GENERALIZED MATHAI-QUILLEN TOPOLOGICAL SIGMA MODELS

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A simple field theoretical approach to Mathai-Quillen topological field theories of maps $X : M_I \rightarrow M_T$ from an internal space to a target space is presented. As an example of applications of our formalism we compute by applying our formulas the action and Q-variations of the fields of two well known topological systems: Topological Quantum Mechanics and type-A topological Sigma Model.

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1. Introduction

Topological Field Theories (TFT's) have been extensively studied during the past recent years (see [1] and references therein). The topological sigma models first introduced in [2] were motivated by the idea of the possibility of understanding gravity as a broken phase of a topological theory. In particular, the string theory would be a broken phase of a topological sigma model. This topological sigma model was obtained in [2] by the so-called topological twisting of a $N = 2$ supersymmetric sigma model, where the internal manifold was bidimensional ($d = 2$). This formalism requires the target manifold to be a Kahler manifold (in order to have $N = 2$ supersymmetry). However, Witten showed that the kahler condition could be relaxed to Hermitean target manifolds. In the present work we introduce a technique to obtain topological sigma models different from that of twisting a supersymmetric theory. This allow us to write topological actions in which the internal manifold can have any dimension $d = m$ and the target manifold can be any smooth manifold (even a real manifold), what constitutes a considerable generalization. The condition that we are going to impose from the begining is the localization of the correlators of the theory on a certain moduli space $\mathcal{M}$ of instantons. Is for this reason that we have called the theories so obtained Mathai-Quillen Topological Sigma Models (MQTSM). As we also show, the topological sigma model of type A ([3,4]) and the topological quantum mechanics ([5]) are particular cases of MQTSM. Mathai-Quillen [6] topological field theories has been previously studied in [7] and [8]. Our approach is more field-theoretical and less geometrical that the ones presented there.

In section two we introduce our definition of Mathai-Quillen Topological Sigma Models. In section three we will introduce a fermionic transformation ∇_Q (closely related to the transformation δ_Q associated to the topological charge Q present on any TFT) which simplifies notably the study of the geometrical properties of the theory in the target space. Important properties of ∇_Q are discussed there. In section four we construct the most general "basic" action of the MQTSM type and give the δ_Q-transformations of the fields. We also comment there the topological character of the theory and discuss the observables and correlators of MQ field theory (leading to intersection numbers on a moduli space of instantons). In section five we apply our formulas to two known examples, the Topological Quantum Mechanics ([5]) and the type-A Topological Sigma Models ([2,4]). Finally, in section six we present our conclusions.

2. The Generalized Mathai-Quillen Topological Sigma Models.

In this section we will introduce the elements defining a Mathai-Quillen Topological Sigma Model (MQTSM).

• First we introduce two smooth C^∞ differentiable manifolds (one m-dimensional internal manifold M_I and one n-dimensional target manifold M_T), together with a continuous map X:

$$X : M_I \longrightarrow M_T. \tag{2.1}$$

Given an atlas $\cup_a(\mathcal{U}_a, \sigma^\mu_{(a)})$ on M_I and an atlas $\cup_A(\mathcal{U}_A, X^i_{(A)})$ on M_T, the functions $X^i_A(\sigma^\mu_a)$ are C^∞-functions in the $\sigma^\mu_{(a)}$ coordinates (here $\mathcal{U}_a$ ($\mathcal{U}_A$) are the open subsets of a covering on M_I (M_T) and $\sigma^\mu_{(a)}$, $\mu : 1, ..., m$ ($X^i_{(A)}$, $i : 1, ...n$) are a system of local coordinates on $\mathcal{U}_a$ ($\mathcal{U}_A$)). From now on we will drop the covering indices a and A to simplify notation. The functions $X^i(\sigma^\mu)$ ($X^i : U^m \to U^n$, being U^m and U^n open subsets of R^m and R^n respectively) give the coordinates of M_I as immersed on M_T. Further, we provide M_I and M_T with Euclidean metrics $g_{\mu\nu}$ and G_{ij} respectively. We will denote by $G_{ij}(X)$ the restriction of the metric G_{ij} to the submanifold $X^i(M_I) \subset M_T$. Under a change of coordinates on M_I ($\sigma'^\mu = \sigma'^\mu(\sigma)$) the functions $X^i(\sigma)$ behave as scalars:

$$X'^i(\sigma') = X^i(\sigma) \tag{2.2}$$

Under a change of coordinates in the target the functions $X^i(\sigma)$ change as usual:

$$X'^i(\sigma) = X'^i(X(\sigma)) \sim X^i(\sigma) + \xi^i(X(\sigma)) \tag{2.3}$$

(where in the last step we have made the coordinate transformation infinitesimal).

• Our second important ingredient for MQTSM formalism (like in any other TFT model) is the existence of a fermionic symmetry transformation δ_Q generated by a fermionic operator Q. By symmetry we mean that the action ($S[\Phi]$), observables ($\mathcal{O}(\Phi)$) and measure ($\mathcal{D}\Phi$) of the theory are invariant under Q:

$$\delta_Q(S[\Phi]) = \delta_Q(\mathcal{D}\Phi) = \delta_Q(\mathcal{O}(\Phi)) = 0 \tag{2.4}$$

(here Φ denotes the field space of the theory). This fermionic operator is taken to be a scalar on M_I (this fixes the properties of Q under target changes of coordinates that will be analyzed in the next section). Moreover, we take this operator to be nilpotent on the field space Φ:

$$\delta_Q^2\Phi = 0 \tag{2.5}$$

Finally, we demand the internal metric $g_{\mu\nu}(\sigma)$ to be invariant under Q:

$$\delta_Q g_{\mu\nu}(\sigma) = 0 \tag{2.6}$$

(in particular, this means that our model is not coupled to topological gravity).

• Our third request is that the action $S[\Phi]$ is taken to be Q-exact:

$$S_t[\Phi] = t\delta_Q(A[\Phi]) \equiv tS[\Phi] \tag{2.7}$$

for some scalar functional $A[\Phi]$ (t is a c-number parameter). The previous equation has an important consequence ([2]). The correlators of the theory are independent of the parameter t:

$$< \mathcal{O}_1 \dots \mathcal{O}_p >_t = \int [\mathcal{D}\Phi]\ \mathcal{O}_1(\Phi) \dots \mathcal{O}_2(\Phi) e^{iS_t[\Phi]} \tag{2.8}$$

$$-i\frac{d}{dt} < \mathcal{O}_1 \dots \mathcal{O}_p >_t = < \mathcal{O}_1 \dots \mathcal{O}_p S[\Phi] >_t = < \delta_Q(\mathcal{O}_1 \dots \mathcal{O}_p A[\Phi]) >_t = 0 \tag{2.9}$$

(in the last step we have used that Q is an exact symmetry of the quantum theory). This observation is very important because allows us to compute correlators in the most convenient value of t (typically one makes the limit $t \to \infty$ where the computation of the path integral reduces to the semiclassical limit). We will make use of this property when discussing the topological character of these theories in section four.

• The fourth and final property of our definition of MQTSM is the one that gives the name "Mathai-Quillen" to the MQTSM's. We will demand the correlators of the theory to be localized on a certain moduli space $\mathcal{M}$ given by ([8]):

$$\mathcal{M} = \{\phi \subset \Phi \mid s = D\phi = 0\}/G \tag{2.10}$$

where Φ is again the space of fields, D is some chosen differential operator, $s = D\phi$ is a section of a vector bundle over M_T and G is some group of symmetries present in the theory. In other words, we want the functional integrals defining the correlators to be localized on the subspace ϕ of the field space Φ satisfying $D\phi = 0$ (modulo symmetry transformations). This localization is obtained by constructing an action of the form:

$$S_t[\Phi] = t\delta_Q(A[\Phi]) = t(\|D\phi\|^2 + \ldots) \tag{2.11}$$

then, it is immediate to see from (2.8) that, in the large t limit, we get localization on $\mathcal{M}$:

$$\lim_{t\to\infty} < \mathcal{O}_1 \ldots \mathcal{O}_p >_t = < \mathcal{O}_1 \ldots \mathcal{O}_p >_{\mathcal{M}} \tag{2.12}$$

We will refer sometimes to ϕ as the "instantons" and $\mathcal{M}$ as the moduli space of instantons.

3. The operator ∇_Q.

The only field of the field space Φ that we have specified so far are the fields $X^i(\sigma)$ ($i : 1, .., n$). Using (2.5) we can deduce the next relations (from now on we will drop the internal coordinates σ^μ and only write them if we consider it clarifying):

$$\delta_Q X^i = \chi^i \; ; \qquad\qquad \delta_Q \chi^i = 0. \tag{3.1}$$

Being Q a fermionic operator and the X^i independent fields we have that the χ^i are n independent fermionic fields. The rest of this section will be devoted to the analysis of the simple relations (3.1). The first observation is that, being both X^i (see (2.2)) and Q scalar objects from the point of view of M_I, the fields χ^i are also scalar fields with respect to change of coordinates on the internal manifold. Now, let us study the behaviour of χ^i under change of coordinates in the target manifold M_T. From (2.3) and (3.1) we get that under an infinitesimal change of coordinates on M_T:

$$\chi'^i = \delta_Q X'^i \sim \chi^i + \partial_j \xi^i(X) \chi^j \tag{3.2}$$

i.e., we conclude that the fields χ^i $(i : 1, ..., n)$ are the components of a vector in the target. Note that this is not a new condition, but a consequence of (2.3) and (3.1). An implication of this observation is that the Q operator is not a (scalar) covariant operator from the point of view of the target space M_T. This is so because Q acting on target coordinate fields (X^i) produces vector-target fields (χ^i). This introduces some problem when analyzing the geometrical aspects of the theory on the target. Let us analyze the problem more carefully. Let us take an arbitrary target-vector $V^i(X)$. Under a change of coordinates like (2.3) we have:

$$V^i(X) \to V'^i(X') \sim V^i(X) + \partial_j \xi^i(X) V^j(X) \tag{3.3}$$

therefore, applying (3.1) we get:

$$\delta_Q V'(X') \sim \delta_Q V^i(X) + \partial_j \xi^i(X) \delta_Q V^j(X) + \partial_k \partial_j \xi^i(X) \chi^j V^k \tag{3.4}$$

We observe then that $\delta_Q V^i(X)$ is not a target vector due to the last term in (3.4) . Now, using the target metric G_{ij} we construct the affine connection Γ^i_{jk} (we will consider torsionless connections). One easily verifies that under an infinitesimal change of coordinates the object $O^i(X) \equiv \Gamma^i_{jk}(X) V_1^j(X) V_2^k(X)$ (here $V_1^i(X)$ and $V_2^i(X)$ are two arbitrary target vectors) transforms as:

$$O'^i(X') \sim O^i(X) + \partial_j \xi^i(X) O^j(X) - \partial_k \partial_j \xi^i(X) V_1^k(X) V_2^j(X). \tag{3.5}$$

Comparing now with (3.4) we are led to naturally introduce the transformation:

$$\nabla_Q V^i(X) = \delta_Q V^i(X) + \Gamma^i_{jk}(X) \chi^j V^k(X) = \chi^j D_j V^i(X) \tag{3.6}$$

which maps target-vectors into target-vectors. ∇_Q, contrary to δ_Q, is then a covariant scalar in the target. This analysis was done for vector fields depending only on X^i (like $V^i(X)$). However, we generalize our definition to vector fields depending on any field $\phi \subset \Phi$. Then we define the ∇_Q transformation by (the analysis for the covariant case can be done in the same way):

$$\nabla_Q V^i(\phi) \equiv \delta_Q V^i(\phi) + \Gamma^i_{jk}(X) \chi^j V^k(\phi) \tag{3.7}$$

$$\nabla_Q V_i(\phi) \equiv \delta_Q V_i(\phi) - \Gamma^k_{ij}(X) \chi^j V_k(\phi) \tag{3.8}$$

We note here that, for a general field ϕ (contrary to the last relation of (3.6) for the special case $\phi = X$) $\nabla_Q V^i(\phi) = \delta_Q V^i(\phi) + \Gamma^i_{jk} \chi^j V^k(\phi) \neq \chi^k D_j V^i(\phi)$. Also, we remark that, due to the fact that Φ contains fermionic fields, the position of the χ^i fields in our previous definitions are important. The generalization of our definitions to tensors with any number of covariant and contravariant indices is trivial. The next properties of the ∇_Q transformation can be straightforwardly checked:

- $\nabla_Q(A(\Phi)B(\Phi)) = (\nabla_Q A(\Phi))B(\Phi) + (-)^{\epsilon_A} A(\Phi)(\nabla_Q B(\Phi))$.
- $\nabla_Q G_{ij}(X) = 0$.
- $\nabla_Q \chi^i = \delta_Q \chi^i = 0$. (3.9)
- $\nabla_Q(A^i(\Phi)B_i(\Phi)) = \delta_Q(A^i(\Phi)B_i(\Phi))$.
- $\nabla_Q^2 A^i(\Phi) = \frac{1}{2} R^i_{jkl}(X) \chi^j \chi^k A^l(\Phi)$.

(the last relation holds if $\nabla_Q A^i(\phi) \neq 0$). Here, ϵ_A is 0 (1) if $A(\Phi)$ is a bosonic (fermionic) operator. $R^i_{jkl}(X)$ is the curvature tensor on the target space. A very important observation should be made from these relations. Looking at the fourth relation in (3.9) we have that the ∇_Q-transformation on any scalar is Q-exact ($\nabla_Q = \delta_Q$ when acting on target-scalars). In particular, the equations (2.4) and (2.7) implies that:

$$\nabla_Q(S[\Phi]) = \nabla_Q(\mathcal{D}\Phi) = \nabla_Q(\mathcal{O}[\Phi]) = 0 \tag{3.10}$$

and

$$S_t[\Phi] = t\delta_Q(A[\Phi]) = t\nabla_Q(A[\Phi]). \tag{3.11}$$

This means that ∇_Q defines also a symmetry of the quantum theory. Finally, let us notice that contrary to δ_Q, ∇_Q is not a nilpotent operator.

4. The Action and Q-transformations for MQTSM's.

In this section, with minimal information, we will construct a rather general action localizing the correlators in some chosen moduli space $\mathcal{M}$. By minimal information we mean that we will not specify the field content of the theory (Φ). To construct concrete examples later, we will have to be more specific in this aspect. Also we write the δ_Q-transformations and ∇_Q-transformations that with this minimal data can be analyzed. A discussion of the topological character and observables of the theory is also presented.

4.1. The Action for MQTSM's.

We already have lot of information for constructing an action $S[\Phi]$ for MQTSM's. First, we know that the action is ∇_Q-exact (3.11) , and second, we want $S[\Phi]$ to depend on a given bosonic section $s[\phi]$ ($\phi \subset \Phi$) of a vector bundle over M_T. The most simple term to start with is:

$$S^1_t[\Phi] = t\nabla_Q\Big(\int d^m\sigma\sqrt{g(\sigma)}\rho^*_i[\Phi]s^i_*[\Phi]\Big) = \int \sqrt{d^m\sigma g(\sigma)}((\nabla_Q\rho^*_i[\Phi])s^i_*[\Phi] - \rho^*_i[\Phi](\nabla_Q s^i_*[\Phi])) \tag{4.1}$$

Here, $\rho^*_i[\Phi]$ is any fermionic function of the field space Φ (the action has to be bosonic). $s^i_*[\Phi]$ is the bosonic section of the vector bundle and "$*$" denotes all internal indices (like indices associated to the internal manifold M_I or gauge indices) which are conveniently contracted to make $S_t[\Phi]$ a scalar functional. To avoid complicate notation we will also drop out the "$*$" from some of the expressions and only restore them whenever we will consider it clarifying. The action that we have obtained in (4.1) is not gaussian in the section as we wish (see (2.11)). But looking at it it is easy to guess what we have to do to get such a gaussian term. Just introduce a "metric" A^{**}_{ij} such that $\rho^*_i[\Phi] = A^{**}_{ij}[\Phi]\rho^i_*[\Phi]$, and add a term $\nabla_Q(\int \sqrt{g(\sigma)}d^m\sigma(\rho^*_i\nabla_Q\rho^i_*))$ to the action. One finds:

$$S_t[\Phi] = t\nabla_Q\Big(\int d^m\sigma\sqrt{g(\sigma)}(\rho^*_i[\Phi]s^i_*[\Phi] + \rho^*_i[\Phi]\nabla_Q\rho^i_*[\Phi])\Big). \tag{4.2}$$

Using the properties in (3.9):

$$
\begin{aligned}
S_t[\Phi] =& t\int d^m\sigma\sqrt{g(\sigma)}\Big\{A_{ij}[\Phi]\big((\nabla_Q\rho^i[\Phi])s^j[\Phi]-\rho^i[\Phi]\nabla_Q(s^j[\Phi])+(\nabla_Q\rho^i[\Phi])(\nabla_Q\rho^j[\Phi])\\
&\qquad -\rho^i[\Phi]\nabla_Q^2\rho^j[\Phi]\big)+(\nabla_Q A_{ij}[\Phi])(\rho^i[\Phi]s^j[\Phi]+\rho^i[\Phi]\nabla_Q\rho^j[\Phi])\Big\} =\\
=& t\int d^m\sigma\sqrt{g(\sigma)}\Big\{A_{ij}[\Phi]\big((\nabla_Q\rho^i[\Phi]+\frac{1}{2}s^i[\Phi])(\nabla_Q\rho^j[\Phi]+\frac{1}{2}s^j[\Phi])-\frac{1}{4}s^i[\Phi]s^j[\Phi]\\
&\qquad -\rho^i[\Phi]\nabla_Q s^j[\Phi]-\frac{1}{2}R^j_{klm}(X)\chi^k\chi^l\rho^i[\Phi]\rho^m[\Phi]\big)\\
&\qquad +(\nabla_Q A_{ij}[\Phi])(\rho^i[\Phi]s^j[\Phi]+\rho^i[\Phi]\nabla_Q\rho^j[\Phi])\Big\}.
\end{aligned}
\tag{4.3}
$$

Defining the “auxiliary” fields H^i_* by:

$$
H^i_*[\Phi]\equiv\nabla_Q\rho^i_*[\Phi]+\frac{1}{2}s^i_*[\Phi]. \tag{4.4}
$$

we finally get the action:

$$
\begin{aligned}
S_t[\Phi] =& t\int d^m\sigma\sqrt{g(\sigma)}\Big\{A_{ij}[\Phi]\big((H^i[\Phi]H^j[\Phi]-\frac{1}{4}s^i[\Phi]s^j[\Phi]-\rho^i[\Phi]\nabla_Q s^j[\Phi]\\
&\qquad -\frac{1}{2}R^j_{klm}(X)\chi^k\chi^l\rho^i[\Phi]\rho^m[\Phi]\big)\\
&\qquad +(\nabla_Q A_{ij}[\Phi])(\rho^i[\Phi]s^j[\Phi]+\rho^i[\Phi]\nabla_Q\rho^j[\Phi])\Big\}.
\end{aligned}
\tag{4.5}
$$

In (4.5) we have obtained a desired action like the one in (2.11) if the tensor $A^{**}_{ij}[\Phi]$ defines a proper norm (i.e., defines a positive-definite quadratic form):

$$
\|D\Phi\|^2 = A^{**}_{ij}[\Phi]s^i_*[\Phi]s^j_*[\Phi]. \tag{4.6}
$$

Using the $t\to\infty$ argument we see that the path integration of a field theory with the previous action is localized on configurations with $s^i_*[\Phi]=0$ as we want. Our analysis is rather general (we have not specify neither the form of the section $s^i_*[\Phi]$ nor the form of the function $\rho^*_i[\Phi]$), but this is the “minimal” form of the action for MQTSM's. “Non-minimal” actions can be obtained by adding ∇_Q-exact (i.e., Q-exact) terms to this “minimal” action.

4.2. Basic Q-Transformations

To write the precise Q-transformations of the fields in the theory requires to specify Φ and the concrete form of $s^i_*[\Phi]$ and $\rho^i_*[\Phi]$. Nevertheless we can already get valuable general information over the structure of such transformations and write expressions for them. Here we collect such expressions:

$$
\begin{aligned}
&\bullet\ \delta_Q X^i = \chi^i\\
&\bullet\ \nabla_Q\chi^i = 0.\\
&\bullet\ \nabla_Q\rho^i_*[\Phi] = H^i_*[\Phi]-\frac{1}{2}s^i_*[\Phi]\\
&\bullet\ \nabla_Q H^i_*[\Phi] = \frac{1}{2}\nabla_Q s^i_*[\Phi]+\frac{1}{2}R^i_{jkl}(X)\chi^j\chi^k\rho^l_*[\Phi]
\end{aligned}
\tag{4.7}
$$

The first two relations were already known in (3.1) and (3.9) and the third in (4.4). Finally, the last one is obtained trivially by applying ∇_Q to the third and using the last property of (3.9). From these relations (4.7) we can derive straightforwardly the δ_Q-transformations, just by using the definition (3.7). One gets:

$$
\begin{aligned}
&\bullet\ \delta_Q X^i = \chi^i. \\
&\bullet\ \delta_Q \chi^i = 0. \\
&\bullet\ \delta_Q \rho^i_*[\Phi] = H^i_*[\Phi] - \frac{1}{2} s^i_*[\Phi] - \Gamma^i_{jk}(X)\chi^j \rho^k_*[\Phi] \\
&\bullet\ \delta_Q H^i_*[\Phi] = \frac{1}{2}\delta_Q s^i_*[\Phi] + \frac{1}{2}\Gamma^i_{jk}(X)\chi^j s^k_*[\Phi] - \Gamma^i_{jk}(X)\chi^j H^k_*[\Phi] \\
&\qquad\qquad + \frac{1}{2} R^i_{jkl}(X)\chi^j\chi^k\rho^l_*[\Phi]
\end{aligned}
\tag{4.8}
$$

One checks that the δ_Q-transformations above are automatically nilpotent (as demanded by (2.5)).

4.3. Topological Character

Let us study the topological character of the theory so far presented. We do not know a rigorous proof to justify the topological character of the theory neither on the internal manifold M_I nor on the target M_T holding at any value of the parameter t. We have an argument that applies in the large $t \to \infty$ limit and then we use (2.9) to generalize it for any value of t. Let us first study the behaviour of the theory under deformations of the metric in the internal manifold $g_{\mu\nu}$. The δ_Q-exactness of the action (4.5) is not enough to guarantee the invariance of the theory under deformations of the metric. This is so due to the possible metric dependence of the transformations (4.8) (this dependence could appear in $s^i_*[\Phi]$, $\rho^i_*[\Phi]$ and $H^i_*[\Phi]$). Actually, only in the case in which the δ_Q-transformations are independent of the metric we have that deformations of the internal metric $\delta g_{\mu\nu}$ and δ_Q-transformations commute and, therefore, that the energy-momentum tensor $T_{\mu\nu}$ is Q-exact. General arguments shows that Q-exact energy momnetums lead to topological theories ([9]) However, if we assume that all the metric dependence in the δ_Q-transformations (4.8) are in the section $s^i[\Phi]$ and the auxiliary field $H^i_*[\Phi]$ (as in the cases that we will consider), then we see that the transformations (4.8) are independent of the metric $g_{\mu\nu}$ if we restrict ourselves to the moduli space $s^i[\Phi] = H^i[\Phi] = 0$ (*i.e.*, $\mathcal{M}$), where the path integral is localized in the limit $t \to \infty$ (thanks to the δ_Q-exactness of the action we can take that limit as correct). We conclude then that, in this case, deformations of $g_{\mu\nu}$ and δ_Q-transformations commute and then the theory described by the action (4.5) is topological with respect to the internal manifold M_I. In fact, in the examples to be consider later we will take $\rho^i_*[\Phi]$ and $H^i_*[\Phi]$ to be independent elemental fields (ρ^i_* and H^i_* respectively), and we have then a topological field theory with respect to M_I.

The analysis of the topological character on the target space M_T is similar. If we observe the ∇_Q-transformations in (4.7) we deduce that in the case in which all the dependence on G_{ij} in (4.7) is in $s^i[\Phi]$ and in $H^i[\Phi]$ then, in the $t \to \infty$ limit (*i.e.*, in the moduli space $\mathcal{M}$ defined by $s^i_*[\Phi] = H^i_*[\Phi] = 0$), all the dependence of the ∇_Q-transformations in the target metric is in the curvature term in the transformation of the field $H^i_*[\Phi]$. But is easy to check that, on $\mathcal{M}$ (here, $\hat{\delta}$ means deformations with respect to the target metric G_{ij}):

$$\begin{aligned}\lim_{t\to\infty}\hat{\delta}\Big(\nabla_Q H^i_*[\Phi]\Big) &= \lim_{t\to\infty}\frac{1}{2}\hat{\delta}(R^i_{jkl})\chi^j\chi^k\rho^l_*[\Phi] = \lim_{t\to\infty}\frac{1}{2}\hat{\delta}\Big\{\delta_Q\Big(\Gamma^i_{kl}\chi^k\rho^l_*[\Phi]\Big)\Big\}\\ &= \lim_{t\to\infty}\frac{1}{2}\nabla_Q\Big\{\hat{\delta}\Big(\Gamma^i_{kl}\chi^k\rho^l_*[\Phi]\Big)\Big\}.\end{aligned} \tag{4.9}$$

We have used that $\hat{\delta}$ commute with δ_Q when acting on X^i and χ^i. Also note that, although Γ^i_{jk} is not a tensor, $\hat{\delta}\Gamma^i_{jk}$ is a tensor and, therefore, the ∇_Q action in the last term in (4.9) is wel defined. In (4.9) we see that deformations $\hat{\delta}$ of the target metric varies the ∇_Q-transformations by a ∇_Q-exact terms. This means that the energy-momentum tensor T_{ij} associated to the target metric G_{ij} is ∇_Q-exact, then leading also to a topological field theory in the target space M_T in the large limit $t\to\infty$. Now we use (2.9) to argue that this topological character should hold for any t.

4.4. Observables.

The observables $\mathcal{O}[\Phi]$ of any topological field theory are metric-independent scalar objects belonging to the cohomology of Q:

$$\mathcal{O}[\Phi]\in\frac{Ker(\delta_Q)}{Im(\delta_Q)}. \tag{4.10}$$

Note that we could replace δ_Q by ∇_Q in the previous (and following) expressions ($\delta_Q=\nabla_Q$ when acting on target scalars). The numerator of (4.10) just says that any observable has to be invariant under the symmetry of the theory (Q). The denominator tells that, due to (2.4), two observables differing by a Q-exact quantity lead to the same correlators (and then, have to be identified as observables). So far, we have specified two of the fields of Φ: X^i and χ^i. With them we can already construct observables satisfying (4.10). In [2] the analysis for the case where the internal manifold was bidimensional ($d=2$) was studied. It is not difficult to generalize the arguments there for the general case $d=m$. The result is the following. Given a a-dimensional homology cycle $\gamma_{a(i)}\in H_a(M_I)$ ($0\le a\le m$ and $i:1,..,b_a=dim(H_a(M_I))$) and a p-form $A(X)\in H^p(M_T;R)$ we define $W^{\gamma_{a(i)}}_A$ by:

$$W^{\gamma_{a(i)}}_A[X,\chi]=\int_{\gamma_{a(i)}}\mathcal{O}^a_A(X,\chi) \tag{4.11}$$

where the object $\mathcal{O}^a_A(X,\chi)$ is given by:

$$\mathcal{O}^a_A(X,\chi)=\binom{p}{a}A(X)_{i_1,..,i_p}dX^{i_1}\wedge\ldots\wedge dX^{i_a}\chi^{i_{a+1}}\ldots\chi^{i_p}\qquad 0\le a\le m. \tag{4.12}$$

$\mathcal{O}^{a-1}_A(X,\chi)$ and $\mathcal{O}^a_A(X,\chi)$ are easily seen to be related by the so-called "topological descendent equations" (use (3.1) and (4.12) to prove this):

$$d\mathcal{O}^{a-1}_A(X,\chi)=\delta_Q(\mathcal{O}^a_A(X,\chi)). \tag{4.13}$$

Similar arguments to those of [2] adapted to the present case show that $W_A^{\gamma_{a(i)}}[X,\chi]$ defines an observable ($\delta_Q W_A^{\gamma_{a(i)}}[X,\chi]=0$) which depends on the homology class of $\gamma_{a(i)}$ and not on the particular cycle chosen. Therefore, to define the observables (4.11) we can take any basis $\gamma_{a(i)}$ of $H_a(M_I)$. An observable (4.11) can then be constructed for any choice of the pair $(A(X),\gamma_{a(i)})$ where $A(X)\in H^p(M_T;R)$ and $\gamma_{a(i)}\in H_a(M_I)$.

The correlators of these observables have the form:

$$< \prod_{A,\gamma_{a(i)}} W_A^{\gamma_{a(i)}} >_t \tag{4.14}$$

The analysis in the limit $t\to\infty$ (see the action (4.5)) localizes these correlators on the moduli space $\mathcal{M}$ ($s^i=0$ and $H^i_*=\rho^*_i\nabla_Q s^i_*=0$). We can not continue this analysis without specifying a concrete model, however, let us just mention that the role of fermionic zero modes ($\rho^*_i\nabla_Q s^i_*=0$) are going to be essential to establish the selection rules for obtaining non-zero outputs from (4.14). These selection rules are dictated by index theorems depending on the manifolds M_I, M_T and the differential operator D defining the section s^i_* ([2,3]).

5. Examples.

In this section we will apply our formulas to derive the action and δ_Q-transformations of two well known examples: Topological Quantum Mechanics ([5]) and Type A Topological Sigma Models ([2,4]).

5.1. Topological Quantum Mechanics.

In this case M_I is taken to be S^1:

$$X : S^1 \to M_T. \tag{5.1}$$

(X can be thought as elements of $\pi_1(M_T)$). The map X can be locally described by functions $X^i(\tau)$ ($i:1,..,n$), being τ a coordinate in S^1. The section is taken to be:

$$s^i(X(\tau)) = \frac{d}{d\tau}X^i + V^i(X) \tag{5.2}$$

where $V^i(X)$ is some smooth vector field on M_T. In this situation we have:

$$\begin{aligned}\nabla_Q s^i(X) &= \delta_Q s^i(X) + \Gamma^i_{jk}(X)\chi^j s^k(X)\\ &= \frac{d}{d\tau}\chi^i + \partial_j V^i(X)\chi^j + \Gamma^i_{jk}(X)\chi^j(\frac{d}{d\tau}X^k + V^k(X))\\ &= \mathcal{D}^i_j\chi^j.\end{aligned} \tag{5.3}$$

where we have used the notation of [5]:

$$\mathcal{D}^i_j = \delta^i_j\frac{d}{d\tau} + \frac{dX^k}{d\tau}\Gamma^i_{kj} + D_j V^i. \tag{5.4}$$

We take the metric $A_{ij}[\Phi] = G_{ij}(X)$ in (4.5) (then, using (3.9) we get $\nabla_Q A_{ij}[\Phi] = 0$). Therefore, the action (4.5) reads in this case:

$$\begin{aligned} S_t[\Phi] =& t\int e(\tau)d\tau G_{ij}(X)\Big\{H^iH^j - \frac{1}{4}(\frac{d}{d\tau}X^i + V^i(X))(\frac{d}{d\tau}X^j + V^j(X)) \\ & - \rho^i\mathcal{D}^j_k\chi^k - \frac{1}{2}R^j_{klm}(X)\chi^k\chi^l\rho^i\rho^m\Big\}. \end{aligned} \tag{5.5}$$

($e(\tau)$ is the "einbein" for S^1 making (5.5) invariant under reparametrizations on τ). This is precisely the action obtained in a different way in [5]. The ∇_Q-transformations can be obtained from (4.8). We get (we write only the two last ∇_Q-transformations of (4.8), because the two first remain the same):

$$\begin{aligned} &\bullet\ \nabla_Q\rho^i = H^i - \frac{1}{2}(\frac{d}{d\tau}X^i + V^i(X)) \\ &\bullet\ \nabla_Q H^i = \frac{1}{2}\mathcal{D}^i_j\chi^j + \frac{1}{2}R^i_{jkl}\chi^j\chi^k\rho^l. \end{aligned} \tag{5.6}$$

Also, the analog of (2.6) for the present case is:

$$\delta_Q e(\tau) = 0 \tag{5.7}$$

The analysis of observables and the computation of the partition function was done in [5] and we refer there for details.

5.2. Type A Topological Sigma Model.

The topological sigma models are well known to be derived from the $N = 2$ supersymmetric sigma models by performing the so-called "topological twisting" ([2,4]). The two basic $N = 2$ multiplets are the chiral and the twisted-chiral multiplets. Twisting the first one gives the topological sigma model of type A. Twisting the second one we get the topological sigma model of type B ([2,4]). There are several differences between type A and B topological sigma models. Perhaps, the most relevant one is that while the type A models can be generalized even to real manifolds (as we will see shortly), the type B models depend strongly on the complex structure and has been formulated so far only for Kahler manifolds. Also, type A and B models formulated on Calabi-Yau spaces are known to be related by mirror symmetry ([3]).

Let us derive the type A topological sigma models by applying our formulas. First, the internal manifold M_I is taken to be a two-dimensional, compact, oriented Riemann surface Σ (endowed with the metric $g_{\alpha\beta}(\sigma)$):

$$X : \Sigma \to M_T. \tag{5.8}$$

Also, we will consider now the case where the target manifold M_T is an Hermitian manifold equipped with a complex structure $J^i{}_j(X)$ and the Hermitian metric $G_{ij}(X)$ (this means that $J_{ij}(X) = G_{ik}(X)J^k_j(X) = -J_{ji}(X)$). Second, the section is taken to be:

$$s^i_\alpha = -2(\partial_\alpha X^i + \epsilon^\beta_\alpha J^i{}_j\partial_\beta X^i) \tag{5.9}$$

(factors are chosen to make contact with the notation of [4]) and the "metric" A_{ij}^{**} in (4.6) is given by:

$$A_{ij}^{\alpha\beta}(X,\sigma) = \frac{1}{4} g^{\alpha\beta}(\sigma) G_{ij}(X) \tag{5.10}$$

From (5.9) and (5.10) one quickly derives:

$$A_{ij}^{\alpha\beta}(X,\sigma) s_\alpha^i s_\beta^j = 2(g^{\alpha\beta} G_{ij} \partial_\alpha X^i \partial_\beta X^j + \epsilon^{\alpha\beta} J_{ij} \partial_\alpha X^i \partial_\beta X^j). \tag{5.11}$$

where we have used that, for Hermitian manifolds, $g^{\alpha\beta}\epsilon_\alpha{}^\tau \epsilon_\beta{}^\mu G_{ij} J^i{}_p J^j{}_k = g^{\tau\mu} G_{pk}$. Also one gets easily:

$$A_{ij}^{\alpha\beta} \rho_\alpha^i \nabla_Q s_\beta^j = -\frac{1}{2}(g^{\alpha\beta} G_{ij} \rho_\alpha^i D_\beta \chi^j + \epsilon^{\alpha\beta} D_k J_{ij} \rho_\alpha^i \chi^k \partial_\beta X^j + \epsilon^{\alpha\beta} J_{ij} \rho_\alpha^i D_\beta \chi^j). \tag{5.12}$$

where $D_\alpha V^i$ is the pull-back of the covariant derivative on the target:

$$D_\alpha V^i = \partial_\alpha V^i + \Gamma^i_{jk} V^j \partial_\alpha X^k. \tag{5.13}$$

Moreover:

$$\frac{1}{2} A_{ij}^{\alpha\beta} R^j_{klm} \chi^k \chi^l \rho_\alpha^i \rho_\beta^m = \frac{1}{8} g^{\alpha\beta} R_{klim} \chi^k \chi^l \rho_\alpha^i \rho_\beta^m. \tag{5.14}$$

Again, $\nabla_Q A_{ij}^{\alpha\beta} = 0$ ((3.9)). Substituting all this information in our expression (4.5) we obtain:

$$\begin{aligned} S_t[\Phi] = -\frac{1}{2} t \int d^2\sigma \sqrt{g} \Big\{ & g^{\alpha\beta} G_{ij} \partial_\alpha X^i \partial_\beta X^j + \epsilon^{\alpha\beta} J_{ij} \partial_\alpha X^i \partial_\beta X^j \\ & - g^{\alpha\beta} G_{ij} \rho_\alpha^i D_\beta \chi^j - \epsilon^{\alpha\beta} D_k J_{ij} \rho_\alpha^i \chi^k \partial_\tau X^j - \epsilon^{\alpha\beta} J_{ij} \rho_\alpha^i D_\beta \chi^j \\ & - \frac{1}{2} g^{\alpha\beta} G_{ij} H_\alpha^i H_\beta^j + \frac{1}{4} R_{klim} \chi^k \chi^l \rho_\alpha^i \rho_\beta^m \Big\}. \end{aligned} \tag{5.15}$$

This expression is slightly different from that on [4], the reason being that we have not demanded here the self-duality conditions on the ρ_α^i and H_α^i fields ($\rho_\alpha^i = \epsilon_\alpha{}^\beta J^i{}_j \rho_\beta^j$ and $H_\alpha^i = \epsilon_\alpha{}^\beta J^i{}_j H_\beta^j$). This means, in particular, that the topological action (5.15) can not be obtained from a twisting of a $N = 2$ supersymmetric sigma model. However, if we desire to make full contact with the action obtained from a twisted $N = 2$ supersymmetry we see straightforwardly that, imposing selfduality on the field ρ_α^i, (5.12) can be written as:

$$A_{ij}^{\alpha\beta} \rho_\alpha^i \nabla_Q s_\beta^j = -(g^{\alpha\beta} G_{ij} \rho_\alpha^i D_\beta \chi^j + \frac{1}{2} \epsilon^{\alpha\beta} D_k J_{ij} \rho_\alpha^i \chi^k \partial_\tau X^j). \tag{5.16}$$

then, on a Kahler manifold (where $D_i J^j{}_k = 0$) we recover exactly the expression for the type A topological sigma model ([2,4]):

$$\begin{aligned} S_t[\Phi] = -\frac{1}{2} t \int d^2\sigma \sqrt{g} \Big\{ & g^{\alpha\beta} G_{ij} \partial_\alpha X^i \partial_\beta X^j + \epsilon^{\alpha\beta} J_{ij} \partial_\alpha X^i \partial_\beta X^j \\ & - 2 g^{\alpha\beta} G_{ij} \rho_\alpha^i D_\beta \chi^j - \frac{1}{2} g^{\alpha\beta} G_{ij} H_\alpha^i H_\beta^j + \frac{1}{4} R_{klim} \chi^k \chi^l \rho_\alpha^i \rho_\beta^m \Big\}. \end{aligned} \tag{5.17}$$

However, is not necessary to impose selfduality to show that our action (5.15) is by its own topological both on the target M_T and on Σ (due to our general considerations in section 4).

Let us now use (4.7) to derive the ∇_Q-transformations for this case (again, the first two transformations in (4.7) remain the same and we just write the two last ones):

$$\begin{aligned} &\bullet \nabla_Q \rho^i_\alpha = H^i_\alpha + \partial_\alpha X^i + \epsilon^\beta_\alpha J^i{}_j \partial_\beta X^j \\ &\bullet \nabla_Q H^i_\alpha = -D_\alpha \chi^i - \epsilon^\beta{}_\alpha J^i{}_j D_\beta \chi^j + \frac{1}{2} R^i_{jkl} \chi^j \chi^k \rho^l_\alpha . \end{aligned} \tag{5.18}$$

The ∇_Q-transformation of ρ^i_α in (5.18) is easily seen to violate selfduality on no-Kahler manifolds. Actually, if:

$$\rho^i_\alpha = \epsilon_\alpha{}^\beta J^i{}_j \rho^j_\beta \tag{5.19}$$

then:

$$\nabla_Q(\rho^i_\alpha) = \epsilon_\alpha^\beta J^i{}_j \nabla_Q(\rho^j_\beta) + \epsilon_\alpha^\beta \chi^k D_k(J^i{}_j)\rho^j_\beta \equiv \epsilon_\alpha^\beta J^i{}_j \nabla_Q(\rho^j_\beta) + A^i_\alpha . \tag{5.20}$$

and self-duality is violated by the ∇_Q-transformations due to the term $A^i_\alpha \equiv \epsilon_\alpha^\beta \chi^k D_k(J^i{}_j)\rho^j_\beta$. This term is obviously zero for a Kahler manifold ($D_k J^i_j = 0$) but not for an arbitrary Hermitian manifold. However, we can check that A^i_α is anti-selfdual:

$$A^i_\alpha = -\epsilon_\alpha^\beta J^i{}_j A^j_\beta \tag{5.21}$$

and consequently, from (5.20), we could naturally define a new operator:

$$\hat{\nabla}_Q \rho^i_\alpha = \nabla_Q \rho^i_\alpha - \frac{1}{2} A^i_\alpha \tag{5.22}$$

which is selfdual. From this we could define, if we want, a selfdual $\hat{Q}$-transformation ([4]) for Hermitean (non-Kahler) manifolds (note that the action (5.15) changes also of form if we define it $\hat{\nabla}_Q$-exact). But we stress again that self-duality is not a necessary condition to have a topological theory in our formalism.

Let us now argue that the theories defined through the actions (5.15) and (5.17) do not depend on the complex structure $J^i{}_j$. The argument is completely similar to the one we used on section 4 to show that the theory is topological on the target manifold M_T. First we note that the ∇_Q-transformations in (5.18) depend on the complex structure $J^i{}_j$ through the section $s^i_\alpha(X, J(X))$. Due to the ∇_Q-exactness of the action we can analyze the theory on the large t limit. There we have localization on the instanton configurations $s^i_\alpha = 0$ and then all the dependence of (5.18) in $J^i{}_j$ disappears. In this case, deformations of the complex structure and ∇_Q transformations commute and the theory is invariant under that deformations of $J^i{}_j$. The introduced reader could be surprised at this point, because the type A topological theories formulated on Kahler manifolds ($dJ = 0$ where $J = J_{ij} dX^i \wedge dX^j$ and $J_{ij} = G_{ik} J^k_j$) are known to depend on the Kahler class J. But we remark here that the action employed by Witten $S^W_t[\Phi]$ in [3] is not (5.17) but:

$$\begin{aligned} S^W_t[\Phi] =& S_t[\Phi] - t \int d^2\sigma \sqrt{g(\sigma)} X^*(J) \\ =& t\nabla_Q\Big(\int d^2\sigma \sqrt{g(\sigma)} (\rho^*_i[\Phi] s^i_*[\Phi] + \rho^*_i \nabla_Q \rho^i_*)\Big) - \int d^2\sigma \sqrt{g(\sigma)} X^*(J) \\ \equiv& S_t[\Phi] - K(J). \end{aligned} \tag{5.23}$$

being:

$$K(J) \equiv \int d^2\sigma \sqrt{g(\sigma)} X^*(J) = \int d^2\sigma \sqrt{g(\sigma)} \epsilon^{\alpha\beta} J_{ij} \partial_\alpha X^i \partial_\beta X^j \tag{5.24}$$

($X^*(J)$ is the pullback of the Kahler form). This term is seen to be invariant under ∇_Q-transformations:

$$\nabla_Q \left(\epsilon^{\alpha\beta} J_{ij} \partial_\alpha X^i \partial_\beta X^j \right) = 2\epsilon^{\alpha\beta} D_\alpha (J_{ij} \chi^i \partial_\beta X^j) \tag{5.25}$$

(we have used the Kahler condition $D_i J^k_{\ l} = 0$), but is not ∇_Q-exact. Therefore the topological character of the theory is not guaranteed (we can not use the large t limit if the action is not δ_Q or ∇_Q-exact). In fact, the term (5.24) added in the action (5.23) is a topological invariant (and consequently, the theory defined by the action (5.23) is still invariant under deformations of the target metric $G_{ij}(X)$ and the internal metric $g_{\mu\nu}$), however, (5.24) depends on the homotopy class of the map X and the cohomology class of the closed form J (and then, is sensible to changes of the Kahler form $J(X) = J_{ij} dX^i \wedge dX^j$). As a consecuence, the theory defined by S_t^W depends on the Kahler form J whereas the one defined by S_t does not.

Before finishing this section, let us do some general comments. The type A topological action that we have constructed here is exactly the same as the one that is obtained by twisting the $N = 2$ supersymmetric sigma models (when the manifold M_T is Kahler and we demand self-duality conditions on the fields ρ^i_α and H^i_α). The generalization to Hermitian manifolds was done by Witten in [2]. In the present context we see that the type A topological sigma models can be generalized even to real manifolds. One just has to take the section (5.9) without the complex structure term and use our formulas (4.5) and (4.8) to derive the acction and Q-transformations. With respect to the observables of the theory, they were analyzed in [3]. Here we note from (5.9) that the localization of correlators are going to take place on the moduli space of holomorphic instantons:

$$-\frac{1}{2} s^i_\alpha = \partial_\alpha X^i + \epsilon_\alpha^{\ \beta} J^i_{\ j} \partial_\beta X^j = 0. \tag{5.26}$$

One has to study again carefully the fermionic zero modes to know which of the observables (4.12) give non-trivial correlators. This depends on Σ, M_T and index theorems. We refer to [3] for details. The result is that, on Kahler manifolds, correlators split on an addition $\sum_k$ of intersection forms on the moduli space of holomorphic instantons of degree k (over instantons of the type (5.26) we have that $K(J) = k$ where, using a proper normalization, k is an integer [3]). If we use the action S_t^W ((5.23)) instead of S_t ((5.17)) these intersection numbers are weighted by exponentials of the degree of the corresponding holomorphic instantons ($e^{-itK(J)} \sim e^{-itk}$). In the case that we work with a real manifold M_T the generalization is straightforward with our formalism. We can just the section s^i_α to be (5.9) without the complex structure term and substitute in our formulae (4.5) and (4.8) to get the action and transformations. A simple analysis gives that correlators, in this case, are just the classical intersection numbers of submanifolds of M_T which are Poincaré duals to the corresponding forms $A(X)$ entering in the observables (4.12) in the correlators (note that the solutions of the moduli equations $\partial_\alpha X^i = 0$ are the constant maps and therefore, $\mathcal{M}$ coincides with the target manifold M_T).

6. Conclusions

We have introduced a wide class of topological field theories of maps $X^i : M_I \to M_T$ from a m-dimensional internal manifold M_I to a n-dimensional target manifold M_T localizing the correlators on a desired moduli space of instantons $\mathcal{M}$. The minimal action is given in (4.5) and the general Q-transformations are given in (4.7). To guarantee the topological character of the theory we were forced to demand the $\rho^i_*[\Phi]$ fields to be independent both of the internal metric $g_{\mu\nu}$ and the target metric G_{ij}. Our models contain previously known topological systems as particular cases, like the topological sigma models of type A. However our approach does not involve the twist procedure of an $N=2$ supersymmetric sigma model. This allowed us to formulate topological matter of type A in real manifolds (to our knowledge, so far, they have been formulated only for Hermitean manifolds [2]). In the case of real manifolds, when the section is chosen to be $s^i_\alpha = \partial_\alpha X^i$, the correlators turn out to be classical intersection numbers of submanifolds of the target space M_T.

It would be interesting to study if the models here introduced contains the other type of known topological sigma models (type B). Also our formalism could be generalized to the case in which the topological charge Q is not nilpotent but closses on a group of symmetry transformations of the theory.

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