{matrix} c \\ a \end{matrix}\; b \,\middle|\, u\right) &= -B_{\rm L}\!\left(\begin{matrix} L+1-c \\ L+1-a \end{matrix}\; L+1-b \,\middle|\, u\right) \\ B_{\rm R}\!\left(b\; \begin{matrix} c \\ a \end{matrix} \,\middle|\, u\right) &= -B_{\rm R}\!\left(L+1-b\; \begin{matrix} L+1-c \\ L+1-a \end{matrix} \,\middle|\, u\right) \end{aligned} \tag{2.29}$$

The boundary weights (2.27) have the diagonal form

$$B_{\rm L}\!\left(\begin{matrix} c \\ a \end{matrix}\; b \,\middle|\, u\right)=B_{\rm L}\!\left(\begin{matrix} c \\ a \end{matrix}\; b \,\middle|\, u\right)\,\delta_{ac}\,,\qquad B_{\rm R}\!\left(b\; \begin{matrix} c \\ a \end{matrix} \,\middle|\, u\right)=B_{\rm R}\!\left(b\; \begin{matrix} c \\ a \end{matrix} \,\middle|\, u\right)\,\delta_{ac} \tag{2.30}$$

This form implies equality of the boundary spins at each end of the double-row $\boldsymbol{D}(u)$ (and hence at each end of the entire lattice). It is convenient to regard these spins as labels for the fixed boundaries and only the internal spins as matrix indices. We therefore define the ABF double-row transfer matrix with fixed left and right boundary spins $a_{\rm L}$ and $a_{\rm R}$, $\boldsymbol{D}(a_{\rm L}a_{\rm R}|u)$, by

$$\langle a_2,\ldots,a_N\,|\,\boldsymbol{D}(a_{\rm L}a_{\rm R}|u)\,|\,b_2,\ldots,b_N\rangle=\langle a_{\rm L},a_2,\ldots,a_N,a_{\rm R}\,|\,\boldsymbol{D}(u)\,|\,a_{\rm L},b_2,\ldots,b_N,a_{\rm R}\rangle \tag{2.31}$$

It is natural in these models to take μ as

$$\mu=\lambda \tag{2.32}$$

With this choice, crossing symmetry of the face weights, (2.23), implies that $\boldsymbol{D}(a_{\rm L}a_{\rm R}|u)$ is symmetric

$$\boldsymbol{D}(a_{\rm L}a_{\rm R}|u) = \boldsymbol{D}(a_{\rm L}a_{\rm R}|u)^t \tag{2.33}$$

Ultimately, we will be interested in the isotropic point, $u = \lambda/2$, at which we now show it is possible to achieve a completely homogeneous lattice, with pure, fixed boundary conditions. If we set $\xi_{\rm L}(a_{\rm L}) = \pm\lambda/2$, $\xi_{\rm R}(a_{\rm R}) = \pm\lambda/2$ and $\mu = \lambda$, then

$$B_{\rm L}\left(\begin{matrix} a_{\rm L} & \\ a_{\rm L} & \end{matrix} a_{\rm L}\mp 1 \,\middle|\, \lambda/2\right) = B_{\rm R}\left(a_{\rm R}\mp 1 \begin{matrix} a_{\rm R} \\ a_{\rm R} \end{matrix} \,\middle|\, \lambda/2\right) = 0$$

so that the transfer matrix $\boldsymbol{D}(a_{\rm L}a_{\rm R}|\lambda/2)$ is simply proportional to the matrix product of two rows of face weights, all with spectral parameter $\lambda/2$, with the three spins on the left boundary fixed to $a_{\rm L}$, $a_{\rm L}\pm 1$, $a_{\rm L}$ and the three spins on the right boundary fixed to $a_{\rm R}$, $a_{\rm R}\pm 1$, $a_{\rm R}$. Similarly, if we set $\xi_{\rm L}(a_{\rm L}) = \pm\lambda/2$ and $\xi_{\rm R}(a_{\rm R}) = \mp\lambda/2$, then $\boldsymbol{D}(a_{\rm L}a_{\rm R}|\lambda/2)$ has the spins on the left boundary fixed to $a_{\rm L}$, $a_{\rm L}\pm 1$, $a_{\rm L}$ and the spins on the right boundary fixed to $a_{\rm R}$, $a_{\rm R}\mp 1$, $a_{\rm R}$.

3. Fusion of IRF Models with Fixed Boundary Conditions

3.1 General Formalism

We now extend our formalism to cover models which have a fusion hierarchy. For these models there is a discrete set of fusion levels, and we assume that each of these is labelled by a single integer, with the original, unfused model corresponding to fusion level 1.

The fused face weights, W^{pq}, are associated with two fusion levels—a horizontal level, p, and a vertical level, q—and the fused boundary weights, K^q, are associated with one fusion level, q. There is now an adjacency matrix, A^q, for each fusion level q, with the adjacency conditions on the fused weights being

$$W^{pq}\left(\begin{matrix} d & c \\ a & b \end{matrix} \,\middle|\, u\right) = A^p_{ab}\, A^q_{bc}\, A^p_{cd}\, A^q_{da}\; W^{pq}\left(\begin{matrix} d & c \\ a & b \end{matrix} \,\middle|\, u\right) \tag{3.1}$$

$$B^q_{\rm L}\left(\begin{matrix} c & \\ a & \end{matrix} b \,\middle|\, u\right) = A^q_{ab}\, A^q_{bc}\; B^q_{\rm L}\left(\begin{matrix} c & \\ a & \end{matrix} b \,\middle|\, u\right) \tag{3.2}$$

$$B^q_{\rm R}\left(b \begin{matrix} c \\ a \end{matrix} \,\middle|\, u\right) = A^q_{ab}\, A^q_{bc}\; B^q_{\rm R}\left(b \begin{matrix} c \\ a \end{matrix} \,\middle|\, u\right) \tag{3.3}$$

The fused double-row transfer matrices are also associated with two fusion levels, and are defined by

$$\langle a_1,\ldots,a_{N+1}|\boldsymbol{D}^{pq}(u)|b_1,\ldots,b_{N+1}\rangle \; = $$

$$\sum_{c_1\ldots c_{N+1}} B^q_{\rm L}\!\left(\begin{matrix} b_1 \\ a_1\end{matrix}\;\; c_1 \;\middle|\; -u-(q-1)\lambda+\mu\right) \;\times \tag{3.4}$$

$$\left[\prod_{j=1}^{N} W^{pq}\!\left(\begin{matrix} c_j & c_{j+1} \\ a_j & a_{j+1}\end{matrix}\;\middle|\; u\right)\; W^{pq}\!\left(\begin{matrix} b_j & b_{j+1} \\ c_j & c_{j+1}\end{matrix}\;\middle|\; -u-(q-1)\lambda+\mu\right)\right]\; B^q_{\rm R}\!\left(c_{N+1}\;\;\begin{matrix} b_{N+1} \\ a_{N+1}\end{matrix}\;\middle|\; u\right)$$

where λ and μ are arbitrary fixed parameters. In this generalised framework, the fused Yang-Baxter equation is

$$\sum_g W^{rq}\!\left(\begin{matrix} f & g \\ a & b\end{matrix}\;\middle|\; u-v\right)\; W^{pq}\!\left(\begin{matrix} g & d \\ b & c\end{matrix}\;\middle|\; u\right)\; W^{pr}\!\left(\begin{matrix} f & e \\ g & d\end{matrix}\;\middle|\; v\right) = \tag{3.5}$$

$$\sum_g W^{pr}\!\left(\begin{matrix} a & g \\ b & c\end{matrix}\;\middle|\; v\right)\; W^{pq}\!\left(\begin{matrix} f & e \\ a & g\end{matrix}\;\middle|\; u\right)\; W^{rq}\!\left(\begin{matrix} e & d \\ g & c\end{matrix}\;\middle|\; u-v\right)$$

the fused inversion relation is

$$\sum_e W^{qr}\!\left(\begin{matrix} d & e \\ a & b\end{matrix}\;\middle|\; u\right)\; W^{rq}\!\left(\begin{matrix} d & c \\ e & b\end{matrix}\;\middle|\; -u\right) = \rho^{qr}(u)\; \rho^{rq}(-u)\; \delta_{ac}\, A^q_{ab}\, A^r_{ad} \tag{3.6}$$

the fused left reflection equation is

$$\rho^{rq}(u-v)\; \rho^{rq}(-u-v-(q-1)\lambda+\mu) \;\times$$

$$\sum_{fg} W^{qr}\!\left(\begin{matrix} c & b \\ f & a\end{matrix}\;\middle|\; u-v+(q-r)\lambda\right)\; W^{qr}\!\left(\begin{matrix} d & c \\ g & f\end{matrix}\;\middle|\; -u-v-(r-1)\lambda+\mu\right)\; B^q_{\rm L}\!\left(\begin{matrix} g \\ a\end{matrix}\;\; f\;\middle|\; u\right)\; B^r_{\rm L}\!\left(\begin{matrix} e \\ g\end{matrix}\;\; d\;\middle|\; v\right)$$

$$= \quad \rho^{qr}(u-v+(q-r)\lambda)\; \rho^{qr}(-u-v-(r-1)\lambda+\mu) \;\times$$

$$\sum_{fg} W^{rq}\!\left(\begin{matrix} e & d \\ f & c\end{matrix}\;\middle|\; u-v\right)\; W^{rq}\!\left(\begin{matrix} f & c \\ g & b\end{matrix}\;\middle|\; -u-v-(q-1)\lambda+\mu\right)\; B^q_{\rm L}\!\left(\begin{matrix} e \\ g\end{matrix}\;\; f\;\middle|\; u\right)\; B^r_{\rm L}\!\left(\begin{matrix} g \\ a\end{matrix}\;\; b\;\middle|\; v\right) \tag{3.7}$$

the fused right reflection equation is

$$\rho^{qr}(u-v+(q-r)\lambda)\; \rho^{qr}(-u-v-(r-1)\lambda+\mu) \;\times$$

$$\sum_{fg} W^{rq}\!\left(\begin{matrix} c & f \\ b & a\end{matrix}\;\middle|\; u-v\right)\; W^{rq}\!\left(\begin{matrix} d & g \\ c & f\end{matrix}\;\middle|\; -u-v-(q-1)\lambda+\mu\right)\; B^q_{\rm R}\!\left(f\;\;\begin{matrix} g \\ a\end{matrix}\;\middle|\; u\right)\; B^r_{\rm R}\!\left(d\;\;\begin{matrix} e \\ g\end{matrix}\;\middle|\; v\right) \tag{3.8}$$

$$= \quad \rho^{rq}(u-v)\; \rho^{rq}(-u-v-(q-1)\lambda+\mu) \;\times$$

$$\sum_{fg} W^{qr}\!\left(\begin{matrix} e & f \\ d & c\end{matrix}\;\middle|\; u-v+(q-r)\lambda\right)\; W^{qr}\!\left(\begin{matrix} f & g \\ c & b\end{matrix}\;\middle|\; -u-v-(r-1)\lambda+\mu\right)\; B^q_{\rm R}\!\left(f\;\;\begin{matrix} e \\ g\end{matrix}\;\middle|\; u\right)\; B^r_{\rm R}\!\left(b\;\;\begin{matrix} g \\ a\end{matrix}\;\middle|\; v\right)$$

the fused left boundary crossing equation is

$$\sum_d B^q_{\mathrm{L}}\!\left(\begin{matrix} c & \\ a & d\end{matrix}\,\middle|\, u\right) W^{qq}\!\left(\begin{matrix} c & b\\ d & a\end{matrix}\,\middle|\, 2u+(q-1)\lambda-\mu\right) = \tag{3.9}$$

$$(-1)^q\, \rho^{qq}(-2u-(q-1)\lambda+\mu)\; B^q_{\mathrm{L}}\!\left(\begin{matrix} c & \\ a & b\end{matrix}\,\middle|\, -u-(q-1)\lambda+\mu\right)$$

and the fused right boundary crossing equation is

$$\sum_d W^{qq}\!\left(\begin{matrix} c & d\\ b & a\end{matrix}\,\middle|\, 2u+(q-1)\lambda-\mu\right) B^q_{\mathrm{R}}\!\left(\begin{matrix} d & c \\ & a\end{matrix}\,\middle|\, u\right) = \tag{3.10}$$

$$(-1)^q\, \rho^{qq}(-2u-(q-1)\lambda+\mu)\; B^q_{\mathrm{R}}\!\left(\begin{matrix} b & c \\ & a\end{matrix}\,\middle|\, -u-(q-1)\lambda+\mu\right)$$

where ρ^{rq} are model-dependent functions. The fused local relations are consistent with the fused initial condition

$$W^{qq}\!\left(\begin{matrix} d & c\\ a & b\end{matrix}\,\middle|\, 0\right) = (-1)^q\, \rho^{qq}(0)\, \delta_{ac}\, A^q_{ab}\, A^q_{ad} \tag{3.11}$$

There are also correspondences between solutions of the fused reflection equations if the fused face weights satisfy certain reflection and rotation symmetries. It can be seen that the fused adjacency conditions, double row transfer matrix, local relations and initial condition reduce to (2.2) and (2.1)–(2.9) for $p = q = r = 1$.

By following a parallel sequence of steps to those of Section 2.3, but now including the fusion levels p and q, we can show that the fused inversion relation and boundary crossing equations, (3.6), (3.9) and (3.10), imply that the fused double-row transfer matrices satisfy crossing symmetry

$$\boldsymbol{D}^{pq}(u) = \boldsymbol{D}^{pq}(-u-(q-1)\lambda+\mu) \tag{3.12}$$

Similarly, by following a parallel sequence of steps to those of Section 2.4, we can show that the fused Yang-Baxter equation, inversion relation, and reflection equations, (3.5)–(3.8), imply that the fused double-row transfer matrices form a commuting family

$$\boldsymbol{D}^{pq}(u)\, \boldsymbol{D}^{pr}(v) = \boldsymbol{D}^{pr}(v)\, \boldsymbol{D}^{pq}(u) \tag{3.13}$$

3.2 ABF Models

3.2.1 Adjacency Conditions

We now return to the case of ABF models and consider their fusion hierarchy [40, 41, 43]. These models are related to the Lie algebra $su(2)$, or more specifically $A^{(1)}_1$. The original ABF models are associated with the spin-$\frac{1}{2}$ representation of $su(2)$ and the higher fusion levels are associated with higher-spin representations of $su(2)$.

For each L, we have $L + 2$ fusion levels, labelled $-1, 0, \ldots, L$. The level q adjacency matrix, A^q, is defined by the condition that a and b are adjacent if and only if

$$a-b \in \{-q,\ -q+2,\ \ldots\ ,\ q-2,\ q\} \tag{3.14}$$

and

$$a+b \in \{q+2,\ q+4,\ \ldots\ ,\ 2L-q-2,\ 2L-q\} \tag{3.15}$$

It can be seen that

$$A^{-1}=0\ , \qquad A^0=I\ , \qquad A^1=A\ , \qquad A^{L-2}=AY\ , \qquad A^{L-1}=Y\ , \qquad A^L=0 \tag{3.16}$$

where I is the $L\times L$ identity matrix, A is given by (2.15), and Y is the $L\times L$ height reversal matrix

$$Y_{ab}=\delta_{L+1-a\,,\,b} \tag{3.17}$$

It can be shown that the fused adjacency matrices satisfy full height reversal symmetry

$$A^q=YA^q\,Y \tag{3.18}$$

partial height reversal symmetry

$$A^q=YA^{L-1-q} \tag{3.19}$$

and the $su(2)$ fusion rules

$$A^qA \;=\; A^{q-1}+A^{q+1}\ , \qquad 0\le q\le L-1 \tag{3.20}$$

$$(A^q)^2 \;=\; I+A^{q-1}\,A^{q+1}\ , \qquad 0\le q\le L-1 \tag{3.21}$$

$$(\tilde{A}^q)^2 \;=\; (I+\tilde{A}^{q-1})\,(I+\tilde{A}^{q+1})\ , \qquad 1\le q\le L-2 \tag{3.22}$$

where

$$\tilde{A}^q=A^{q-1}\,A^{q+1}\ , \qquad 0\le q\le L-1 \tag{3.23}$$

For what follows, it is useful to define a set, P^q_{ab}, of $q-1$-point paths between a and b, as[†]

$$P^q_{ab}=\begin{cases}\{1,\ldots,L\}^{q-1} & ,\ A^q_{ab}=0\\ \Big\{(c_1,\ldots,c_{q-1})\in\{1,\ldots,L\}^{q-1}\,\Big|\,A_{ac_1}\,A_{c_1c_2}\,\ldots\,A_{c_{q-2}c_{q-1}}\,A_{c_{q-1}b}=1\Big\}, & A^q_{ab}=1\end{cases} \tag{3.24}$$

3.2.2 Face and Boundary Weights

We now define ABF fused face weights, W^{pq}, and fused boundary weights, K^q. These definitions will involve the fusion normalisation function

$$\theta^q_k(u) \;=\; \frac{\prod_{j=0}^{q-1}\theta(u+k\lambda-j\lambda)}{\theta(\lambda)^q} \tag{3.25}$$

[†]It can be shown that, for $A^q_{ab}=1$, the number of paths in P^q_{ab} is $(A)^q{}_{ab}=\dfrac{q\,!}{\left(\frac{q+a-b}{2}\right)!\left(\frac{q+b-a}{2}\right)!}$

and the fusion gauge factors

$$X^q_{ab} \quad = \quad \begin{cases} 1 & , \quad A^q_{ab}=0 \\ \dfrac{\displaystyle\prod_{j=\frac{a+b-q}{2}}^{\frac{a+b+q}{2}} \theta(j\lambda) \prod_{j=2}^{\frac{a-b+q}{2}} \theta(j\lambda) \prod_{j=2}^{\frac{b-a+q}{2}} \theta(j\lambda)}{\theta(\lambda)^{2q-1}} & , \quad A^q_{ab}=1 \end{cases} \tag{3.26}$$

and

$$G^q_{a_0,a_1,\dots,a_{q-1},a_q} \quad = \quad X^q_{a_0 a_q} \, \frac{\theta(\lambda)^{q+1}}{\displaystyle\prod_{j=0}^{q} \theta(a_j\lambda)} \tag{3.27}$$

where, as before, λ is given by (2.16) and θ is given by (2.18). Throughout this section, a product $\displaystyle\prod_{j=j'}^{j''} P(j)$ is taken to be 1 if $j''<j'$.

For weights involving fusion level -1, we must, in order to satisfy the adjacency condition, define

$$W^{p,-1}\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = W^{-1,q}\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = B^{-1}_{\mathrm{L}}\left(\begin{matrix} c & \\ a & \end{matrix} b \,\middle|\, u\right) = B^{-1}_{\mathrm{R}}\left(b \begin{matrix} c \\ a \end{matrix}\,\middle|\, u\right) = 0 \tag{3.28}$$

For weights involving fusion level 0, we define

$$\begin{aligned} W^{p,0}\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) &= \theta^p_{-1}(u)\,\delta_{ad}\,\delta_{bc}\,A^p_{ab} \\ W^{0,q}\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) &= \delta_{ab}\,\delta_{cd}\,A^q_{ad} \\ B^0_{\mathrm{L}}\left(\begin{matrix} c & \\ a & \end{matrix} b \,\middle|\, u\right) = B^0_{\mathrm{R}}\left(b \begin{matrix} c \\ a \end{matrix}\,\middle|\, u\right) &= \delta_{ab}\,\delta_{bc} \end{aligned} \tag{3.29}$$

For fusion level 1, the non-zero ABF weights are defined as

$$\begin{aligned} W^{11}\left(\begin{matrix} a\pm1 & a \\ a & a\mp1 \end{matrix}\,\middle|\, u\right) &= \frac{\theta(\lambda-u)}{\theta(\lambda)} \\ W^{11}\left(\begin{matrix} a & a\pm1 \\ a\mp1 & a \end{matrix}\,\middle|\, u\right) &= -\frac{\theta((a\pm1)\lambda)\,\theta(u)}{\theta(a\lambda)\,\theta(\lambda)} \\ W^{11}\left(\begin{matrix} a & a\pm1 \\ a\pm1 & a \end{matrix}\,\middle|\, u\right) &= \frac{\theta(a\lambda\pm u)}{\theta(a\lambda)} \\ B^1_{\mathrm{L}}\left(\begin{matrix} a & \\ a & \end{matrix} a\mp1 \,\middle|\, u\right) &= \mp\frac{\theta((a\mp1)\lambda)\,\theta(u+\frac{\lambda-\mu}{2}\mp\xi_{\mathrm{L}}(a))\,\theta(u\pm a\lambda+\frac{\lambda-\mu}{2}\pm\xi_{\mathrm{L}}(a))}{\theta(\lambda)^3} \\ B^1_{\mathrm{R}}\left(a\mp1 \begin{matrix} a \\ a \end{matrix}\,\middle|\, u\right) &= \mp\frac{\theta(u+\frac{\lambda-\mu}{2}\mp\xi_{\mathrm{R}}(a))\,\theta(u\pm a\lambda+\frac{\lambda-\mu}{2}\pm\xi_{\mathrm{R}}(a))}{\theta(a\lambda)\,\theta(\lambda)} \end{aligned} \tag{3.30}$$

where, as before, $\xi_{\rm L}(a)$ and $\xi_{\rm R}(a)$ are arbitrary constants. These weights are related to the standard ABF weights, (2.17) and (2.27), by the gauge transformation

$$\begin{aligned}
W^{11}\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) &= \epsilon_a\,\epsilon_c\,\sqrt{\frac{\theta(c\lambda)}{\theta(a\lambda)}}\;W\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right)\\
B^1_{\rm L}\!\left(\begin{matrix} c & \\ a & \end{matrix} b\,\middle|\, u\right) &= \epsilon_{a+1}\,\epsilon_b\,\frac{\sqrt{\theta(a\lambda)\,\theta(b\lambda)}}{\theta(\lambda)}\;B_{\rm L}\!\left(\begin{matrix} c & \\ a & \end{matrix} b\,\middle|\, u\right)\\
B^1_{\rm R}\!\left(b \begin{matrix} c \\ a \end{matrix}\,\middle|\, u\right) &= \epsilon_{a+1}\,\epsilon_b\,\frac{\theta(\lambda)}{\sqrt{\theta(a\lambda)\,\theta(b\lambda)}}\;B_{\rm R}\!\left(b \begin{matrix} c \\ a \end{matrix}\,\middle|\, u\right)
\end{aligned}\qquad (3.31)$$

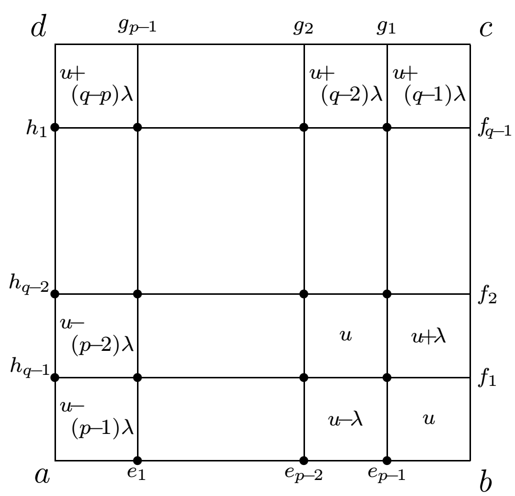
where ϵ_a are factors whose required properties are[†]

$$(\epsilon_a)^2 = 1\;, \qquad\qquad \epsilon_a\,\epsilon_{a+2} = -1 \qquad (3.32)$$

The Yang-Baxter, inversion, reflection and boundary crossing equations, (2.3)–(2.8), are still satisfied by these level 1 weights since the gauge factors corresponding to the internal spins of these equations cancel, while the gauge factors corresponding to the external spins are the same on both sides of each equation. However, the face weights no longer satisfy the reflection symmetry (2.11). We note that the face and boundary weights which appear in all subsequent diagrams are the level 1 ABF weights of (3.30).

We now proceed to ABF weights involving higher fusion levels, which are defined in terms of sums of products of the level 1 weights of (3.30) as follows:

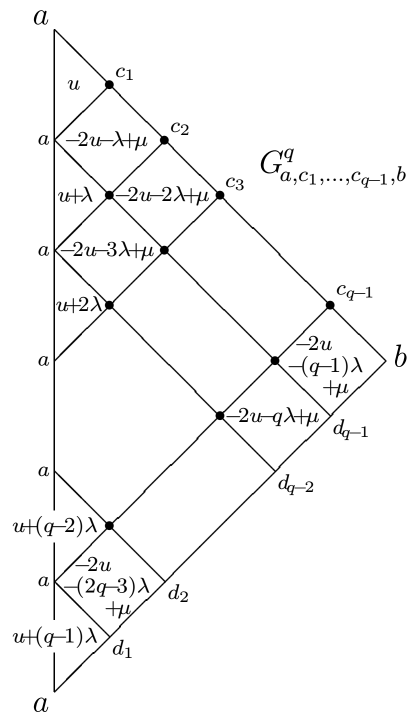
$$W^{pq}\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = \frac{A^p_{ab}\,A^q_{ad}}{\prod_{j=0}^{q-2}\theta^p_j(u)}\sum_{e_1\ldots e_{p-1}}\;\sum_{h_1\ldots h_{q-1}} \qquad (3.33)$$



[†]For example, one of the (four possible) choices for ϵ is $\epsilon_a = \begin{cases} 1, & a = 0 \text{ or } 1 \pmod 4 \\ -1, & a = 2 \text{ or } 3 \pmod 4 \end{cases}$

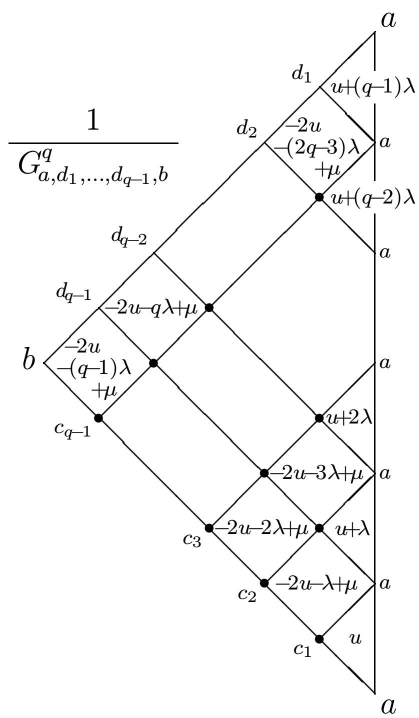
where $(f_1,\ldots,f_{q-1})\in P^q_{bc}$ and $(g_1,\ldots,g_{p-1})\in P^p_{cd}$,

$$B^q_{\mathrm{L}}\!\left(\begin{matrix} c & b \\ a & \end{matrix}\,\middle|\, u\right) \;=\; \frac{\delta_{ac}\;A^q_{ab}}{\prod_{j=0}^{q-2}\theta^{j+1}_{2j+1}(2u-\mu)} \sum_{c_1\ldots c_{q-1}} \tag{3.34}$$



where $(d_1,\ldots,d_{q-1})\in P^q_{ab}$, and

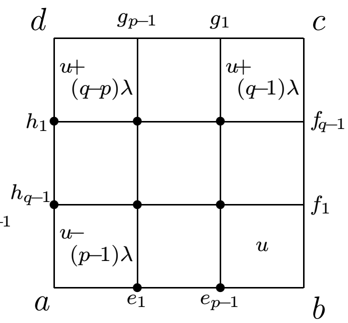
$$B^q_{\mathrm{R}}\!\left(\begin{matrix} b & c \\ & a \end{matrix}\,\middle|\, u\right) \;=\; \frac{\delta_{ac}\;A^q_{ab}}{\prod_{j=0}^{q-2}\theta^{j+1}_{2j+1}(2u-\mu)} \sum_{c_1\ldots c_{q-1}} \tag{3.35}$$



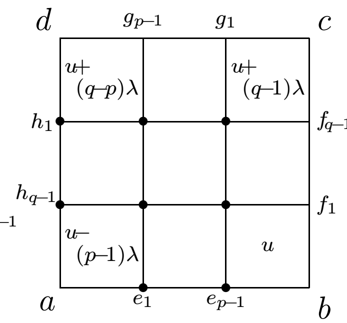
where $(d_1, \ldots, d_{q-1}) \in P^q_{ab}$. It is shown in [43] that $W^{pq}\begin{pmatrix} d & c \\ a & b \end{pmatrix} u \Bigg)$ is independent of the choice of $(f_1, \ldots, f_{q-1}) \in P^q_{bc}$ and $(g_1, \ldots, g_{p-1}) \in P^p_{cd}$, and it can be shown similarly that $B^q_{\mathrm{L}}\left(\begin{matrix} a & \\ & b \\ a & \end{matrix} \middle| u \right)$ and $B^q_{\mathrm{R}}\left(\begin{matrix} & a \\ b & \\ & a \end{matrix} \middle| u \right)$ are each independent of the choice of $(d_1, \ldots, d_{q-1}) \in P^q_{ab}$.

It can also be shown that $W^{pq}\left(\begin{matrix} d & c \\ a & b \end{matrix} \middle| u \right)$ satisfies (3.1) even though the adjacency condition is only explicitly applied on the edges being summed. This can be regarded as a push-through property of the entries of the adjacency matrix,

$$A^p_{ab}\, A^q_{bc}\, A^p_{cd}\, A^q_{da} \times \sum_{e_1 \ldots e_{p-1}} \sum_{h_1 \ldots h_{q-1}}$$



$$= A^p_{ab}\, A^q_{da} \times \sum_{e_1 \ldots e_{p-1}} \sum_{h_1 \ldots h_{q-1}}$$

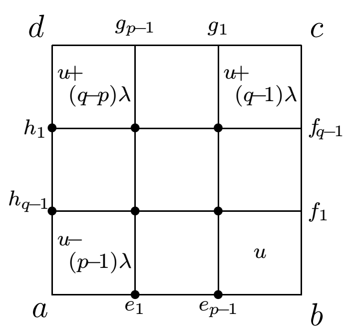


(3.36)

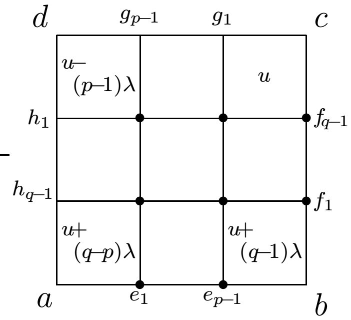
Furthermore, it can be shown that the configuration of level 1 weights in $W^{pq}\left(\begin{matrix} d & c \\ a & b \end{matrix} \middle| u \right)$ can be re-oriented, as follows:

$$\left(\prod_{j=0}^{q-2} \theta^p_j(u) \right) \quad W^{pq}\left(\begin{matrix} d & c \\ a & b \end{matrix} \middle| u \right)$$

$$= \quad A^p_{ab}\, A^q_{da} \sum_{e_1 \ldots e_{p-1}} \sum_{h_1 \ldots h_{q-1}} \qquad (3.37)$$



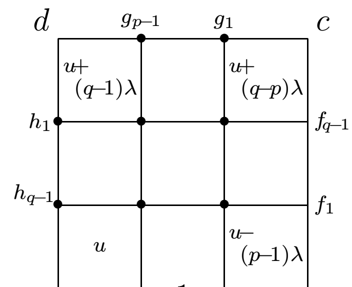
$$= A^p_{ab}\, A^q_{bc} \sum_{e_1\ldots e_{p-1}} \sum_{f_1\ldots f_{q-1}} \frac{1}{G^q_{d,h_1,\ldots,h_{q-1},a}}$$



$$G^q_{b,f_1,\ldots,f_{q-1},c} \qquad (3.38)$$

$$G^p_{c,g_1,\ldots,g_{p-1},d}$$

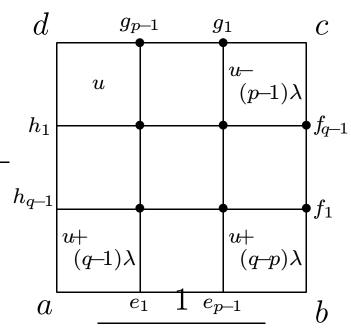
$$= A^p_{cd}\, A^q_{da} \sum_{g_1\ldots g_{p-1}} \sum_{h_1\ldots h_{q-1}} \qquad (3.39)$$



$$\frac{1}{G^p_{a,e_1,\ldots,e_{p-1},b}}$$

$$G^p_{c,g_1,\ldots,g_{p-1},d}$$

$$= A^q_{bc}\, A^p_{cd} \sum_{f_1\ldots f_{q-1}} \sum_{g_1\ldots g_{p-1}} \frac{1}{G^q_{d,h_1,\ldots,h_{q-1},a}}$$



$$G^q_{b,f_1,\ldots,f_{q-1},c} \qquad (3.40)$$

$$\frac{1}{G^p_{a,e_1,\ldots,e_{p-1},b}}$$

In these expressions, the external edge spins which are not summed are arbitrary, as long as we have $(e_1,\ldots,e_{p-1}) \in P^p_{ab}$ in (3.39) and (3.40), $(f_1,\ldots,f_{q-1}) \in P^q_{bc}$ in (3.37) and (3.39), $(g_1,\ldots,g_{p-1}) \in P^p_{cd}$ in (3.37) and (3.38), and $(h_1,\ldots,h_{q-1}) \in P^q_{da}$ in (3.38) and (3.40). One way to prove (3.36) and (3.38)–(3.40) is to use the fusion projection operators of [48], which satisfy a push-through property relative to the fused face weights and whose entries are proportional to the gauge factors G.

In [43], it is shown that for the ABF fused face weights, the summation over multiple spins in (3.33) can always be reduced either to a single term or to a summation over a single index, and that $\prod_{j=0}^{q-2}\theta_j^p(u)$ always arises as a common factor. The resulting expressions for the weights are presented, and from these we find that we have crossing symmetry

$$W^{pq}\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = (-1)^{p(q-1)}\,\epsilon_a\,\epsilon_b\,\epsilon_c\,\epsilon_d\,\frac{\theta(a\lambda)}{\theta(d\lambda)}\,\frac{X^p_{cd}}{X^p_{ab}}\,W^{pq}\left(\begin{matrix} a & b \\ d & c \end{matrix}\,\middle|\, -u+(p-q+1)\lambda\right) \tag{3.41}$$

partial height reversal symmetry

$$W^{pq}\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = (-1)^{pL}\,\epsilon_{a+d}\,\epsilon_{b+c}\,\frac{X^q_{bc}}{X^q_{ad}}\,W^{p,L-1-q}\left(\begin{matrix} d & c \\ L+1-a & L+1-b \end{matrix}\,\middle|\, u+(q+1)\lambda\right) \tag{3.42}$$

full height reversal symmetry

$$W^{pq}\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = W^{pq}\left(\begin{matrix} L+1-d & L+1-c \\ L+1-a & L+1-b \end{matrix}\,\middle|\, u\right) \tag{3.43}$$

and an initial condition

$$W^{qq}\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, 0\right) = \theta^q_q(0)\;\delta_{ac}\;A^q_{ab}\;A^q_{ad} \tag{3.44}$$

Properties (3.41) and (3.43) can be proved alternatively by applying the corresponding properties of the level 1 weights directly in (3.33).

Using techniques similar to those used in [43] to derive explicit formulae for the ABF fused face weights, it can be also shown that, for the boundary weights, the summations over multiple spins in (3.34) and (3.35) always reduce to a single term, with $\prod_{j=0}^{q-2}\theta^{j+1}_{2j+1}(2u-\mu)$ as a factor. This gives, for $A^q_{ab}=1$,

$$\begin{aligned} B^q_{\mathrm{L}}\left(\begin{matrix} a \\ a \end{matrix}\; b\,\middle|\, u\right) &= \\ &\prod_{j=1}^{\frac{a-b+q}{2}} \frac{\theta(j\lambda)\,\theta((a-j)\lambda)\,\theta(-u-(q-j)\lambda-\frac{\lambda-\mu}{2}+\xi_{\mathrm{L}}(a))\,\theta(u+(q-j+a)\lambda+\frac{\lambda-\mu}{2}+\xi_{\mathrm{L}}(a))}{\theta(\lambda)^4} \\ &\times \prod_{j=1}^{\frac{b-a+q}{2}} \frac{\theta(j\lambda)\,\theta((a+j)\lambda)\,\theta(u+(q-j)\lambda+\frac{\lambda-\mu}{2}+\xi_{\mathrm{L}}(a))\,\theta(u+(q-j-a)\lambda+\frac{\lambda-\mu}{2}-\xi_{\mathrm{L}}(a))}{\theta(\lambda)^4} \end{aligned} \tag{3.45}$$

and

$$\begin{aligned} B^q_{\mathrm{R}}\left(b\; \begin{matrix} a \\ a \end{matrix}\,\middle|\, u\right) &= \prod_{j=1}^{\frac{a-b+q}{2}} \frac{\theta(-u-(q-j)\lambda-\frac{\lambda-\mu}{2}+\xi_{\mathrm{R}}(a))\,\theta(u+(q-j+a)\lambda+\frac{\lambda-\mu}{2}+\xi_{\mathrm{R}}(a))}{\theta(j\lambda)\,\theta((b+j)\lambda)} \\ &\times \prod_{j=1}^{\frac{b-a+q}{2}} \frac{\theta(u+(q-j)\lambda+\frac{\lambda-\mu}{2}+\xi_{\mathrm{R}}(a))\,\theta(u+(q-j-a)\lambda+\frac{\lambda-\mu}{2}-\xi_{\mathrm{R}}(a))}{\theta(j\lambda)\,\theta((b-j)\lambda)} \end{aligned} \tag{3.46}$$

It follows from these expressions that the ABF fused boundary weights satisfy partial height reversal symmetry

$$\begin{aligned}
&\theta^{L-q}_{-2}(u+\tfrac{\lambda-\mu}{2}-\xi_{\mathrm{L}}(a))\,\theta^{L-q}_{-2}(u+\tfrac{\lambda-\mu}{2}+\xi_{\mathrm{L}}(a))\,B^q_{\mathrm{L}}\!\left(\begin{smallmatrix} a & \\ & b \\ a & \end{smallmatrix}\middle|\, u\right) \;=\\
&\quad -\left(\frac{X^q_{ab}}{\theta^L_L(0)}\right)^2 \theta^{q+1}_{q-1-a}(u+\tfrac{\lambda-\mu}{2}-\xi_{\mathrm{L}}(a))\,\theta^{q+1}_{q-1+a}(u+\tfrac{\lambda-\mu}{2}+\xi_{\mathrm{L}}(a))\,B^{L-1-q}_{\mathrm{L}}\!\left(\begin{smallmatrix} a & \\ & L+1-b \\ a & \end{smallmatrix}\middle|\, u+(q+1)\lambda\right)
\end{aligned} \tag{3.47}$$

$$\begin{aligned}
&\theta^{L-q}_{-2}(u+\tfrac{\lambda-\mu}{2}-\xi_{\mathrm{R}}(a))\,\theta^{L-q}_{-2}(u+\tfrac{\lambda-\mu}{2}+\xi_{\mathrm{R}}(a))\,B^q_{\mathrm{R}}\!\left(\begin{smallmatrix} & a \\ b & \\ & a \end{smallmatrix}\middle|\, u\right) \;=\\
&\quad -\left(\frac{\theta^L_L(0)}{X^q_{ab}}\right)^2 \theta^{q+1}_{q-1-a}(u+\tfrac{\lambda-\mu}{2}-\xi_{\mathrm{R}}(a))\,\theta^{q+1}_{q-1+a}(u+\tfrac{\lambda-\mu}{2}+\xi_{\mathrm{R}}(a))\,B^{L-1-q}_{\mathrm{R}}\!\left(\begin{smallmatrix} & a \\ L+1-b & \\ & a \end{smallmatrix}\middle|\, u+(q+1)\lambda\right)
\end{aligned}$$

and, provided that (2.28) is satisfied, full height reversal symmetry

$$\begin{aligned}
B^q_{\mathrm{L}}\!\left(\begin{smallmatrix} c & \\ & b \\ a & \end{smallmatrix}\middle|\, u\right) &= B^q_{\mathrm{L}}\!\left(\begin{smallmatrix} L+1-c & \\ & L+1-b \\ L+1-a & \end{smallmatrix}\middle|\, u\right)\\
B^q_{\mathrm{R}}\!\left(\begin{smallmatrix} & c \\ b & \\ & a \end{smallmatrix}\middle|\, u\right) &= B^q_{\mathrm{R}}\!\left(\begin{smallmatrix} & L+1-c \\ L+1-b & \\ & L+1-a \end{smallmatrix}\middle|\, u\right)
\end{aligned} \tag{3.48}$$

3.2.3 Local Relations

We now consider the fused local relations, (3.5)–(3.10). It is shown in [43] that the ABF fused face weights satisfy the fused Yang-Baxter equation (3.5). A proof proceeds as follows: if fusion level -1 is involved, then each side of (3.5) is zero, if fusion level 0 is involved, then each side of (3.5) immediately reduces to a product of the same terms, and if higher fusion levels only are involved, then (3.5) can be verified by seting internal arbitrary spins equal to adjoining summed spins, using (3.36) to push all explicit occurrences of the fused adjacency condition to external edges, and applying the original Yang Baxter equation, (2.3), pqr times.

It can also be shown that the ABF fused face weights satisfy the fused inversion relation (3.6) with

$$\rho^{qr}(u) = \theta^q_{-1}(u) \tag{3.49}$$

Again, if fusion level -1 is involved, then each side of (3.6) is zero, if fusion level 0 is involved, then the left side of (3.6) immediately reduces to the same product of terms as the right side, and if higher fusion levels only are involved, then (3.6) can be verified by setting internal arbitrary spins equal to adjoining summed spins, using (3.36) to push all explicit occurrences of the fused adjacency condition to external edges, and applying the original inversion relation, (2.4), qr times.

Finally, in Appendix B we show that the ABF fused face and boundary weights, together with ρ^{qr} given by (3.49), also satisfy the fused reflection equations, (3.7) and (3.8), and fused boundary crossing equations, (3.9) and (3.10), where λ in these equations is taken as the crossing parameter (2.16) and μ is arbitrary.

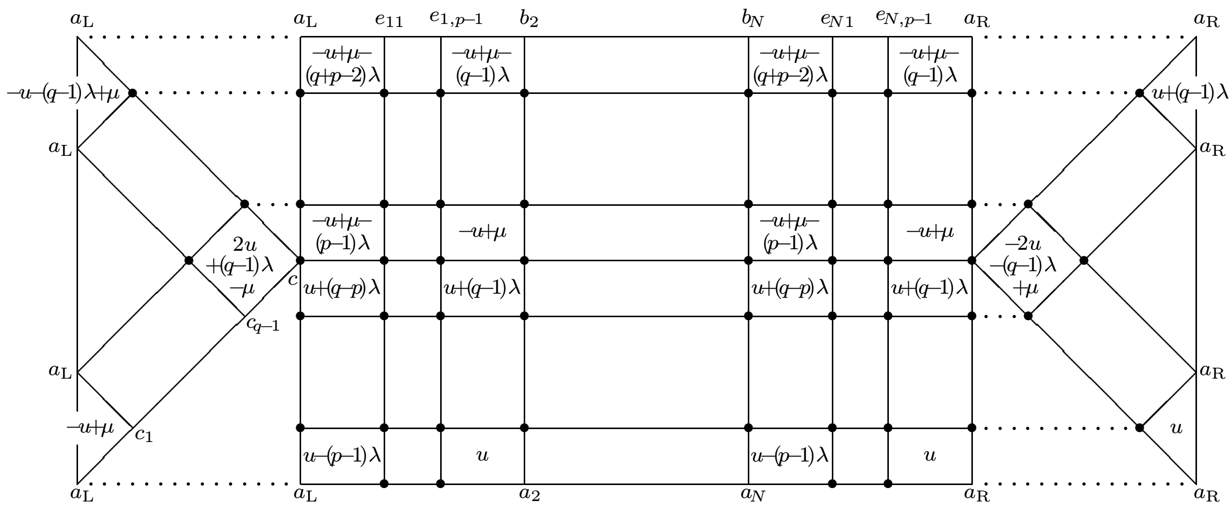
3.2.4 Double-Row Transfer Matrices

We now consider the ABF fused double-row transfer matrices $\boldsymbol{D}^{pq}(u)$, which are defined by (3.4), with λ given by (2.16), and the ABF fused double-row transfer matrices with fixed left and right boundary spins $a_{\rm L}$ and $a_{\rm R}$, $\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)$, given by

$$\langle a_2,\ldots,a_N\,|\,\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)\,|\,b_2,\ldots,b_N\rangle = \langle a_{\rm L},a_2,\ldots,a_N,a_{\rm R}\,|\,\boldsymbol{D}^{pq}(u)\,|\,a_{\rm L},b_2,\ldots,b_N,a_{\rm R}\rangle \quad (3.50)$$

By re-configuring the fused face weights in the top row according to (3.38), using (3.36) repeatedly to push all explicit entries of A^p to the lower edges, and all explicit entries of A^q to a single internal edge, setting as many internal arbitrary spins as possible equal to adjoining summed spins, and cancelling all of the gauge factors G which appear along the top row, we find that $\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)$ can be written as

$$\langle a_2,\ldots,a_N\,|\,\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)\,|\,b_2,\ldots,b_N\rangle \;=\; \frac{A^p_{a_{\rm L}a_2}\ldots A^p_{a_N a_{\rm R}}}{K^{pq}(u)}\;\sum_c A^q_{a_{\rm L}c}\times \quad (3.51)$$



where

$$K^{pq}(u) = \prod_{j=0}^{q-2}\Big(\theta^p_j(u)\;\theta^p_{-j-1}(-u+\mu)\Big)^N\;\theta^{j+1}_{2j+1}(2u-\mu)\;\theta^{j+1}_{-j-1}(-2u+\mu) \quad (3.52)$$

and we must have

$$(e_{11},\ldots,e_{1,p-1})\in P^p_{a_{\rm L}b_2},\;\ldots\;,(e_{N1},\ldots,e_{N,p-1})\in P^p_{b_N a_{\rm R}}$$

and, for each c in the sum,

$$(c_1,\ldots,c_{q-1})\in P^q_{a_{\rm L}c}$$

We note that the spins $c_1,\ldots,c_{q-1}$ can not be set equal to the adjoining summed spins, as they only become arbitrary *after* the summation has occured.

Since the required fused local relations are satisfied, we have, from (3.13), commutativity

$$\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)\;\boldsymbol{D}^{pr}(a_{\rm L}a_{\rm R}|v)=\boldsymbol{D}^{pr}(a_{\rm L}a_{\rm R}|v)\;\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)\tag{3.53}$$

and, from (3.12), crossing symmetry

$$\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)=\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|-u-(q-1)\lambda+\mu)\tag{3.54}$$

It follows, from (3.42) and (3.47), that the ABF fused double-row transfer matrices also satisfy partial height reversal symmetry

$$\alpha_{-2}^{L-q}(a_{\rm L}a_{\rm R}|u)\;\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)\;=\;(-1)^{pN}\;\beta_{q-1}^{q+1}(a_{\rm L}a_{\rm R}|u)\;\boldsymbol{D}^{p,L-1-q}(a_{\rm L}a_{\rm R}|u+(q+1)\lambda)\tag{3.55}$$

or, equivalently,

$$\beta_{-2}^{L-q}(a_{\rm L}a_{\rm R}|u)\;\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)\;=\;(-1)^{pN}\;\alpha_{q-1}^{q+1}(a_{\rm L}a_{\rm R}|u)\;\boldsymbol{D}^{p,L-1-q}(a_{\rm L}a_{\rm R}|u+(q+1)\lambda)\tag{3.56}$$

where

$$\begin{aligned}\alpha_k^r(a_{\rm L}a_{\rm R}|u)\;&=\;\theta_k^r(u+\tfrac{\lambda-\mu}{2}-\xi_{\rm L}(a_{\rm L}))\;\theta_k^r(u+\tfrac{\lambda-\mu}{2}+\xi_{\rm L}(a_{\rm R}))\\&\qquad\times\;\;\theta_k^r(u+\tfrac{\lambda-\mu}{2}-\xi_{\rm R}(a_{\rm R}))\;\theta_k^r(u+\tfrac{\lambda-\mu}{2}+\xi_{\rm R}(a_{\rm R}))\end{aligned}\tag{3.57}$$

$$\begin{aligned}\beta_k^r(a_{\rm L}a_{\rm R}|u)\;&=\;\theta_{k-a_{\rm L}}^r(u+\tfrac{\lambda-\mu}{2}-\xi_{\rm L}(a_{\rm L}))\;\theta_{k+a_{\rm L}}^r(u+\tfrac{\lambda-\mu}{2}+\xi_{\rm L}(a_{\rm L}))\\&\qquad\times\;\;\theta_{k-a_{\rm R}}^r(u+\tfrac{\lambda-\mu}{2}-\xi_{\rm R}(a_{\rm R}))\;\theta_{k+a_{\rm R}}^r(u+\tfrac{\lambda-\mu}{2}+\xi_{\rm R}(a_{\rm R}))\end{aligned}\tag{3.58}$$

Considering $q=-1$, 0, $L-1$ and L in (3.55) or (3.56), we have, using (3.28) and (3.29),

$$\boldsymbol{D}^{p,-1}(a_{\rm L}a_{\rm R}|u)\;=\;\boldsymbol{0}\tag{3.59}$$

$$\boldsymbol{D}^{p,L}(a_{\rm L}a_{\rm R}|u)\;=\;\boldsymbol{0}\tag{3.60}$$

$$\boldsymbol{D}^{p,0}(a_{\rm L}a_{\rm R}|u)\;=\;f^p(u-\lambda)\;\boldsymbol{I}^p(a_{\rm L}a_{\rm R})\tag{3.61}$$

$$\boldsymbol{D}^{p,L-1}(a_{\rm L}a_{\rm R}|u)\;=\;(-1)^{pN}\;\frac{\alpha_{L-2}^{L}(a_{\rm L}a_{\rm R}|u)}{\beta_{-2}^{1}(a_{\rm L}a_{\rm R}|u)}\;f^p(u-2\lambda)\;\boldsymbol{I}^p(a_{\rm L}a_{\rm R})\tag{3.62}$$

$$=\;(-1)^{pN}\;\frac{\beta_{L-2}^{L}(a_{\rm L}a_{\rm R}|u)}{\alpha_{-2}^{1}(a_{\rm L}a_{\rm R}|u)}\;f^p(u-2\lambda)\;\boldsymbol{I}^p(a_{\rm L}a_{\rm R})\tag{3.63}$$

where

$$f^p(u)=\left(\theta_0^p(u)\;\theta_{-1}^p(-u+\mu)\right)^N\tag{3.64}$$

and $\boldsymbol{I}^p(a_{\rm L}a_{\rm R})$ is the adjacency-inclusive identity

$$\begin{aligned}\langle a_2,a_3,\ldots,a_{N-1},a_N\,|\,\boldsymbol{I}^p(a_{\rm L}a_{\rm R})\,|\,b_2,b_3,\ldots,b_{N-1},b_N\rangle\quad&=\\ \delta_{a_2b_2}\ldots\delta_{a_Nb_N}\;A^p_{a_{\rm L}a_2}\,A^p_{a_2a_3}\ldots A^p_{a_{N-1}a_N}\;A^p_{a_Na_{\rm R}}&\end{aligned}\tag{3.65}$$

It follows from (3.43) and (3.48), with (2.28), that the ABF fused double-row transfer matrices satisfy full height reversal symmetry

$$\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)=\boldsymbol{Y}\;\boldsymbol{D}^{pq}(L{+}1{-}a_{\rm L},L{+}1{-}a_{\rm R}|u)\;\boldsymbol{Y} \tag{3.66}$$

where

$$\langle a_2,\ldots,a_N\,|\,\boldsymbol{Y}\,|\,b_2,\ldots,b_N\rangle=\delta_{L+1-a_2\,,\,b_2}\;\ldots\;\delta_{L+1-a_N\,,\,b_N} \tag{3.67}$$

For the ABF fused models, a natural choice for μ in $\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)$ is

$$\mu=p\lambda \tag{3.68}$$

We note that p-dependence, as opposed to q-dependence, of μ in $\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)$ does not destroy commutativity or crossing symmetry. With this choice, crossing symmetry of the fused face weights, (3.41), implies that $\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)$ is similar to its transpose

$$\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)=\boldsymbol{S}^{p}(a_{\rm L}a_{\rm R})^{-1}\;\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)^{t}\;\boldsymbol{S}^{p}(a_{\rm L}a_{\rm R}) \tag{3.69}$$

where

$$\begin{aligned}\langle a_2,a_3,\ldots,a_{N-1},a_N\,|\,\boldsymbol{S}^{p}(a_{\rm L}a_{\rm R})\,|\,b_2,b_3,\ldots,b_{N-1},b_N\rangle\quad&=\\ \delta_{a_2b_2}\ldots\delta_{a_Nb_N}\;X^p_{a_{\rm L}a_2}\,X^p_{a_2a_3}\ldots X^p_{a_{N-1}a_N}\;X^p_{a_Na_{\rm R}}\;&\frac{\theta(\lambda)^{N-1}}{\theta(a_2\lambda)\ldots\theta(a_N\lambda)}\end{aligned} \tag{3.70}$$

3.2.5 Functional Equations

In Appendix C, we show that the ABF fused double-row transfer matrices satisfy functional equations whose structure reflects that of the $su(2)$ fusion rule (3.20) satisfied by the adjacency matrices. There are two families of functional equations,

$$\begin{aligned}g^0_q(2u{-}\lambda)\;\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)\;\boldsymbol{D}^{p1}(a_{\rm L}a_{\rm R}|u{-}\lambda)\quad&=\\ \alpha^1_{-1}(a_{\rm L}a_{\rm R}|u)\;\beta^1_{-1}(a_{\rm L}a_{\rm R}|u)\;g^{-1}_q(2u{-}\lambda)\;f^p(u{-}2\lambda)\;&\boldsymbol{D}^{p,q-1}(a_{\rm L}a_{\rm R}|u{+}\lambda)\\ +\quad g^1_q(2u{-}\lambda)\;f^p(u{-}\lambda)\;&\boldsymbol{D}^{p,q+1}(a_{\rm L}a_{\rm R}|u{-}\lambda)\end{aligned} \tag{3.71}$$

and

$$\begin{aligned}g^0_q(2u{+}q\lambda)\;\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)\;\boldsymbol{D}^{p1}(a_{\rm L}a_{\rm R}|u{+}q\lambda)\quad&=\\ \alpha^1_{q-1}(a_{\rm L}a_{\rm R}|u)\;\beta^1_{q-1}(a_{\rm L}a_{\rm R}|u)\;g^{-1}_q(2u{+}q\lambda)\;f^p(u{+}q\lambda)\;&\boldsymbol{D}^{p,q-1}(a_{\rm L}a_{\rm R}|u)\\ +\quad g^1_q(2u{+}q\lambda)\;f^p(u{+}(q{-}1)\lambda)\;&\boldsymbol{D}^{p,q+1}(a_{\rm L}a_{\rm R}|u)\end{aligned} \tag{3.72}$$

where $-1\le p\le L$, $0\le q\le L{-}1$, and

$$g^k_q(u)=\frac{\theta(u{+}(k{-}1)\lambda{-}\mu)\;\theta(u{+}(q{-}k)\lambda{-}\mu)}{\theta(\lambda)^2} \tag{3.73}$$

The importance of these equations is that they describe the essential content of the fusion hierarchy, since we see that either family, together with $\boldsymbol{D}^{p0}(a_{\rm L}a_{\rm R}|u)$ and $\boldsymbol{D}^{p1}(a_{\rm L}a_{\rm R}|u)$, can be used to determine recursively the higher fusion level double-row transfer matrices.

It can be shown, using induction as done in [45] for the periodic-boundary case, that (3.59), (3.61), and either (3.71) or (3.72), imply that the fused double-row transfer matrices also satisfy equations which correspond to (3.21), and from which follow the generalised inversion identity, which corresponds to (3.22),

$$\begin{aligned}\boldsymbol{d}^{pq}(a_{\rm L}a_{\rm R}|u)\,\boldsymbol{d}^{pq}(a_{\rm L}a_{\rm R}|u+\lambda) &= \Big(\boldsymbol{I}^p(a_{\rm L}a_{\rm R})+\boldsymbol{d}^{p,q-1}(a_{\rm L}a_{\rm R}|u+\lambda)\Big)\\ &\qquad\times\quad\Big(\boldsymbol{I}^p(a_{\rm L}a_{\rm R})+\boldsymbol{d}^{p,q+1}(a_{\rm L}a_{\rm R}|u)\Big)\end{aligned}\tag{3.74}$$

where $-1\le p\le L$, $1\le q\le L-2$, and

$$\begin{aligned}\boldsymbol{d}^{pq}(a_{\rm L}a_{\rm R}|u) \quad &= \\ &\frac{\theta(2u+q\lambda-\mu)^2\;\boldsymbol{D}^{p,q-1}(a_{\rm L}a_{\rm R}|u+\lambda)\;\boldsymbol{D}^{p,q+1}(a_{\rm L}a_{\rm R}|u)}{\alpha^q_{q-1}(a_{\rm L}a_{\rm R}|u)\;\beta^q_{q-1}(a_{\rm L}a_{\rm R}|u)\;\theta(2u-\lambda-\mu)\;\theta(2u+(2q+1)\lambda-\mu)\;f^p(u-\lambda)f^p(u+q\lambda)}\end{aligned}\tag{3.75}$$

From (3.54), it follows that

$$\boldsymbol{d}^{pq}(a_{\rm L}a_{\rm R}|u)=\boldsymbol{d}^{pq}(a_{\rm L}a_{\rm R}|-u-q\lambda+\mu)\tag{3.76}$$

and, if μ is given by (3.68), then (3.69) implies that

$$\boldsymbol{d}^{pq}(a_{\rm L}a_{\rm R}|u)=\boldsymbol{S}^p(a_{\rm L}a_{\rm R})^{-1}\;\boldsymbol{d}^{pq}(a_{\rm L}a_{\rm R}|u)^t\;\boldsymbol{S}^p(a_{\rm L}a_{\rm R})\tag{3.77}$$

4. Discussion

We have presented a general formalism for applying fixed boundary conditions to IRF models and have specialised to the case of ABF models and their fusion hierarchy. In future work, we intend both to continue our study of ABF models and to proceed with the application of fixed boundary conditions to other IRF models.

With regard to the ABF models, we note that the functional equations (3.71) and (3.72) have the same $su(2)$ structure as those satisfied by the ABF row transfer matrices with periodic boundary conditions, and as first presented in [44]. We therefore plan to use the same approach as in [44], to obtain Bethe ansatz equations for the eigenspectra of the double-row transfer matrices. Subsequently, we hope to calculate the boundary free energy of these models, and to use the technique of [53, 54, 45] to calculate analytically the central charges and conformal weights of the conformal field theories associated with the models at criticality. Other directions in which our treatment of ABF models could be developed further would be to investigate the existence of boundary weights of a non-diagonal form, and to explore the connection between ABF boundary weights and known K matrices [16, 17, 18] for the eight-vertex model.

With regard to other models, the ABF models at criticality correspond to the A series within the classical A–D–E models [55], and we plan to study other members of this group.

At criticality, the face weights of these models satisfy the Yang-Baxter equation through the defining relations of the Temperley-Lieb algebra alone and it would be valuable to find boundary weights which similarly satisfy the IRF reflection equations through this algebra. The Temperley-Lieb-related A–D–E models are all based on the untwisted affine Lie algebra $A_1^{(1)}$, and we also hope to consider the dilute A–D–E models [56, 57] which are based on the twisted affine Lie algebra $A_2^{(2)}$. In these models, the unfused adjacency matrices allow identical spins to be adjacent, which would therefore enable us to consider the case of fixed boundaries of the form $a, a, a, \ldots$, whereas for the level 1 ABF models only the form $a, a\pm 1, a, \ldots$ is possible. Finally, it would be worthwhile to study the application of fixed boundary conditions to higher-rank IRF models, such as those based on the untwisted affine Lie algebras $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$ [58].

Appendix A: Derivation of ABF Boundary Weights

In this appendix, we find boundary weights which, together with the ABF face weights (2.17), satisfy the reflection equations (2.5) and (2.6). We then show that these weights also satisfy the boundary crossing equations (2.7) and (2.8).

Since the ABF weights satisfy the symmetry (2.11), the left and right reflection equations are effectively the same, so it suffices to solve them together. We assume that there are solutions which have the diagonal form

$$B_{\mathrm{L}}\left(\begin{matrix} c \\ a \end{matrix}\; b \,\middle|\, u\right) = B_{\mathrm{R}}\left(b\; \begin{matrix} c \\ a \end{matrix} \,\middle|\, u\right) = k(a,b|u)\,\delta_{ac} \tag{A.1}$$

The reflection equations then become

$$\begin{aligned}\sum_f W\left(\begin{matrix} c & f \\ b & a \end{matrix} \,\middle|\, u-v\right) W\left(\begin{matrix} d & a \\ c & f \end{matrix} \,\middle|\, \mu-u-v\right) k(a,f|u)\,k(a,d|v) = \\ \sum_f W\left(\begin{matrix} a & f \\ d & c \end{matrix} \,\middle|\, u-v\right) W\left(\begin{matrix} f & a \\ c & b \end{matrix} \,\middle|\, \mu-u-v\right) k(a,f|u)\,k(a,b|v)\end{aligned} \tag{A.2}$$

This equation is trivially satisfied if $A_{ab}\,A_{bc}\,A_{cd}\,A_{da} = 0$. Proceeding to $A_{ab}\,A_{bc}\,A_{cd}\,A_{da} = 1$, we see that if $b = d = a\pm 1$ and $c = a$ or $c = a\pm 2$, then both sides of (A.2) are automatically equal, since the ABF weights satisfy the symmetry (2.10). The only class of assignments remaining is $b = a\pm 1$, $c = a$ and $d = a\mp 1$. This gives $L-2$ pairs of identical equations which we find, after substituting the explicit face weights (2.17) into (A.2), are given by

$$\begin{aligned} &\sqrt{\frac{\theta((a+1)\lambda)}{\theta((a-1)\lambda)}}\,\theta(u-v)\,\theta(u+v-a\lambda+\lambda-\mu)\,k(a,a-1|u)\,k(a,a-1|v) \\ &\qquad - \theta(u+v+\lambda-\mu)\,\theta(u-v-a\lambda)\,k(a,a-1|u)\,k(a,a+1|v) \\ &\qquad - \theta(u+v+\lambda-\mu)\,\theta(u-v+a\lambda)\,k(a,a+1|u)\,k(a,a-1|v) \\ &+ \sqrt{\frac{\theta((a-1)\lambda)}{\theta((a+1)\lambda)}}\,\theta(u-v)\,\theta(u+v+a\lambda+\lambda-\mu)\,k(a,a+1|u)\,k(a,a+1|v) = 0 \end{aligned} \tag{A.3}$$

where $a=2,\dots,L{-}1$. We note that the boundary weights $k(1,2|u)$ and $k(L,L{-}1|u)$ do not appear in any of these equations and can therefore be set to arbitrary functions, $g_1(u)$ and $g_L(u)$ respectively. Returning to $a=2,\dots,L{-}1$, we now assume that there are constants $\xi(a)$ for which $k(a,a-1|\xi(a)-\frac{\lambda-\mu}{2})=0$. Taking $v=\xi(a)-\frac{\lambda-\mu}{2}$ in (A.3), we find that solutions must have the form

$$\begin{aligned} k(a,a{-}1|u) &= \sqrt{\theta((a{-}1)\lambda)}\ \theta(u{+}\tfrac{\lambda-\mu}{2}{-}\xi(a))\ \theta(u{+}a\lambda{+}\tfrac{\lambda-\mu}{2}{+}\xi(a))\ g_a(u) \\ k(a,a{+}1|u) &= \sqrt{\theta((a{+}1)\lambda)}\ \theta(u{+}\tfrac{\lambda-\mu}{2}{+}\xi(a))\ \theta(u{-}a\lambda{+}\tfrac{\lambda-\mu}{2}{-}\xi(a))\ g_a(u) \end{aligned} \tag{A.4}$$

for some functions g_a.

We now verify that these are in fact solutions for arbitrary constants $\xi(a)$ and arbitrary functions g_a. Substituting (A.4) into the left side of (A.3) gives

$$\begin{aligned} &\sqrt{\theta((a{+}1)\lambda)\,\theta((a{-}1)\lambda)}\ g_a(u)\ g_a(v)\ \Big(Q_a(u+\tfrac{\lambda-\mu}{2},v+\tfrac{\lambda-\mu}{2})-Q_a(u+\tfrac{\lambda-\mu}{2},-v-\tfrac{\lambda-\mu}{2}) \\ &\qquad -Q_a(-u-\tfrac{\lambda-\mu}{2},v+\tfrac{\lambda-\mu}{2})+Q_a(-u-\tfrac{\lambda-\mu}{2},-v-\tfrac{\lambda-\mu}{2})\Big) \end{aligned}$$

where

$$Q_a(u,v)=\theta(u{-}v)\ \theta(u{+}v{-}a\lambda)\ \theta(u{-}\xi(a))\ \theta(u{+}a\lambda{+}\xi(a))\ \theta(v{-}\xi(a))\ \theta(v{+}a\lambda{+}\xi(a)) \tag{A.5}$$

We now find that

$$\begin{aligned} Q_a(u,v)-Q_a(u,-v) &= \theta(u{-}\xi(a))\ \theta(u{+}a\lambda{+}\xi(a))\times \\ &\quad \Big(\theta(u{-}v)\ \theta(u{+}v{-}a\lambda)\ \theta(v{-}\xi(a))\ \theta(v{+}a\lambda{+}\xi(a)) \\ &\qquad -\ \theta(u{+}v)\ \theta(u{-}v{-}a\lambda)\ \theta({-}v{-}\xi(a))\ \theta({-}v{+}a\lambda{+}\xi(a))\Big) \\ &= \theta(u{-}\xi(a))\ \theta(u{+}a\lambda{+}\xi(a))\ \theta(u{-}a\lambda{-}\xi(a))\ \theta(u{+}\xi(a))\ \theta(a\lambda)\ \theta(2v) \end{aligned}$$

where we have used the identity (2.21) with

$$s=u-\frac{a\lambda}{2},\quad x=-v+\frac{a\lambda}{2},\quad t=v+\frac{a\lambda}{2},\quad y=-\frac{a\lambda}{2}-\xi(a) \tag{A.6}$$

We can now see that $Q_a(u,v)-Q_a(u,-v)$ is even in u and therefore that the left side of (A.3) vanishes as required. The solution (A.4) obtained here matches that of (2.27) if we take

$$\begin{aligned} g_1(u) &= \sqrt{\frac{\theta(2\lambda)}{\theta(\lambda)}}\ \frac{\theta(u{+}\frac{\lambda-\mu}{2}{+}\xi(1))\ \theta(u{-}\ \lambda{+}\frac{\lambda-\mu}{2}{-}\xi(1))}{\theta(\lambda)^2} \\ g_a(u) &= \frac{1}{\sqrt{\theta(a\lambda)}\ \theta(\lambda)^2} \\ g_L(u) &= \sqrt{\frac{\theta((L{-}1)\lambda)}{\theta(L\lambda)}}\ \frac{\theta(u{+}\frac{\lambda-\mu}{2}{-}\xi(L))\ \theta(u{+}L\lambda{+}\frac{\lambda-\mu}{2}{+}\xi(L))}{\theta(\lambda)^2} \end{aligned} \tag{A.7}$$

and set $\xi\mapsto\xi_{\mathrm{L}}$ and $\xi\mapsto\xi_{\mathrm{R}}$ for the left and right boundary weights respectively.

Finally, we consider the boundary crossing equations, (2.7) and (2.8), with the ABF face weights, the boundary weights found here, and ρ given by (2.26). These equations are satisfied since if $\delta_{ac}\, A_{ab}=0$, then both sides of the equations are zero, if $a=c=1,\ L$ and $b=2,\ L-1$, then the left sides are single terms which we immediately find are equal to the terms on the right side, and if $2\le a=c\le L-1$ and $b=a\pm 1$, then the left sides are sums of two terms which we find can be reduced to the terms on the right side using a single application of (2.21).

Appendix B: ABF Fused Reflection and Boundary Crossing Equations

In this appendix, we show that the fused right reflection and boundary crossing equations, (3.8) and (3.10), are satisfied by the ABF fused weights. The proofs for the fused left reflection and boundary crossing equations, (3.7) and (3.9), are similar.

We begin with (3.8). If $q=-1$ or $r=-1$ then, due to (3.28), each side of (3.8) is zero, and if $q=0$ or $r=0$ then, using (3.28) and (3.49), we find that each side of (3.8) reduces to a product of the same terms.

We now proceed to the case $q\ge 1$ and $r\ge 1$. Having substituted the ABF fused weights, (3.33) and (3.35), and ρ given by (3.49), into (3.8), we then re-configure the central fused face weights on each side according to (3.38) and the upper fused face weight on the right side according to (3.40), set internal arbitrary spins equal to adjoining summed spins, use (3.36) to push all explicit occurrences of the fused adjacency condition to external edges, cancel internal gauge factors G, and take external arbitrary spins to be the same on each side of the equation. After these steps, we find that the left side of (3.8) is given by

$$\frac{\theta^q_{-1}(u-v+(q-r)\lambda)\,\theta^q_{-1}(-u-v-(r-1)\lambda+\mu)}{\prod_{j=0}^{q-2}\theta^r_j(u-v)\,\theta^r_j(-u-v-(q-1)\lambda+\mu)\,\theta^{j+1}_{2j+1}(2u-\mu)\,\prod_{j=0}^{r-2}\theta^{j+1}_{2j+1}(2v-\mu)}\times$$

$$\frac{\delta_{ae}\,A^r_{ab}\,A^q_{bc}}{G^q_{c,h_1,\ldots,h_{q-1},d}\,G^r_{d,i_1,\ldots,i_{r-1},a}}\sum_{f_1\ldots f_{r-1}}\sum_{g_1\ldots g_{q-1}}\mathcal{L}^{qr}(u,v)_{a,b,c,d,a,f_1,\ldots,f_{r-1},g_1,\ldots,g_{q-1},h_1,\ldots,h_{q-1},i_1,\ldots,i_{r-1}}$$

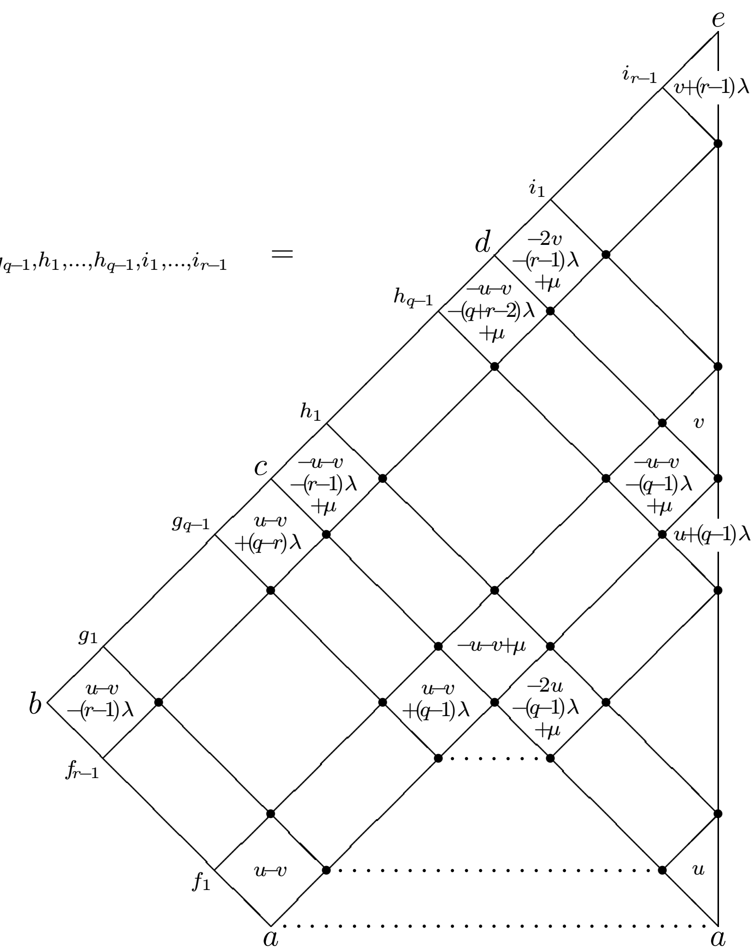
and that the right side of (3.8) is given by

$$\frac{\theta^r_{-1}(u-v)\,\theta^r_{-1}(-u-v-(q-1)\lambda+\mu)}{\prod_{j=0}^{r-2}\theta^q_j(u-v+(q-r)\lambda)\,\theta^q_j(-u-v-(r-1)\lambda+\mu)\,\theta^{j+1}_{2j+1}(2v-\mu)\,\prod_{j=0}^{q-2}\theta^{j+1}_{2j+1}(2u-\mu)}\times$$

$$\frac{\delta_{ae}\,A^r_{ab}\,A^q_{bc}}{G^q_{c,h_1,\ldots,h_{q-1},d}\,G^r_{d,i_1,\ldots,i_{r-1},a}}\sum_{f_1\ldots f_{r-1}}\sum_{g_1\ldots g_{q-1}}\mathcal{R}^{qr}(u,v)_{a,b,c,d,a,f_1,\ldots,f_{r-1},g_1,\ldots,g_{q-1},h_1,\ldots,h_{q-1},i_1,\ldots,i_{r-1}}$$

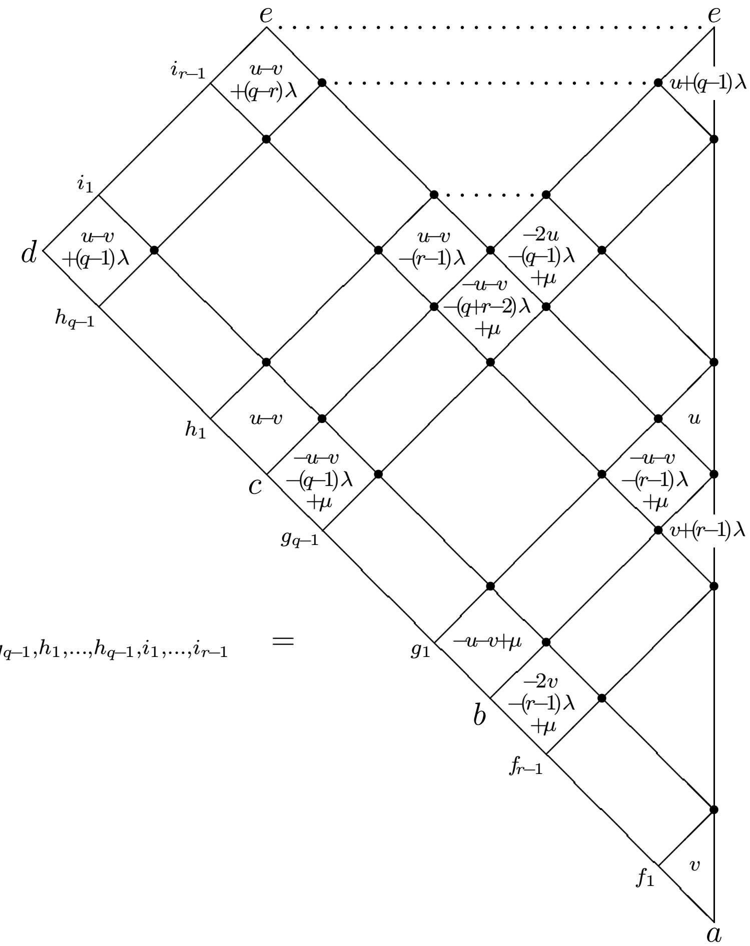
where we must have $(h_1,\ldots,h_{q-1})\in P^q_{cd}$ and $(i_1,\ldots,i_{r-1})\in P^r_{da}$, and where

$$\mathcal{L}^{qr}(u,v)_{a,b,c,d,e,f_1,\ldots,f_{r-1},g_1,\ldots,g_{q-1},h_1,\ldots,h_{q-1},i_1,\ldots,i_{r-1}} \quad =$$



and

$$\mathcal{R}^{qr}(u,v)_{a,b,c,d,e,f_1,\ldots,f_{r-1},g_1,\ldots,g_{q-1},h_1,\ldots,h_{q-1},i_1,\ldots,i_{r-1}} =$$



We now claim that, for any model in which the original Yang Baxter equation, (2.3), and right reflection equation, (2.6), are satisfied, and for arbitrary λ, we in fact have

$$\mathcal{L}^{qr}(u,v)_{a,b,c,d,e,f_1,\ldots,f_{r-1},g_1,\ldots,g_{q-1},h_1,\ldots,h_{q-1},i_1,\ldots,i_{r-1}} = \mathcal{R}^{qr}(u,v)_{a,b,c,d,e,f_1,\ldots,f_{r-1},g_1,\ldots,g_{q-1},h_1,\ldots,h_{q-1},i_1,\ldots,i_{r-1}} \tag{B.1}$$

This can be proved by induction, which consists of showing that

$$\mathcal{L}^{1,1}(u,v) = \mathcal{R}^{1,1}(u,v)$$

that

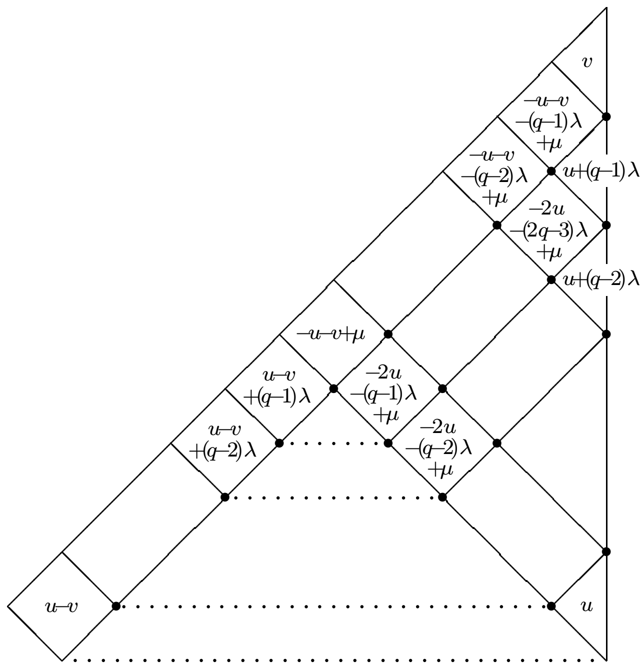
$$\mathcal{L}^{1,1}(u,v) = \mathcal{R}^{1,1}(u,v) \quad \text{and} \quad \mathcal{L}^{q-1,1}(u,v) = \mathcal{R}^{q-1,1}(u,v) \quad \text{imply that} \quad \mathcal{L}^{q,1}(u,v) = \mathcal{R}^{q,1}(u,v)$$

and, finally, that

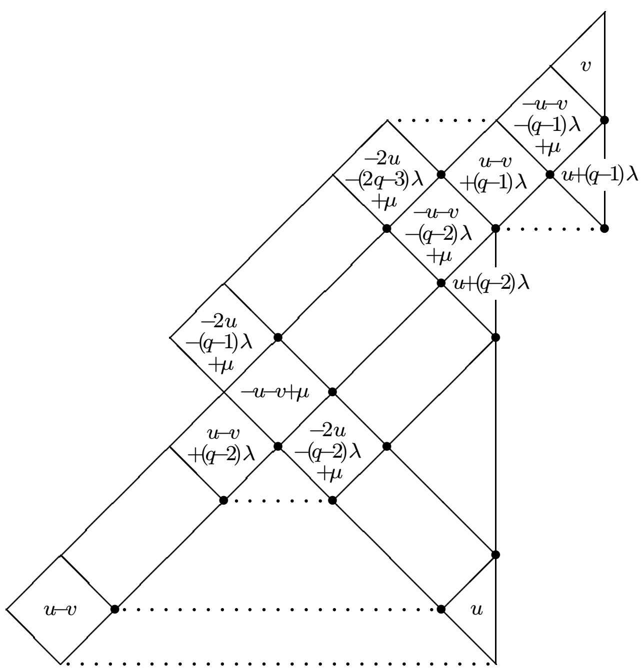
$$\mathcal{L}^{q,1}(u,v) = \mathcal{R}^{q,1}(u,v) \quad \text{and} \quad \mathcal{L}^{q,r-1}(u,v) = \mathcal{R}^{q,r-1}(u,v) \quad \text{imply that} \quad \mathcal{L}^{q,r}(u,v) = \mathcal{R}^{q,r}(u,v)$$

We know the first statement holds, since it is simply the original right reflection equation (2.6). We shall only explicitly demonstrate the second statement, since the third can be demonstrated similarly. We have, for $q \geq 2$,

$\mathcal{L}^{q,1}(u,v) \quad =$

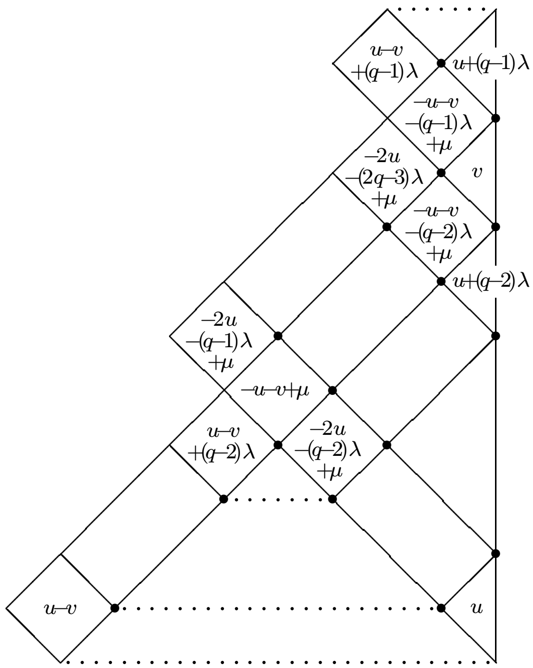


$=$



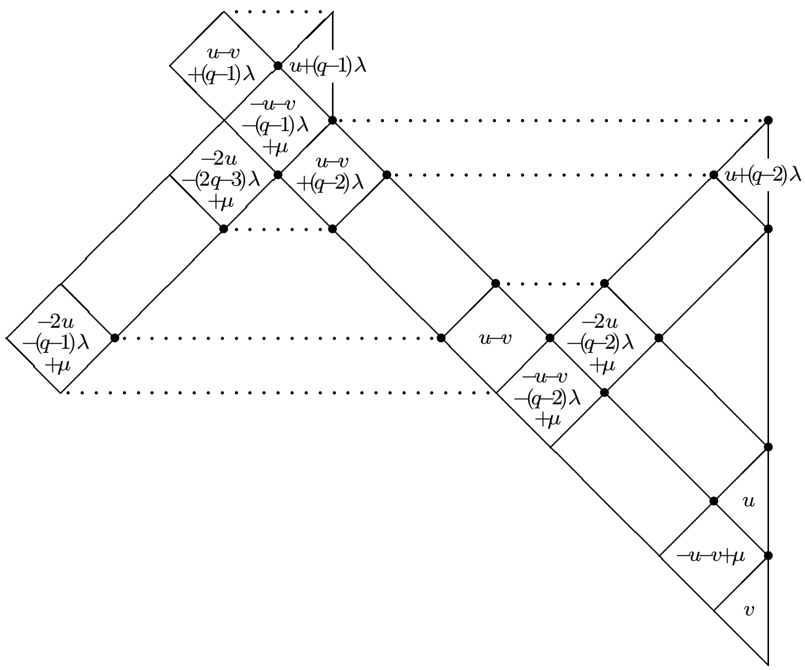
using the Yang-Baxter equation $q-1$ times

$=$

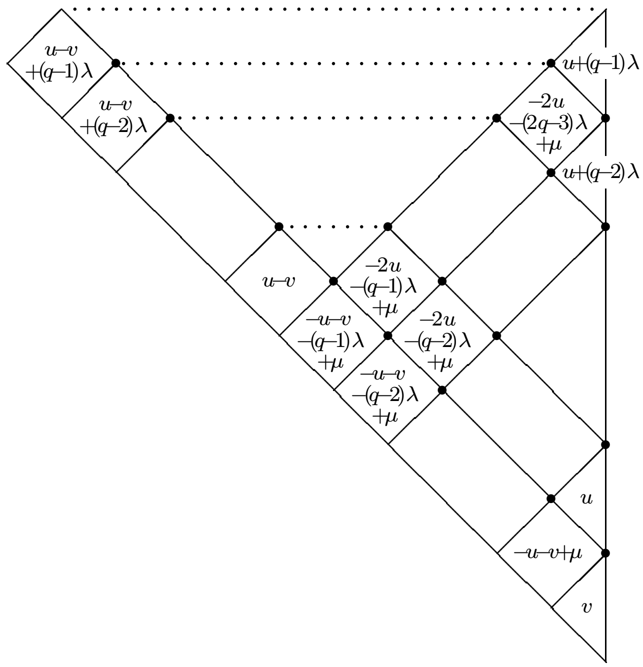


applying
$\mathcal{L}^{1,1}(u,v) =$
$\mathcal{R}^{1,1}(u,v)$

$=$



applying
$\mathcal{L}^{q-1,1}(u,v) =$
$\mathcal{R}^{q-1,1}(u,v)$



$=$ using the Yang-Baxter equation $q-1$ times

$$= \qquad \mathcal{R}^{q,1}(u,v)$$

Having established (B.1), it is straightforward to verify that the u, v dependent factors on each side of (3.8) are equal, which completes our proof that (3.8) is satisfied.

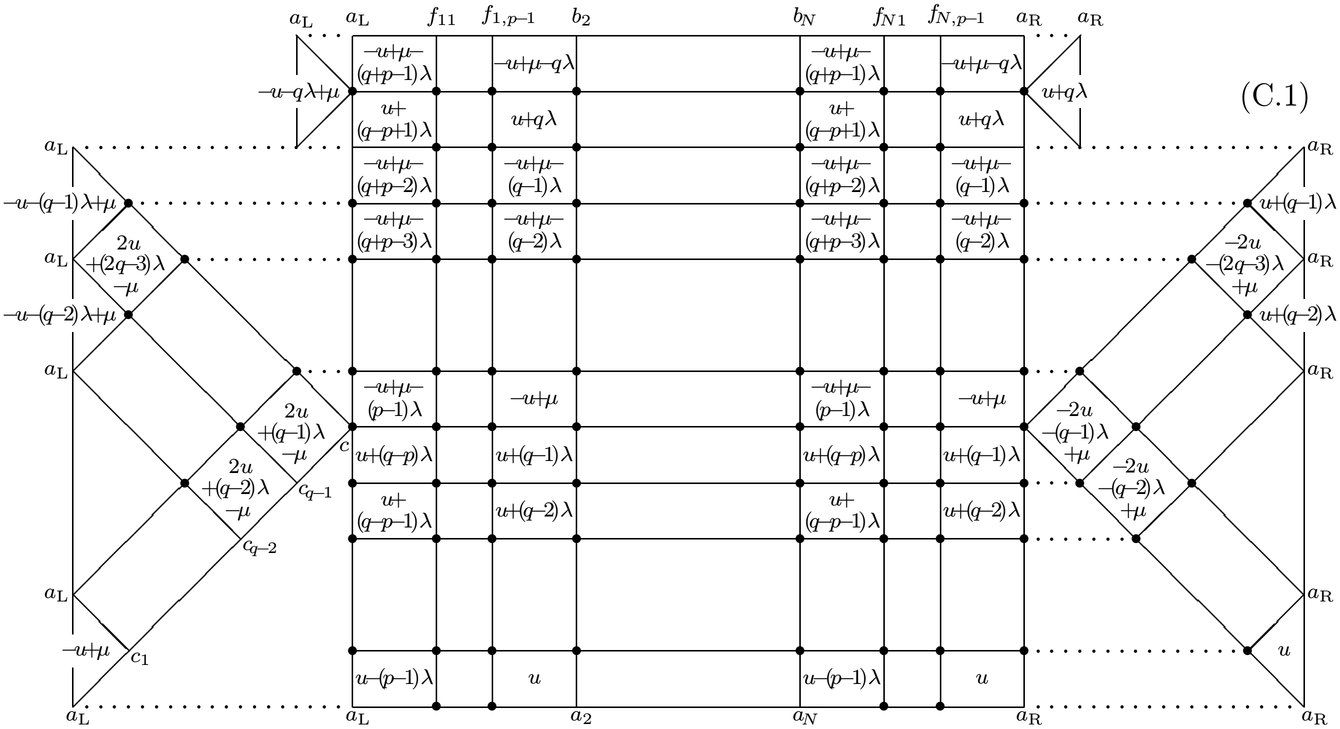
The proof that the fused right boundary crossing equation, (3.10), is also satisfied by the ABF weights corresponds closely to that for the fused right reflection equation. If $q=-1$ then each side of (3.10) is zero, and if $q=0$, then each side of (3.10) is given by $\delta_{ab}\,\delta_{bc}$. For $q\geq 1$, we re-configure the fused face weight on the left side of (3.10) according to (3.39), set internal arbitrary spins equal to adjoining summed spins, use (3.36) to push explicit occurrences of the fused adjacency condition to external edges, cancel internal gauge factors G, and take the external arbitrary spins to be the same on each side of the equation. After these steps, we find that each side of (3.8) is proportional to a sum of products of level 1 face and boundary weights, and that the proof can be completed by using induction on q to show that face and boundary weight components of each side are proportional, and then verifying that the overall proportionality factors on each side are the same. The induction argument here is valid for any model in which the original inversion relation, (2.4), and right boundary crossing equation, (2.8), are satisfied.

Appendix C: Proof of ABF Functional Equations

In this appendix, we prove that the ABF fused double-row transfer matrices satisfy the functional equations (3.72). The proof of (3.71) is similar.

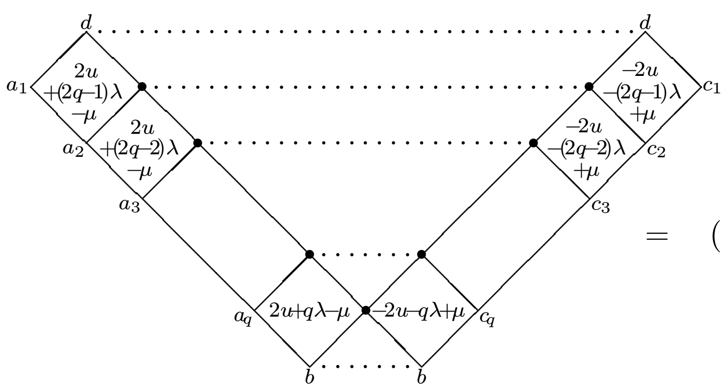
We note that for $q=0$, (3.72) is immediately satisfied due to (3.59) and (3.61). We therefore proceed to the case $q\geq 1$ and begin by considering an entry of $\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)$ $\boldsymbol{D}^{p1}(a_{\rm L}a_{\rm R}|u+q\lambda)$. Using, (3.51), we find that

$$\langle a_2,\ldots,a_N\,|\,\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)\;\boldsymbol{D}^{p1}(a_{\rm L}a_{\rm R}|u+q\lambda)\,|\,b_2,\ldots,b_N\rangle \;=\; \frac{A^p_{a_{\rm L}a_2}\ldots A^p_{a_N a_{\rm R}}}{K^{pq}(u)}\;\sum_c A^q_{a_{\rm L}c}\;\times$$



(C.1)

where we must have $(c_1,\ldots,c_{q-1})\in P^q_{a_{\rm L}c}$, for each c in the sum, and $(f_{11},\ldots,f_{1,p-1})\in P^p_{a_{\rm L}b_2}$, $\ldots$, $(f_{N1},\ldots,f_{N,p-1})\in P^p_{b_N a_{\rm R}}$. We now use the identity



$$= \quad (-1)^q\;\theta^q_{2q-2}(2u-\mu)\;\theta^q_{2q}(2u-\mu) \qquad \text{(C.2)}$$
$$\times\;\delta_{a_1c_1}\ldots\delta_{a_qc_q}\;A_{da_1}\ldots A_{a_qb}$$

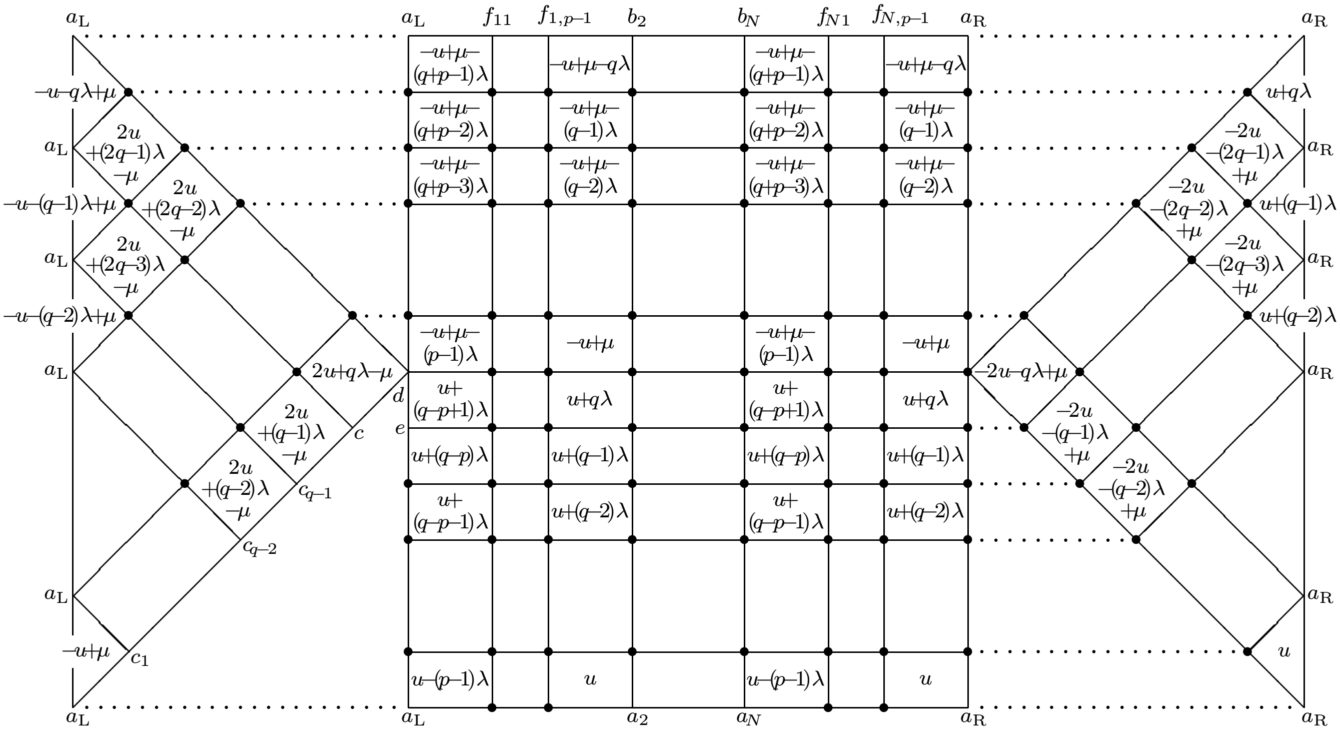
which can be obtained by using the inversion relation q times. By inserting (C.2) just below the top row of faces in (C.1), and then using the Yang-Baxter equation qpN times, we find that

$$(-1)^q\;\theta^q_{2q-2}(2u-\mu)\;\theta^q_{2q}(2u-\mu)\;\langle a_2,\ldots,a_N\,|\,\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)\;\boldsymbol{D}^{p1}(a_{\rm L}a_{\rm R}|u+q\lambda)\,|\,b_2,\ldots,b_N\rangle \;=$$

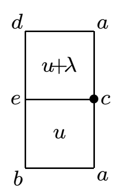
$$\frac{1}{K^{pq}(u)}\;\sum_d\;\sum_c\;A^q_{a_{\rm L}c}\;\mathcal{D}(a_{\rm L},c_1,\ldots,c_{q-2},c_{q-1},c,d,c)\Big|_{(c_1,\ldots,c_{q-1})\in P^q_{a_{\rm L}c}} \qquad \text{(C.3)}$$

where we define

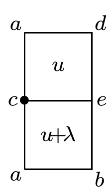
$$\mathcal{D}(a_{\rm L}, c_1, \ldots, c_{q-2}, c_{q-1}, c, d, e) \quad = \quad A^p_{a_{\rm L}a_2} \ldots A^p_{a_N a_{\rm R}} \times \tag{C.4}$$



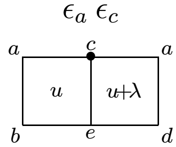
We note that the dependence of $\mathcal{D}$ on p, q, u and all of the external spins except $a_{\rm L}$ has been suppressed. The next step in the proof of (3.72), will be to decompose the sum over c, in (C.3), into antisymmetric and symmetric sums. However, in order to do so, we shall need several subsidiary results. We begin with the following local identities:

$$\sum_c \epsilon_b\,\epsilon_e$$ 

$$\epsilon_a\,\epsilon_c \quad = \quad \frac{\theta(a\lambda)}{\theta(b\lambda)}\;\frac{\theta(\lambda-u)\,\theta(\lambda+u)}{\theta(\lambda)^2}\;\delta_{bd}\,A_{ab}\,A_{be} \tag{C.5a}$$

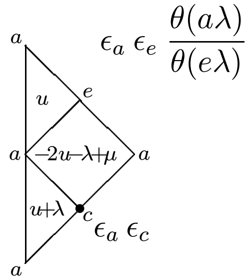
$$\sum_c \epsilon_a\,\epsilon_c\,\frac{\theta(c\lambda)}{\theta(a\lambda)}$$ 

$$\epsilon_d\,\epsilon_e\,\frac{\theta(d\lambda)}{\theta(e\lambda)} \quad = \quad \frac{\theta(d\lambda)}{\theta(a\lambda)}\;\frac{\theta(\lambda-u)\,\theta(\lambda+u)}{\theta(\lambda)^2}\;\delta_{bd}\,A_{ab}\,A_{be} \tag{C.5b}$$

$$\sum_c \quad \epsilon_a\,\epsilon_c \;/\; \epsilon_d\,\epsilon_e$$ 

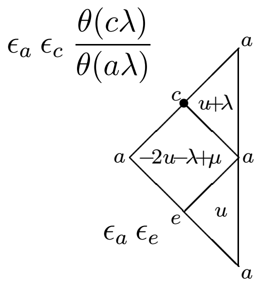
$$= \quad \frac{\theta(a\lambda)}{\theta(d\lambda)}\;\frac{\theta(\lambda-u)\,\theta(\lambda+u)}{\theta(\lambda)^2}\;\delta_{bd}\,A_{ab}\,A_{be} \tag{C.5c}$$

$\sum_c$



$$= \left(\frac{\theta(a\lambda)}{\theta(\lambda)}\right)^2 \frac{\theta(2u+3\lambda-\mu)}{\theta(\lambda)}\, \gamma_{\rm L}(a|u)\, A_{ae} \qquad \text{(C.5d)}$$

$\sum_c$



$$= \left(\frac{\theta(\lambda)}{\theta(a\lambda)}\right)^2 \frac{\theta(2u+3\lambda-\mu)}{\theta(\lambda)}\, \gamma_{\rm R}(a|u)\, A_{ae} \qquad \text{(C.5e)}$$

where

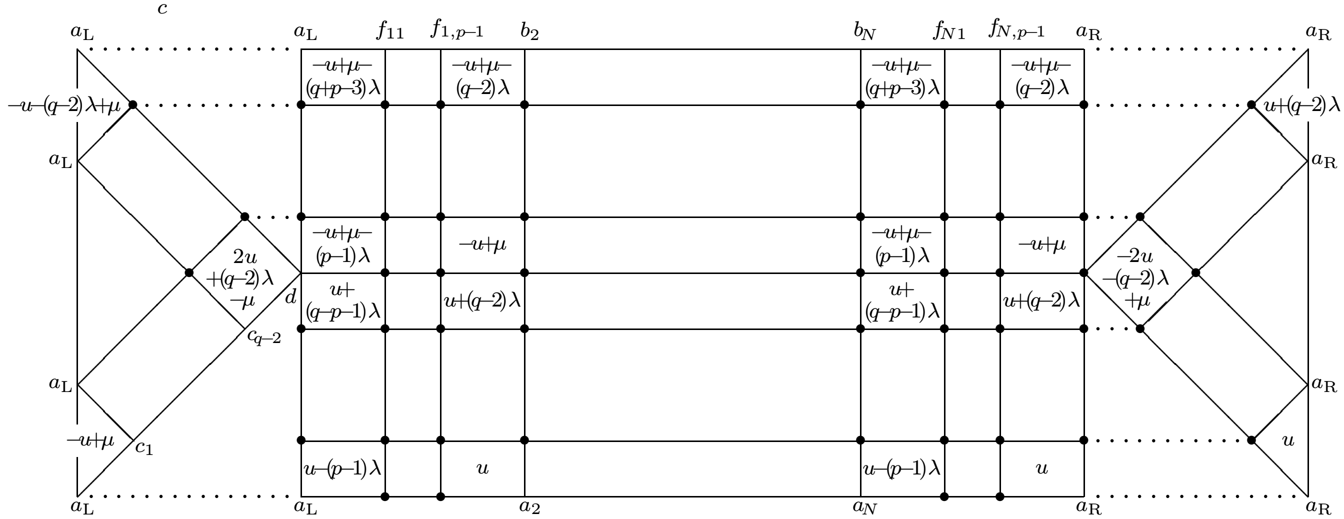
$$\gamma_{\rm L/R}(a|u) \quad = \qquad \text{(C.6)}$$
$$\frac{\theta(u+\frac{\lambda-\mu}{2}-\xi_{\rm L/R}(a))\,\theta(u+\frac{\lambda-\mu}{2}+\xi_{\rm L/R}(a))\,\theta(u-a\lambda+\frac{\lambda-\mu}{2}-\xi_{\rm L/R}(a))\,\theta(u+a\lambda+\frac{\lambda-\mu}{2}+\xi_{\rm L/R}(a))}{\theta(\lambda)^4}$$

Identities (C.5a)– (C.5c) can each be proved as follows: if the external spins do not satisfy the adjacency conditions, then both sides of the equation are zero; if $b = d = a \pm 1$ and $e = a \pm 2$, or else $a = 1$ or L, then the left side is a single term, which we immediately find is equal to the term on the right side; if $b = a \pm 1$, $d = a \mp 1$ and $e = a$, then the left side is a sum of two terms, which immediately cancel, as required by the delta function on the right; finally, if $2 \le a \le L-1$, $b = d = a \pm 1$ and $e = a$, then the left side is a sum of two terms which can be reduced to the term on the right side using a single application of (2.21).

The proofs of (C.5d) and (C.5e) are similar: if a and e do not satisfy the adjacency condition, then both sides of the equation are zero; if $a = 1$ or L then the left side is a single term; and, if $2 \le a \le L-1$ and $e = a \pm 1$, then the left side is a sum of two terms which can be reduced to the term on the right side using (2.21).

We now use these identities repeatedly in (C.4). By starting at c and proceeding in a clockwise loop, using (C.5a) $q-1$ times, (C.5d) once, (C.5b) pN times, (C.5e) once, (C.5c) $q-1$ times, and (C.5a) pN times, we find that, for $q \ge 2$,

$$\epsilon_e \sum_c \epsilon_c\, \mathcal{D}(a_{\rm L}, c_1, \ldots, c_{q-2}, d, c, d, e) \quad = \quad A_{de}\, M^{pq}(a_{\rm L} a_{\rm R}|u)\, \times \tag{C.7a}$$



while, for $q = 1$,

$$\epsilon_e \sum_c \epsilon_c\, \mathcal{D}(a_{\rm L}, c, a_{\rm L}, e) = A_{a_{\rm L} e}\, M^{p1}(a_{\rm L} a_{\rm R}|u)\, \langle a_2, \ldots, a_N \,|\, \boldsymbol{I}^p(a_{\rm L} a_{\rm R}) \,|\, b_2, \ldots, b_N \rangle \tag{C.7b}$$

where

$$\begin{aligned} M^{pq}(a_{\rm L} a_{\rm R}|u) \quad = \quad & \\ & \theta_{-1}^{q-1}(2u+(2q-3)\lambda-\mu)\; \theta_{1}^{q-1}(2u+(2q-3)\lambda-\mu)\; \theta_{0}^{1}(-2u-(2q-3)\lambda+\mu)\; \gamma_{\rm L}(a_{\rm L}|-u-q\lambda+\mu) \\ & \times \Big(\theta_{-1}^{p}(-u-q\lambda+\mu)\; \theta_{1}^{p}(-u-q\lambda+\mu)\Big)^N \\ & \times\, \theta_{0}^{1}(2u+(2q+1)\lambda-\mu)\; \gamma_{\rm R}(a_{\rm R}|u+(q-1)\lambda)\; \theta_{-1}^{q-1}(-2u-q\lambda+\mu)\; \theta_{1}^{q-1}(-2u-q\lambda+\mu) \\ & \times \Big(\theta_{-1}^{p}(u+(q-1)\lambda)\; \theta_{1}^{p}(u+(q-1)\lambda)\Big)^N \end{aligned} \tag{C.8}$$

We now assert the following properties of $\mathcal{D}$:

$$A^q_{a_{\rm L} e}\, \mathcal{D}(a_{\rm L}, c_1, \ldots, c_{q-2}, c_{q-1}, c, d, e) \quad = \quad A^q_{a_{\rm L} e} A^q_{a_{\rm L} c}\, \mathcal{D}(a_{\rm L}, c_1, \ldots, c_{q-2}, c_{q-1}, c, d, e), \quad \text{for } (c_1, \ldots, c_{q-1}) \in P^q_{a_{\rm L} c} \tag{C.9a}$$

$$A^q_{a_{\rm L} e}\, \mathcal{D}(a_{\rm L}, c_1, \ldots, c_{q-2}, c_{q-1}, c, d, e) \text{ is independent of } (c_1, \ldots, c_{q-1}) \in P^q_{a_{\rm L} c} \tag{C.9b}$$

$$\epsilon_e \sum_c \epsilon_c\, \mathcal{D}(a_{\rm L}, c'_1, \ldots, c'_{q-2}, d, c, d, e) \text{ is independent of } e, \text{ for } A_{de} = 1 \tag{C.9c}$$

$$A^{q-1}_{a_{\rm L} d}\; \epsilon_e \sum_c \epsilon_c\, \mathcal{D}(a_{\rm L}, c'_1, \ldots, c'_{q-2}, d, c, d, e) \text{ is independent of } (c'_1, \ldots, c'_{q-2}) \in P^{q-1}_{a_{\rm L} d} \tag{C.9d}$$

$$A^{q+1}_{a_{\rm L} d} \sum_e \mathcal{D}(a_{\rm L}, c''_1, \ldots, c''_{q-2}, c''_{q-1}, c''_q, d, e) \text{ is independent of } (c''_1, \ldots, c''_q) \in P^{q+1}_{a_{\rm L} d} \tag{C.9e}$$

Properties (C.9a) and (C.9b) follow by considering $A^q_{a_{\rm L}e}\,\mathcal{D}(a_{\rm L},c_1,\ldots,c_{q-2},c_{q-1},c,d,e)$, with $(c_1,\ldots,c_{q-1})\in P^q_{a_{\rm L}c}$, as a linear combination of terms of the form

$$W^{pq}\!\left(\begin{matrix} e & g_2 \\ a_{\rm L} & a_2\end{matrix}\,\middle|\, u\right)\ldots W^{pq}\!\left(\begin{matrix} g_N & g_{N+1} \\ a_N & a_{\rm R}\end{matrix}\,\middle|\, u\right) B^q_{\rm R}\!\left(g_{N+1}\ \begin{matrix} a_{\rm R} \\ a_{\rm R}\end{matrix}\,\middle|\, u\right)$$

$$\times\ W^{1q}\!\left(\begin{matrix} i_{N+1} & a_{\rm R} \\ h_{N+1} & g_{N+1}\end{matrix}\,\middle|\, -2u-(2q-1)\lambda+\mu\right)$$

$$\times\ W^{pq}\!\left(\begin{matrix} i_N & i_{N+1} \\ h_N & h_{N+1}\end{matrix}\,\middle|\, -u-(q-1)\lambda+\mu\right)\ldots W^{pq}\!\left(\begin{matrix} i_1 & i_2 \\ d & h_2\end{matrix}\,\middle|\, -u-(q-1)\lambda+\mu\right)$$

$$\times\ W^{q1}\!\left(\begin{matrix} i_1 & d \\ a_{\rm L} & c\end{matrix}\,\middle|\, 2u+(2q-1)\lambda-\mu\right) B^q_{\rm L}\!\left(\begin{matrix} a_{\rm L} \\ a_{\rm L}\end{matrix}\ c\,\middle|\, -u-(q-1)\lambda+\mu\right)$$

Property (C.9c) follows immediately from (C.7a) and (C.7b), while property (C.9d) follows from (C.7a) by considering $A^{q-1}_{a_{\rm L}d}\ \epsilon_e \sum_c \epsilon_c\,\mathcal{D}(a_{\rm L},c'_1,\ldots,c'_{q-2},d,c,d,e)$, with $A_{de}=1$ and $(c'_1,\ldots,c'_{q-2})\in P^{q-1}_{a_{\rm L}d}$, as proportional to a sum of terms of the form

$$W^{p,q-1}\!\left(\begin{matrix} g_1 & g_2 \\ a_{\rm L} & a_2\end{matrix}\,\middle|\, u\right)\ldots W^{p,q-1}\!\left(\begin{matrix} g_N & g_{N+1} \\ a_N & a_{\rm R}\end{matrix}\,\middle|\, u\right) B^{q-1}_{\rm R}\!\left(g_{N+1}\ \begin{matrix} a_{\rm R} \\ a_{\rm R}\end{matrix}\,\middle|\, u\right)$$

$$\times\ W^{p,q-1}\!\left(\begin{matrix} b_N & a_{\rm R} \\ g_N & g_{N+1}\end{matrix}\,\middle|\, -u-(q-2)\lambda+\mu\right)\ldots W^{p,q-1}\!\left(\begin{matrix} a_{\rm L} & b_2 \\ g_1 & g_2\end{matrix}\,\middle|\, -u-(q-2)\lambda+\mu\right)$$

$$\times\ B^{q-1}_{\rm L}\!\left(\begin{matrix} a_{\rm L} \\ a_{\rm L}\end{matrix}\ g_1\,\middle|\, -u-(q-2)\lambda+\mu\right)$$

Finally, property (C.9e) follows by considering $A^{q+1}_{a_{\rm L}d}\sum_e \mathcal{D}(a_{\rm L},c''_1,\ldots,c''_{q-2},c''_{q-1},c''_q,d,e)$, with $(c''_1,\ldots,c''_q)\in P^{q+1}_{a_{\rm L}d}$, as a sum of terms of the form

$$W^{p,q+1}\!\left(\begin{matrix} g_1 & g_2 \\ a_{\rm L} & a_2\end{matrix}\,\middle|\, u\right)\ldots W^{p,q+1}\!\left(\begin{matrix} g_N & g_{N+1} \\ a_N & a_{\rm R}\end{matrix}\,\middle|\, u\right) B^{q+1}_{\rm R}\!\left(g_{N+1}\ \begin{matrix} a_{\rm R} \\ a_{\rm R}\end{matrix}\,\middle|\, u\right)$$

$$\times\ W^{p,q+1}\!\left(\begin{matrix} b_N & a_{\rm R} \\ g_N & g_{N+1}\end{matrix}\,\middle|\, -u-q\lambda+\mu\right)\ldots W^{p,q+1}\!\left(\begin{matrix} a_{\rm L} & b_2 \\ g_1 & g_2\end{matrix}\,\middle|\, -u-q\lambda+\mu\right)$$

$$\times\ B^{q+1}_{\rm L}\!\left(\begin{matrix} a_{\rm L} \\ a_{\rm L}\end{matrix}\ g_1\,\middle|\, -u-q\lambda+\mu\right)$$

We now return to the sum over c in (C.3), which we claim can be decomposed into antisymmetric and symmetric sums,

$$\sum_c A^q_{a_{\rm L}c}\,\mathcal{D}(a_{\rm L},c_1,\ldots,c_{q-2},c_{q-1},c,d,c)\quad= \tag{C.10}$$

$$A^{q-1}_{a_{\rm L}d}\ \epsilon_e\sum_c \epsilon_c\,\mathcal{D}(a_{\rm L},c'_1,\ldots,c'_{q-2},d,c,d,e)\quad+\quad A^{q+1}_{a_{\rm L}d}\sum_e \mathcal{D}(a_{\rm L},c''_1,\ldots,c''_{q-2},c''_{q-1},c''_q,d,e)$$

In this decomposition we assume the following: that $(c_1,\ldots,c_{q-1}) \in P^q_{a_{\rm L}c}$ for each c in the sum of the left side; that e satisfies $A_{de}=1$ and that $(c'_1,\ldots,c'_{q-2}) \in P^{q-1}_{a_{\rm L}d}$ in the antisymmetric sum; and that $(c''_1,\ldots,c''_q) \in P^{q+1}_{a_{\rm L}d}$ in the symmetric sum. Therefore, due to (C.9b)–(C.9e), all of these spins are arbitrary.

We now proceed to prove (C.10). We begin by constructing the following table of values of the adjacency matrix entries which appear in (C.10) (as well as in the $su(2)$ fusion rule (3.20)):

	$d{-}a\notin \{-q{-}1,-q{+}1,-q{+}3,\ldots,q{-}3,q{-}1,q{+}1\}$			$d{-}a=-q{-}1$		$d{-}a=q{+}1$		$d{-}a\in\{-q{+}1,-q{+}3,\ldots,q{-}3,q{-}1\}$								
								$d{+}a\le q{-}1$		$d{+}a=q{+}1$		$q{+}3\le d{+}a \le 2L{-}q{-}1$	$d{+}a=2L{-}q{+}1$		$d{+}a\ge 2L{-}q{+}3$	
	$d=1$	$2\le d\le L{-}1$	$d=L$	$d=1$	$2\le d\le L{-}1$	$2\le d\le L{-}1$	$d=L$	$d=1$	$2\le d\le L{-}1$	$d=1$	$2\le d\le L{-}1$	$2\le d\le L{-}1$	$2\le d\le L{-}1$	$d=L$	$2\le d\le L{-}1$	$d=L$
$A^q_{a,d-1}$	–	0	0	–	0	1	1	–	0	–	0	1	1	1	0	0
$A^q_{a,d+1}$	0	0	–	1	1	0	–	0	0	1	1	1	0	–	0	–
$A^{q-1}_{a,d}$	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0
$A^{q+1}_{a,d}$	0	0	0	1	1	1	1	0	0	0	0	1	0	0	0	0

The entries in this table can all be obtained directly from the fused adjacency conditions, (3.14) and (3.15). We now denote the left side of (C.10) by $\mathcal{L}$ and the right side of (C.10) by $\mathcal{R}$, and consider cases corresponding to those listed in the table.

(I) $d-a_{\rm L} \notin \{-q-1,-q+1,\ldots,q-1,q+1\}$, $d+a_{\rm L}\le q-1$ or $d+a_{\rm L}\ge 2L-q+3$

In these cases, $\mathcal{L}$ and $\mathcal{R}$ are each zero.

(II) $d-a_{\rm L}=\pm(q+1)$

In these cases, $\mathcal{L}$ and $\mathcal{R}$ are each given by the single term

$$\mathcal{L}=\mathcal{R}=\mathcal{D}(a_{\rm L},a_{\rm L}\pm1,\ldots,a_{\rm L}\pm q,a_{\rm L}\pm(q+1),a_{\rm L}\pm q)$$

(III) $d-a_{\rm L}\in\{-q+1,\ldots,q-1\}$

In these cases, we can satisfy $(c_1,\ldots,c_{q-1})\in P^q_{a_{\rm L}c}$, for each c in the sum in $\mathcal{L}$, by taking $c_{q-1}=d$ and $(c_1,\ldots,c_{q-2})\in P^{q-1}_{a_{\rm L},d}$. We use this choice in each of the following subcases:

(i) $d+a_{\rm L}=q+1$

In this case, $\mathcal{L}$ is comprised of the single term

$$\mathcal{L}=\mathcal{D}(a_{\rm L},c_1,\ldots,c_{q-2},d,d+1,d,d+1)$$

Meanwhile, $\mathcal{R}$ is comprised of the antisymmetric sum only, for which we choose $e = d+1$. If $d = 1$, we have a single term, which immediately matches $\mathcal{L}$. For $d \geq 2$, we have

$$\mathcal{R} = \mathcal{D}(a_{\mathrm{L}}, c'_1, \ldots, c'_{q-2}, d, d+1, d, d+1) - \mathcal{D}(a_{\mathrm{L}}, c'_1, \ldots, c'_{q-2}, d, d-1, d, d+1)$$

but, by taking $c = d-1$ and $e = d+1$ in (C.9a), we find that the second of these terms vanishes and, therefore, we again have a single term which matches $\mathcal{L}$.

(ii) $d+a_{\mathrm{L}} = 2L-q+1$

This case is similar to the previous one, with $\mathcal{L}$ now comprised of the single term

$$\mathcal{L} = \mathcal{D}(a_{\mathrm{L}}, c_1, \ldots, c_{q-2}, d, d-1, d, d-1)$$

Again, $\mathcal{R}$ is comprised of the antisymmetric sum only, for which we now choose $e = d-1$. If $d = L$ we immediately have a term which matches $\mathcal{L}$, while for $d \leq L-1$, we have

$$\mathcal{R} = \mathcal{D}(a_{\mathrm{L}}, c'_1, \ldots, c'_{q-2}, d, d-1, d, d-1) - \mathcal{D}(a_{\mathrm{L}}, c'_1, \ldots, c'_{q-2}, d, d+1, d, d-1)$$

but, as before, we find that the second of these terms vanishes by taking $c = d+1$ and $e = d-1$ in (C.9a).

(iii) $q+3 \leq d+a_{\mathrm{L}} \leq 2L-q-1$

In this case, we have

$$\mathcal{L} = \mathcal{D}(a_{\mathrm{L}}, c_1, \ldots, c_{q-2}, d, d-1, d, d-1) + \mathcal{D}(a_{\mathrm{L}}, c_1, \ldots, c_{q-2}, d, d+1, d, d+1)$$

For $\mathcal{R}$, we choose, in the antisymmetric sum, $e = d\pm1$, and, in the symmetric sum, $c''_q = d\mp1$, $c''_{q-1} = d$, and $(c''_1, \ldots, c''_{q-2}) \in P^{q-1}_{a_{\mathrm{L}}d}$, giving

$$\begin{aligned}\mathcal{R} \;=\;& \mathcal{D}(a_{\mathrm{L}}, c'_1, \ldots, c'_{q-2}, d, d\pm1, d, d\pm1) - \mathcal{D}(a_{\mathrm{L}}, c'_1, \ldots, c'_{q-2}, d, d\mp1, d, d\pm1)\\ &+ \mathcal{D}(a_{\mathrm{L}}, c''_1, \ldots, c''_{q-2}, d, d\mp1, d, d\pm1) + \mathcal{D}(a_{\mathrm{L}}, c''_1, \ldots, c''_{q-2}, d, d\mp1, d, d\mp1)\end{aligned}$$

We see that the two middle terms of $\mathcal{R}$ cancel, while the two outer terms match those of $\mathcal{L}$. This completes our proof of (C.10)

We now substitute (C.10) and (C.7a) or (C.7b) into (C.3), and use (C.4), (3.51) and (3.52) to give

$$(-1)^q\, \theta^q_{2q-2}(2u-\mu)\, \theta^q_{2q}(2u-\mu)\, \langle a_2, \ldots, a_N \,|\, \boldsymbol{D}^{pq}(a_{\mathrm{L}}a_{\mathrm{R}}|u)\, \boldsymbol{D}^{p1}(a_{\mathrm{L}}a_{\mathrm{R}}|u+q\lambda) \,|\, b_2, \ldots, b_N\rangle \;= \tag{C.11}$$

$$\frac{M^{pq}(a_{\mathrm{L}}a_{\mathrm{R}}|u)}{\Big(\theta^p_{q-2}(u)\, \theta^p_{-q+1}(-u+\mu)\Big)^N\, \theta^{q-1}_{2q-3}(2u-\mu)\, \theta^{q-1}_{-q+1}(-2u+\mu)}\;\; \langle a_2, \ldots, a_N \,|\, \boldsymbol{D}^{p,q-1}(a_{\mathrm{L}}a_{\mathrm{R}}|u) \,|\, b_2, \ldots, b_N\rangle$$

$$+ \Big(\theta^p_{q-1}(u)\, \theta^p_{-q}(-u+\mu)\Big)^N\, \theta^q_{2q-1}(2u-\mu)\, \theta^q_{-q}(-2u+\mu)\;\; \langle a_2, \ldots, a_N \,|\, \boldsymbol{D}^{p,q+1}(a_{\mathrm{L}}a_{\mathrm{R}}|u) \,|\, b_2, \ldots, b_N\rangle$$

By using (C.8) and then cancelling the common factor $(-1)^q\, \theta^{q-1}_{2q-2}(2u-\mu)\, \theta^{q-1}_{2q-1}(2u-\mu)$ from each side of (C.11), it is relatively straightforward to show that the coefficients of each term in (C.11) match those in (3.72), which completes our proof of (3.72).

References

[1] Baxter, R. J., *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).

[2] Sklyanin, E. K., *J. Phys. A* **21** (1988), 2375-2389. *Boundary Conditions for Integrable Quantum Systems.*

[3] Baxter, R. J., *Ann. Phys.* **76** (1973), 25-47. *Eight-Vertex Model in Lattice Statistics and One-Dimensional Anisotropic Heisenberg Chain. II. Equivalence to a Generalised Ice-Type Lattice Model.*

[4] Cardy, J. L., *Nucl. Phys. B* **275** (1986), 200-218. *Effect of Boundary Conditions on the Operator Content of Two-Dimensional Conformally Invariant Theories.*

[5] Andrews, G. E., Baxter, R. J. and Forrester, P. J., *J. Stat. Phys.* **35** (1984), 193-266. *Eight-Vertex SOS Model and Generalized Rogers-Ramanujan-Type Identities.*

[6] Cherednik, I. V., *Teor. Mat. Fiz.* **61** (1984), 35-44. *Factorizing Particles on a Half Line and Root Systems.*

[7] Mezincescu, L. and Nepomechie, R. I., *J. Phys. A* **24** (1991), L17-L23. *Integrable Open Spin Chains with Non-Symmetric R-Matrices.*

[8] Mezincescu, L. and Nepomechie, R. I., *Int. J. Mod. Phys. A* **7** (1992), 5657-5659. *Addendum: Integrability of Open Spin Chains with Quantum Algebra Symmetry.*

[9] Yue, R. H. and Chen, Y. X., *J. Phys. A* **26** (1993), 2989-2994. *Integrable $Z_n \times Z_n$ Belavin Model with Non-Trivial Boundary Terms.*

[10] Kulish, P. P., *Preprint* hep-th/9507070. *Yang-Baxter Equation and Reflection Equations in Integrable Models.*

[11] Kulish, P. P. and Sklyanin, E. K., *J. Phys. A* **25** (1992), 5963-5975. *Algebraic Structures Related to Reflection Equations.*

[12] Kulish, P. P., Sasaki, R. and Schwiebert, C., *J. Math. Phys.* **34** (1993), 286-304. *Constant Solutions of Reflection Equations and Quantum Groups.*

[13] de Vega, H. J. and González-Ruiz, A., *J. Phys. A* **26** (1993), L519-L524. *Boundary K-Matrices for the Six Vertex and the $n(2n-1)$ A_{n-1} Vertex Models.*

[14] Ghoshal, S. and Zamolodchikov, A., *Int. J. Mod. Phys. A* **9** (1994), 3841-3885. *Boundary S Matrix and Boundary States in Two-Dimensional Integrable Quantum Field Theory.*

[15] Cuerno, R. and González-Ruiz, A., *J. Phys. A* **26** (1993), L605-L610. *Free Fermionic Elliptic Reflection Matrices and Quantum Group Invariance.*

[16] de Vega, H. J. and González-Ruiz, A., *J. Phys. A* **27** (1994), 6129-6137. *Boundary K-Matrices for the XYZ, XXZ and XXX Spin Chains.*

[17] Hou, B. Y. and Yue, R. H., *Phys. Lett. A* **183** (1993), 169-174. *General Solution of the Reflection Equations for the Eight-Vertex Model.*

[18] Inami, T. and Konno, H., *J. Phys. A* **27** (1994), L913-L918. *Integrable XYZ Spin Chain with Boundaries.*

[19] Mezincescu, L., Nepomechie, R. I. and Rittenberg, V., *Phys. Lett. A* **147** (1990), 70-78. *Bethe Ansatz Solution of the Fateev-Zamolodchikov Quantum Spin Chain with Boundary Terms.*

[20] Abad, J. and Rios, M., *Preprint* hep-th/9502129. *Non-Diagonal Solutions to Reflection Equations in $su(n)$ Spin Chains.*

[21] Mezincescu, L. and Nepomechie, R. I., *Int. J. Mod. Phys. A* **6** (1991), 5231-5248. *Integrability of Open Spin Chains with Quantum Algebra Symmetry.*

[22] Kim, J. D., *Preprint* hep-th/9412192. *Boundary K Matrix for the Quantum Mikhailov-Shabat Model.*

[23] Yung, C. M. and Batchelor, M. T., *Phys. Lett. A* **198** (1995), 395. *Diagonal K-Matrices and Transfer Matrix Eigenspectra Associated with the $G_2^{(1)}$ R-Matrix.*

[24] González-Ruiz, A., *Nucl. Phys. B* **424** (1994), 468-486. *Integrable Open Boundary Conditions for the Supersymmetric t-J Model: The Quantum Group Invariant Case.*

[25] Kulish, P. P. and Sklyanin, E. K., *J. Phys. A* **24** (1991), L435-L439. *The General $U_q(sl(2))$ Invariant XXZ Integrable Quantum Spin Chain.*

[26] Mezincescu, L. and Nepomechie, R. I., *Nucl. Phys. B* **372** (1992), 597-621. *Analytical Bethe Ansatz for Quantum-Algebra-Invariant Spin Chains.*

[27] Artz, S., Mezincescu, L. and Nepomechie, R. I., *Preprint* hep-th/9409130. *Spectrum of Transfer Matrix for $U_q(B_n)$-Invariant $A_{2n}^{(2)}$ Open Spin Chain.*

[28] Artz, S., Mezincescu, L. and Nepomechie, R. I., *Preprint* hep-th/9504085. *Analytical Bethe Ansatz for $A_{2n-1}^{(2)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$ Quantum-Algebra-Invariant Open Spin Chains.*

[29] Destri, C. and de Vega, H. J., *Nucl. Phys. B* **374** (1992), 692-719. *Bethe Ansatz and Quantum Groups: The Light-Cone Lattice Approach (I). Six Vertex and SOS Models.*

[30] Destri, C. and de Vega, H. J., *Nucl. Phys. B* **385** (1992), 361-391. *Bethe Ansatz and Quantum Groups: The Light-Cone Approach (II). From RSOS$(p+1)$ Models to p-Restricted Sine-Gordon Field Theories.*

[31] de Vega, H. J. and González-Ruiz, A., *Nucl. Phys. B* **417** (1994), 553-578. *Exact Solution of the $SU_q(n)$-Invariant Quantum Spin Chains.*

[32] de Vega, H. J. and González-Ruiz, A., *Mod. Phys. Lett. A* **9** (1994), 2207-2216. *Exact Bethe Ansatz Solution for A_{n-1} Chains with Non-$SU_q(n)$-Invariant Open Boundary Conditions.*

[33] Yung, C. M. and Batchelor, M. T., *Nucl. Phys. B* **435** (1995), 430-462. *Integrable Vertex and Loop Models on the Square Lattice with Open Boundaries via Reflection Matrices.*

[34] Foerster, A. and Karowski, M., *Nucl. Phys. B* **408** (1993), 512-534. *The Supersymmetric t-J Model with Quantum Group Invariance.*

[35] Zhou, Y. K., *Preprint* hep-th/9502053. *Fusion Hierarchy and Finite-Size Corrections of $U_q[sl(2)]$-Invariant Vertex Models with Open Boundary Conditions.*

[36] Jimbo, M., Kedem, R., Kojima, T., Konno, H. and Miwa, T., *Nucl. Phys. B* **441** (1995), 437-470. *XXZ Chain with a Boundary.*

[37] Ghoshal, S. and Zamolodchikov, A., *Int. J. Mod. Phys. A* **9** (1994), 4353. *Erratum: Boundary S Matrix and Boundary States in Two-Dimensional Integrable Quantum Field Theory.*

[38] Fring, A. and Köberle, R., *Nucl. Phys. B* **421** (1994), 159-172. *Factorised Scattering in the Presence of Reflecting Boundaries.*

[39] Kulish, P. P., Reshetikhin, N. Y. and Sklyanin, E. K., *Lett. Math. Phys.* **5** (1981), 393-403. *Yang-Baxter Equation and Representation Theory: I.*

[40] Date, E., Jimbo, M., Miwa, T. and Okado, M., *Lett. Math. Phys.* **12** (1986), 209-215. *Fusion of the Eight Vertex SOS Model.*

[41] Date, E., Jimbo, M., Miwa, T. and Okado, M., *Lett. Math. Phys.* **14** (1987), 97. *Erratum and Addendum: Fusion of the Eight Vertex SOS Model.*

[42] Date, E., Jimbo, M., Kuniba, A., Miwa, T. and Okado, M., *Nucl. Phys. B* **290** (1987), 231-273. *Exactly Solvable SOS Models: Local Height Probabilities and Theta Function Identities.*

[43] Date, E., Jimbo, M., Kuniba, A., Miwa, T. and Okado, M., *Adv. Stud. Pure Math.* **16** (1988), 17-122. *Exactly Solvable SOS Models II: Proof of the Star-Triangle Relation and Combinatorial Identities.*

[44] Bazhanov, V. V. and Reshetikhin, N. Y., *Int. J. Mod. Phys. A* **4** (1989), 115-142. *Critical RSOS Models and Conformal Field Theory.*

[45] Klümper, A. and Pearce, P. A., *Physica A* **183** (1992), 304-350. *Conformal Weights of RSOS Lattice Models and their Fusion Hierarchies.*

[46] Jimbo, M., Kuniba A., Miwa, T. and Okado, M., *Comm. Math. Phys.* **119** (1988), 543-565. *The $A_n^{(1)}$ Face Models.*

[47] Bazhanov, V. V. and Reshetikhin, N. Y., *J. Phys. A* **23** (1990), 1477-1492. *Restricted Solid-on-Solid Models Connected with Simply Laced Algebras and Conformal Field Theory.*

[48] Zhou, Y. K. and Pearce, P. A., *Int. J. Mod. Phys. B* **8** (1994), 3531-3577. *Fusion of A–D–E Lattice Models.*

[49] Zhou, Y. K., Pearce, P. A. and Grimm, U., *Preprint* hep-th/9506108. *Fusion of Dilute A_L Lattice Models.*

[50] Mezincescu, L. and Nepomechie, R. I., *J. Phys. A* **25** (1992), 2533-2543. *Fusion Procedure for Open Chains.*

[51] Yue, R. H., *J. Phys. A* **27** (1994), 1633-1644. *Integrable High Spin Chains Related to the Elliptic Solution of the Yang-Baxter Equation.*

[52] Saleur, H. and Bauer, M., *Nucl. Phys. B* **320** (1989), 591-624. *On Some Relations Between Local Height Probabilities and Conformal Invariance.*

[53] Pearce, P. A. and Klümper, A., *Phys. Rev. Lett.* **66** (1991), 974-977. *Finite-Size Corrections and Scaling Dimensions of Solvable Lattice Models: An Analytic Method.*

[54] Klümper, A. and Pearce, P. A., *J. Stat. Phys.* **64** (1991), 13-76. *Analytic Calculation of Scaling Dimensions: Tricritical Hard Squares and Critical Hard Hexagons.*

[55] Pasquier, V., *Nucl. Phys. B* **285** (1987), 162-172. *Two-Dimensional Critical Systems Labelled by Dynkin Diagrams.*

[56] Roche, P., *Phys. Lett. B* **285** (1992), 49-53. *On the Construction of Integrable Dilute A–D–E Models.*

[57] Warnaar, S. O., Nienhuis, B. and Seaton, K. A., *Phy. Rev. Lett.* **69** (1992), 710-712. *New Construction of Solvable Lattice Models Including an Ising Model in a Field.*

[58] Jimbo, M., Miwa, T. and Okado, M., *Comm. Math. Phys.* **116** (1988), 507-525. *Solvable Lattice Models Related to the Vector Representation of Classical Simple Lie Algebras.*