

A q -deformation of the Coulomb Problem

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ABSTRACT

The algebra of observables of $SO_q(3)$ -symmetric quantum mechanics is extended to include the inverse $\frac{1}{R}$ of the radial coordinate and used to obtain eigenvalues and eigenfunctions of a q -deformed Coulomb Hamiltonian.

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1 Introduction

Much work has been done recently to explore the $SO_q(3)$ -symmetric quantum mechanics developed in [1] and [2]. In particular, a lot is known about the q -deformations of the harmonic oscillator. The other nontrivial soluble problem in ordinary quantum mechanics is the Coulomb problem, but for that one needs some notion of an inverse radius. Weich [3] considered a q -deformed Coulomb potential, defining $\frac{1}{R}$ in a manner dependent upon a particular Hilbert space representation. This differs from the more standard "wave-function" type approaches used in investigations of the oscillator (Refs. [4] - [6] for example).

Here we approach this problem by defining $\frac{1}{R}$ as an actual element of the algebra of observables, thereby achieving representation-independence. Since $X^2 = R^2$ is already defined, we can then also define R as well as all its integral powers. A study of the action of momentum operators on powers of R then helps to bring out the interpretation of these operators as symmetric q -derivatives.

Using this definition of $\frac{1}{R}$, a self-adjoint symmetric q -deformation of the Coulomb Hamiltonian can be found which shares the n^2 -fold degeneracy of the undeformed Hamiltonian for the q -analog of bound states. As in [3], we obtain a Balmer-type spectrum for these states with

$$E_n = - \left(\frac{\alpha}{[n]_q} \right)^2, \quad (1)$$

where the symmetric q -analog of n is defined as

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \quad (2)$$

In addition, we also obtain positive-energy wave functions and a candidate q -Coulomb S -matrix.

The paper is structured as follows. After a brief review of $SO_q(3)$ -symmetric quantum mechanics (Section 2), we set up the formalism for dealing with the q -Coulomb problem (Sections 3-5), which is then treated in Section 6. The detailed proofs of some statements made in the text are deferred to five Appendices.

2 $SO_q(3)$ -symmetric quantum mechanics

We build upon the algebra of observables as it is defined in Refs. [5] and [3]. The q -deformed metric and Levi-Civita tensors are defined as follows:

$$\gamma^{ij} \equiv \gamma_{ij} \equiv \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{q}} \\ 0 & 1 & 0 \\ \sqrt{q} & 0 & 0 \end{bmatrix}$$

$$\epsilon_1^{ij} \equiv \epsilon_{ij}^1 \equiv \begin{bmatrix} 0 & \frac{1}{\sqrt{q}} & 0 \\ -\sqrt{q} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\epsilon_2^{ij} \equiv \epsilon_{ij}^2 \equiv \begin{bmatrix} 0 & 0 & -1 \\ 0 & \frac{1}{\sqrt{q}} - \sqrt{q} & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\epsilon_3^{ij} \equiv \epsilon_{ij}^3 \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{q}} \\ 0 & -\sqrt{q} & 0 \end{bmatrix}$$

These give rise to an R-matrix:

$$\tilde{R}_{kl}^{ij} = q\delta_k^i\delta_l^j - \epsilon_a^{ij}\epsilon_{kl}^a + \left(\frac{1}{q} - 1\right)\gamma^{ij}\gamma_{kl}$$

which is a solution to the Yang-Baxter equation

$$\tilde{R}_{ab}^{ij}\tilde{R}_{cn}^{bk}\tilde{R}_{lm}^{ac} = \tilde{R}_{de}^{jk}\tilde{R}_{lf}^{id}\tilde{R}_{mn}^{fe},$$

and has the inverse

$$\tilde{R}^{-1}{}^{ij}{}_{kl} = \frac{1}{q}\delta_k^i\delta_l^j - \epsilon_a^{ij}\epsilon_{kl}^a + (q-1)\gamma^{ij}\gamma_{kl}.$$

We use the metric and Levi-Civita tensors to define scalar and vector products as for the undeformed tensors: $A \cdot B = \gamma^{ij}A_iB_j$ and $[A \times B]_k = \epsilon_k^{ij}A_iB_j$.

$\mathcal{A}_{R_q^3}^X$ is the $SO_q(3)$ -covariant *-algebra defined by the generators X_1, X_2, X_3 subject to the relations

$$[X \times X]_k = 0$$

and

$$X_i^* = \gamma^{ij}X_j.$$

$X^2 \equiv X \cdot X$ is then real and central in this algebra. In the $q = 1$ limit, X_1 and X_3 correspond to $\frac{1}{\sqrt{2}}(X \pm Y)$ while X_2 is Z . The space of wave functions in harmonic-oscillator treatments of $SO_q(3)$ -symmetric quantum mechanics is an appropriate subspace of $\mathcal{A}_{R_q^3}^X$.

One also considers an $SO_q(3)$ -covariant *-algebra $\mathcal{D}_{R_q^3}^X$ of operators on $\mathcal{A}_{R_q^3}^X$, whose generators are the X_i , derivative operators $\bar{\partial}_i$, and a scaling operator μ . The X_i act on $\mathcal{A}_{R_q^3}^X$ by left multiplication. μ is defined such that $\mu(1) = 1$; and for all $f \in \mathcal{A}_{R_q^3}^X$, $\mu(X_i f) = qX_i\mu(f)$. $\bar{\partial}_i$ is defined such that $\bar{\partial}_i(1) = 0$; and for all $f \in \mathcal{A}_{R_q^3}^X$,

$$\bar{\partial}_i(X_j f) = [\gamma_{ij} + \frac{1}{q}\tilde{R}_{ij}^{kl}X_k\bar{\partial}_l]f.$$

The generators of $\mathcal{D}_{R_q^3}^X$ then obey the relations:

$$\mu X_i = qX_i\mu,$$

$$\mu \bar{\partial}_i = \frac{1}{q} \bar{\partial}_i \mu,$$

$$\bar{\partial}_i X_j = \gamma_{ij} + \frac{1}{q} \tilde{R}_{ij}^{kl} X_k \bar{\partial}_l$$

$$[\bar{\partial} \times \bar{\partial}]_k = 0.$$

One also defines an inverse of μ :

$$\mu^{-1} \equiv \mu[1 + q^{-2}(1 - q^2)X \cdot \bar{\partial} + q^{-3}(1 - q)^2 X^2 \bar{\partial}^2].$$

In addition, there is a conjugate set of derivative operators in $\mathcal{D}_{R_q^3}^X$,

$$\partial_i \equiv \mu^2[\bar{\partial}_i + (q^{-2} - q^{-1})X_i \bar{\partial}^2].$$

∂_i then satisfy the relations

$$[\partial \times \partial]_k = 0$$

and

$$\partial_i X_j = \gamma_{ij} + q \tilde{R}^{-1}_{ij}{}^{kl} X_k \partial_l$$

The *-operation on μ and $\bar{\partial}_i$ is defined as $\mu^* \equiv q^{-3} \mu^{-1}$ and $(\bar{\partial}_i)^* \equiv -q^3 \gamma^{ij} \partial_j$.

Neither triplet of derivative operators has a subalgebra isomorphic to $\mathcal{A}_{R_q^3}^X$, but a linear combination of the two does. This linear combination is then the triplet of q -momentum operators

$$P_i = \frac{\partial_i + q^{-3} \bar{\partial}_i}{i(1 + q^{-3})}.$$

Then $(P_i)^* = \gamma^{ij} P_j$, $[P \times P]_k = 0$, and P^2 is a real scalar that commutes with the P_i .

The X_i and P_j satisfy q -deformed versions of the Heisenberg relations:

$$i(P_a X_b - q \tilde{R}^{-1}_{ab}{}^{cd} X_c P_d) = \mu^{-1} \left(\gamma_{ab} W + \frac{q-1}{qK} \epsilon_{ab}{}^m L_m \right)$$

and

$$-i(X_a P_b - q \tilde{R}^{-1}_{ab}{}^{cd} P_c X_d) = q^3 \mu \left(\gamma_{ab} W + \frac{q-1}{qK} \epsilon_{ab}{}^m L_m \right),$$

where $W \equiv \mu[1 + q^{-2}(1 - q)X \cdot \bar{\partial}]$ is a real scalar, $L_i \equiv \frac{1}{q} \mu[X \times \bar{\partial}]_i$, and

$$K \equiv q - 1 + \frac{1}{q}. \quad (3)$$

The L_i and W generate a q -deformed angular momentum algebra. Vectors Z_i (such as the X_i , ∂_j , and P_k) satisfy the following relations with the L_i and W , generalizing the role of the L_i as generators of rotations:

$$L_i Z_j = -\epsilon_i^{cd} \epsilon_{dj}^e Z_c L_e + \epsilon_{ij}^a Z_a W,$$

$$L \cdot Z = Z \cdot L = 0,$$

$$W Z_j = K Z_j W - (K - 1) \epsilon_j^{rs} Z_r L_s.$$

In addition Z^2 commutes with the L_i and W . The L_i and W also satisfy the following relations:

$$L^2 = \frac{W^2 - 1}{K - 1}$$

and

$$[L \times L]_k = L_k W = W L_k.$$

$\mathcal{A}_{R_q^3}^X$ modulo powers of X^2 can be shown to be a direct sum, indexed by nonnegative integers l , of irreducible representations of the angular momentum algebra. The l th representation is then $(2l+1)$ -dimensional, and W is a Casimir operator with eigenvalue

$$w_l = \frac{q^{l+1} + q^{-l}}{q + 1}. \quad (4)$$

For $q = 1$, the eigenvalue of L^2 in the l th representation becomes the familiar $l(l+1)$.

As a last preliminary result, the momentum operators can be expressed in terms of W , μ , and X^2 :

$$X^2 P_i = \frac{1}{iK(q - \frac{1}{q})(q - 1)} \left[\frac{1}{q} X_i W \mu^{-1} - W X_i \mu^{-1} + q^2 X_i W \mu - q W X_i \mu \right], \quad (5)$$

and

$$X^2 P^2 = \frac{1}{K^2(q - \frac{1}{q})^2} \left[\frac{(q+1)^2}{q} W^2 - q\mu^2 - 2 - \frac{1}{q} (\mu^{-1})^2 \right]. \quad (6)$$

These identities will be essential for calculations in the representation we will introduce later. We have included the numerical K factors on the left hand side of these identities, as we will define the hamiltonian scaled by a K^2 factor, which goes to one when $q \rightarrow 1$.

3 Definition of $\frac{1}{R}$

$\frac{1}{R}$ is already a well-defined concept in the space of undeformed, complex functions on R^3 . Its essential properties are that it is a real, scalar function and

that $X^2(\frac{1}{R})^2 = (\frac{1}{R})^2 X^2 = 1$. The simplest generalization of these properties is then to define $\frac{1}{R}$ to be a real, scalar corepresentation of $SO_q(3)$. $\hat{\mathcal{A}}_{R^3}^X$ is then the *-algebra generated by the X_i and $\frac{1}{R}$, where the X_i obey the same relations as in $\mathcal{A}_{R^3}^X$, $\frac{1}{R}$ commutes with the X_i , and

$$X^2(\frac{1}{R})^2 = 1. \quad (7)$$

$\hat{\mathcal{D}}_{R^3}^X$ is the $SO_q(3)$ -covariant *-algebra of operators on $\hat{\mathcal{A}}_{R^3}^X$, where we add $\frac{1}{R}$ to the generators of $\hat{\mathcal{D}}_{R^3}^X$. For this definition to be complete we must have a set of relations involving $\frac{1}{R}$, μ , and the derivatives. These must be consistent with the relations between $\frac{1}{R}$ and the X_i . Clearly since $\frac{1}{R}$ has dimensions of an inverse length, we should have

$$\mu \frac{1}{R} = q^{-1} \frac{1}{R} \mu.$$

Equation 7 implies $\partial_i(\frac{1}{R})$ may be.

$$\partial_i X^2 (\frac{1}{R})^2 = \partial_i(1) = 0$$

It follows from the algebra of $\mathcal{D}_{R^3}^X$ that

$$\partial_i X^2 = (q^{-1} + 1)X_i + q^2 X^2 \partial_i.$$

Thus

$$(q^{-1} + 1)X_i \frac{1}{R^2} + q^2 X^2 \partial_i (\frac{1}{R^2}) = 0,$$

and therefore

$$\partial_i \left(\frac{1}{R^2} \right) = -q^{-2} (q^{-1} + 1) \frac{1}{R^4} X_i \quad (8)$$

Note that we are using the notation that if $A \in \hat{\mathcal{D}}_{R^3}^X$, and $f \in \hat{\mathcal{A}}_{R^3}^X$ then $A(f) \in \hat{\mathcal{A}}_{R^3}^X$ is the result of evaluating the effect of A on f , whereas $Af \in \hat{\mathcal{D}}_{R^3}^X$ is the product of A and f as operators.

The simplest solution for $\partial_i \frac{1}{R}$ is

$$\partial_i \frac{1}{R} = q^{-1} \frac{1}{R} \partial_i - q^{-2} \frac{1}{R^3} X_i. \quad (9)$$

Repeated application of Equation (9) indeed gives (8). Similarly one finds that

$$\bar{\partial}_i \frac{1}{R} = q \frac{1}{R} \bar{\partial}_i - q^2 \frac{1}{R^3} X_i. \quad (10)$$

Equation (9) must be checked for consistency with the algebra of $\hat{\mathcal{A}}_{R^3}^X$ before we can conclude that $\hat{\mathcal{D}}_{R^3}^X$ is a consistent operator algebra. In particular we must

show that $\partial_i(\frac{1}{R}f) = \partial_i(f\frac{1}{R})$ and $\partial_i(X^2\frac{1}{R^2}) = \partial_i(\frac{1}{R^2}X^2) = 0$. In addition if $\frac{1}{R}$ is truly a scalar, it should commute with the L_i and W . The proofs that these conditions are satisfied are given in Appendix A.

Having defined $\frac{1}{R}$, we can now also define the q -deformed radius $R \equiv \frac{1}{R}X^2$. This has the following commutation relation with ∂_i :

$$\begin{aligned}\partial_i R &= \partial_i \frac{1}{R} X^2 \\ &= \left[q^{-1} \frac{1}{R} \partial_k - q^{-2} \frac{1}{R^3} X_k \right] X^2 \\ &= q^{-1} \frac{1}{R} [(q^{-1} + 1) X_i + q^2 X^2 \partial_i] - q^{-2} \frac{1}{R} X_i \\ &= q^{-1} \frac{1}{R} X_i + q R \partial_i\end{aligned}$$

Induction over positive and negative n gives

$$\partial_i R^n = q^{-1}(n)_q R^{n-2} X_i + q^n R^n \partial_i$$

where

$$(n)_q \equiv \frac{q^n - 1}{q - 1}$$

is the asymmetric q -analog. One can in fact develop a theory using ∂ or $\bar{\partial}$ as momentum operators and rewrite q -deformed harmonic oscillator theories in the language of R and integral powers of R . However since ∂^2 and $\bar{\partial}^2$ are not self-adjoint, we will concentrate on the action of the P_i on elements of $\hat{\mathcal{A}}_{R^3}^X$.

4 Separation of Variables

Using the formalism developed in the last section, we can now consider the q -analog of the separation of variables problem for the kinetic term of the hamiltonian. To this end we first introduce q -analogs of spherical harmonics. These are defined, up to a—for us irrelevant—normalization as elements Y_q^l of $\hat{\mathcal{A}}_{R^3}^X$ which obey the following two conditions:

- (i) Left multiplication of Y_q^l by L^2 or by L_2 yields Y_q^l multiplied by a real eigenvalue, or in other words, Y_q^l is an “eigenfunction” of both L^2 and L_2 .
- (ii) All Y_q^l commute with μ .

For illustration, we give the explicit expressions of Y_q^l for $l = 0, 1, 2$

$$\begin{aligned}Y_q^0 &= 1 \\ Y_q^1 &= \frac{1}{R} X_1 \\ Y_q^1 &= \frac{1}{R} X_2\end{aligned}$$

$$\begin{aligned}
Y_{q1}^1 &= \frac{1}{R} X_3 \\
Y_{q-2}^2 &= \frac{1}{R^2} X_1^2 \\
Y_{q-1}^2 &= \frac{1}{R^2} X_1 X_2 \\
Y_{q0}^2 &= \frac{1}{R^2} [qX_1 X_3 - (\sqrt{q} + \frac{1}{\sqrt{q}}) X_2 X_2 + \frac{1}{q} X_3 X_1] \\
Y_{q1}^2 &= \frac{1}{R^2} X_2 X_3 \\
Y_{q2}^2 &= \frac{1}{R^2} X_3^2
\end{aligned}$$

Now, by multiplying equations (5) and (6) on the left by $\frac{1}{R}$ and $\frac{1}{R^2}$ respectively, we obtain

$$K P_i = \frac{1}{i(q - \frac{1}{q})(q-1)} \frac{1}{R} \left[\frac{1}{q} X_i W \mu^{-1} - W X_i \mu^{-1} + q^2 X_i W \mu - q W X_i \mu \right], \quad (11)$$

and

$$K^2 P^2 = \frac{1}{(q - \frac{1}{q})^2} \frac{1}{R^2} \left[\frac{(q+1)^2}{q} W^2 - q \mu^2 - 2 - \frac{1}{q} (\mu^{-1})^2 \right]. \quad (12)$$

Since μ commutes with the $Y_{q^l m}$ and W commutes with powers of R , if we expand functions in terms of $R^n Y_{q^l m}$, Equation (12) is ready made for calculating their momentum squared.

$$\begin{aligned}
K^2 P^2 R^n Y_{q^l m} &= \frac{1}{(q - \frac{1}{q})^2} \frac{1}{R^2} \left[\frac{(q+1)^2}{q} w_l^2 - q^{2n+1} - 2 - q^{-2n-1} \right] R^n Y_{q^l m} \\
&= \frac{q^{2l+1} + q^{-2l-1} - q^{2n+1} - q^{-2n-1}}{(q - \frac{1}{q})^2} R^{n-2} Y_{q^l m}.
\end{aligned}$$

In terms of the symmetric q -analog of n of equation (2), we can write this in the simplified form

$$K^2 P^2 R^n Y_{q^l m} = -[n+l+1]_q [n-l]_q R^{n-2} Y_{q^l m}. \quad (13)$$

This is a clear generalization of the result from ordinary real calculus that

$$\nabla [r^n Y_{lm}(\theta, \phi)] = (n+l+1)(n-l) r^{n-2} Y_{lm}(\theta, \phi),$$

and it is the main result of this section.

Equation (11) is not nearly as useful as its counterpart because the action of the X_i on $Y_{q^l m}$ is nontrivial. However for powers of R , one can obtain the simple result that

$$i K P_i (R^n) = [n]_q R^{n-2} X_i = [n]_q R^{n-1} \frac{X_i}{R}. \quad (14)$$

5 The free particle

Let us consider a system with Hamiltonian

$$H = K^2 P^2$$

with the convenient normalization factor K^2 determined by equation (3). Then the Schrödinger equation for this system is the q -deformed Helmholtz equation

$$K^2 P^2 \psi = k^2 \psi.$$

The solutions to this are of the form $j_{[q]l}(kR)Y_q^l$ and $n_{[q]l}(kR)Y_q^l$ where $j_{[q]l}$ and $n_{[q]l}$ are respectively the q -spherical Bessel and Neumann functions.

$$j_{[q]l}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n [2n+2l]_q!!}{[2n]_q!! [2n+2l+1]_q!} x^{2n+l} \quad (15)$$

$$\begin{aligned} n_{[q]l}(x) = & - \sum_{n=0}^{l-1} \frac{[2l-2n]_q!}{[2n]_q!! [2l-2n]_q!!} x^{2n-l-1} \\ & + (-1)^{l+1} \sum_{n=l}^{\infty} \frac{(-1)^n [2n-2l]_q!!}{[2n]_q!! [2n-2l]_q!} x^{2n-l-1} \end{aligned} \quad (16)$$

where for nonnegative integers n

$$[n]_q! \equiv \begin{cases} \prod_{k=1}^n [k]_q & n > 0 \\ 1 & n = 0 \end{cases} \quad (17)$$

and

$$[2n]_q!! \equiv \begin{cases} \prod_{k=1}^n [2k]_q & n > 0 \\ 1 & n = 0 \end{cases} \quad (18)$$

That Equations (15) and (16) give rise to solutions is easily seen by applying Equation (13). This eliminates the zeroth element, and then reindexing gives the desired result as would be the case for the $q = 1$ differential equation.

We can also obtain a q -deformed generalization of the Rayleigh formulas, which provide an alternative definition of these Bessel and Neumann functions. This will be a more convenient form for obtaining q -spherical Hankel functions,

$$h_{[q]l}^{(1)}(x) \equiv j_{[q]l}(x) + in_{[q]l}(x) \quad (19)$$

and

$$h_{[q]2}^{(1)}(x) \equiv j_{[q]l}(x) - in_{[q]l}(x), \quad (20)$$

corresponding to incoming and outgoing spherical waves. These results are discussed in Appendix B.

It is interesting to note that, because the P_i do not commute, it is not possible to have a plane wave with definite momentum as in ordinary quantum mechanics. The best we could do is specify the component of the momentum in the direction of propagation. Since, moreover, P_2 for example does not commute with L_2 , the problem of expanding even these quasiplane waves as a sum of $R^n Y_q^l$ terms is nontrivial.

6 The q -Coulomb Problem

In ordinary quantum mechanics, the Coulomb Hamiltonian is

$$H = \frac{p^2}{2m} - \frac{\alpha}{r};$$

or if we rescale this by $2m$ and incorporate the mass into α ,

$$H = p^2 - \frac{2\alpha}{r}.$$

There are several possible ways to q -deform this. We are interested in a self-adjoint Hamiltonian which preserves the properties of the ordinary Hamiltonian that make it amenable to finding eigenfunctions. That is to say we require the existence of a q -deformed Lenz vector which commutes with the q -deformed Hamiltonian so that there continue to be degeneracies between solutions with different angular momentum quantum numbers.

Following [3], we define our q -deformed Coulomb Hamiltonian to be:

$$H = K^2 P^2 - \alpha q \left(\frac{1}{R} \mu + \mu^* \frac{1}{R} \right)$$

which is clearly self-adjoint (for convenience the normalization of the kinetic term has again been chosen with a prefactor K^2 determined by equation (3)). Noting that $\mu^* = q^{-3}\mu^{-1}$, this can be written in the simpler form

$$H = K^2 P^2 - \alpha \frac{1}{R} (q\mu + q^{-1}\mu^{-1}). \quad (21)$$

This Hamiltonian also commutes with the Lenz vector

$$\begin{aligned} A_k &\equiv \frac{[W, iK P_k]}{K-1} + \frac{\alpha X_k}{R} \\ &= iK(P_k W - (P \times L)_k) + \frac{\alpha X_k}{R}. \end{aligned}$$

The Lenz vector along with the angular momentum operators then generate the algebra given in [3].

If we write the eigenvalues E of the Hamiltonian (21) in the form

$$E = - \left(\frac{\alpha}{[\gamma]_q} \right)^2, \quad (22)$$

then the corresponding ‘‘eigenfunctions’’ are (see Appendix C for their derivation)

$$\psi_{\gamma lm} = \sum_{p=0}^{\infty} A_p(\gamma) R^l \left(\frac{\alpha q^\gamma R}{[\gamma]_q} \right)^p \exp_q \left(\frac{-q^{l+1+p-\gamma} \alpha R}{[\gamma]_q} \right) Y_q^l{}_m \quad (23)$$

with

$$A_p(\gamma) = q^{p(l+\frac{1}{2}p(p+1))} \frac{(1-q^2)^p (q^{2(l+1-\gamma)}; q^2)_p (q^{4(l+1)}; q^4)_p}{(q^2; q^2)_p (q^{2(2l+2)}; q^2)_p (q^{2(l+1)}; q^2)_p}. \quad (24)$$

Here

$$(a; u)_p \equiv \begin{cases} \prod_{m=0}^{p-1} (1 - au^m) & p = 1, 2, 3, \dots \\ 1 & p = 0 \end{cases}, \quad (25)$$

is the q -deformed Pochhammer symbol [7], with $u = q^2$ and q^4 in Eq. (24), and

$$\exp_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} \quad (26)$$

is the q -deformed exponential, where we have used the notation of Eq. (17). q^γ is obtained in terms of the energy E from the quadratic equation (22). This has two solutions

$$q^{-\gamma_\pm} = \frac{\eta \pm \sqrt{\eta^2 - E}}{\sqrt{-E}}, \quad (27)$$

with η given by

$$\eta = \frac{(q^{-1} - q)\alpha}{2}. \quad (28)$$

(We assume $q < 1$ to insure convergence; had we instead set $q > 1$, we would have to everywhere change $q \rightarrow q^{-1}$)

Note that for a given energy, we actually only have one solution since if we write the solutions as power series of R , the difference equation admits only one solution that behaves as R^l for small R . Whether we write it in terms of γ_+ or γ_- simply gives us two expressions for the same result.

In ordinary quantum mechanics, a decaying exponential multiplied by a polynomial in r is normalizable and gives rise to a bound state. Something very similar happens in the $q \neq 1$ limit. Consider the wave function (23), which is a sum of terms each of which is a nonnegative power of R multiplying a q -exponential of the form $\exp_q(-c_q(p)R)$. As can be seen from (23), the $c_q(p)$ become independent of p as $q \rightarrow 1$, thus reproducing the usual result. Then we define a bound state in the q -deformed case to be a wave function of the type (23) for which the sum over p truncates at the finite value $p = n - l - 1$. On account of the factor $(q^{2(l+1-\gamma)}; q^2)_p$ in the numerator of (24), such a truncation occurs if γ equals a positive integer $n > l$. From Eq. (22), we then obtain the q -Balmer formula, and we find the corresponding wave functions given by (23) and (24). The q -Balmer formula already appears in Ref. [3], where, as noted in the introduction, the operator $\frac{1}{R}$ is treated differently. This difference is reflected in our wave functions.

In principle these same wave functions (23) should also cover the continuum part of the spectrum, and one should be able to extract an S -matrix from them. How to do this in a rigorous fashion remains to be seen. Here we consider the candidate S -matrix suggested by equation (24)

$$S_l^{(q)}(E) = (1 - q^2)^{(\gamma_- - \gamma_+)} \frac{\Gamma_{q^2}(l + 1 - \gamma_+)}{\Gamma_{q^2}(l + 1 - \gamma_-)} = \prod_{n=0}^{\infty} \frac{1 - q^{2(l+1-\gamma_-+n)}}{1 - q^{2(l+1-\gamma_++n)}}, \quad (29)$$

where the q -gamma function is defined as in [7]:

$$\Gamma_{q^2}(x) := \frac{(q^2; q^2)_\infty}{(q^{2x}; q^2)_\infty} (1 - q^2)^{1-x}. \quad (30)$$

The S -matrix (29) appears to have all the right features:

A. For $q \rightarrow 1$ this S -matrix reproduces the familiar Coulomb S -matrix. In this limit the prefactor goes to one, as can be seen from Eq. (27), and the q -gamma functions become precisely the ordinary gamma functions which appear in the ordinary Coulomb S -matrix.

B. For integer $\gamma_+ \geq l + 1$, $S_l^{(q)}(E)$ has a pole corresponding to a q -Balmer state. Both the the location and the residue of this pole differ from those of its ordinary ($q = 1$) Balmer limit.

C. As can be seen from Eqs. (29) and (27) the branchpoint of $S_l^{(q)}(E)$, the scattering threshold, is now located $E = \eta^2$, with η given in Eq. (28) and not at $E = 0$ as in the ordinary case. This is the most dramatic departure from the ordinary case: the scattering region starts at $E = \eta^2$. For $q = 1$ this reduces to the expected threshold $E = 0$.

A Properties of $\frac{1}{R}$

We have already satisfied $\partial_i(X^2 \frac{1}{R^2}) = 0$. This definition of $\bar{\partial}_i \frac{1}{R}$ also satisfies the other half of the second constraint.

$$\begin{aligned} \partial_i \frac{1}{R^2} &= q^{-2} \frac{1}{R^2} \partial_i - q^{-2}(q^{-1} + 1) \frac{1}{R^4} X_i \\ \partial_i \left(\frac{1}{R^2} X^2 \right) &= \left[q^{-2} \frac{1}{R^2} \partial_i + q^2 X^2 \partial_i \right] (1) - q^{-2}(q^{-1} + 1) \frac{1}{R^2} X_i \\ &= 0 \end{aligned}$$

Trivially $\partial_i 1 \frac{1}{R} = \partial_i \frac{1}{R} 1 = \partial_i \frac{1}{R}$. Suppose that $\partial_i f \frac{1}{R} = \partial_i \frac{1}{R} f$ where $f \in \hat{\mathcal{A}}_{R^3}^X$.

$$\begin{aligned} \partial_i X_j f \frac{1}{R} &= [\gamma_{ij} + q \tilde{R}^{-1}{}_{ij}{}^{kl} X_k \partial_l] f \frac{1}{R} \\ &= \gamma_{ij} f \frac{1}{R} + q \tilde{R}^{-1}{}_{ij}{}^{kl} X_k \partial_l f \frac{1}{R} \\ &= \gamma_{ij} f \frac{1}{R} + q \tilde{R}^{-1}{}_{ij}{}^{kl} X_k \left(q^{-1} \frac{1}{R} \partial_l - q^{-2} \frac{1}{R^3} X_l \right) f \\ &= \gamma_{ij} f \frac{1}{R} \\ &\quad - q^{-2} \delta_i^k \delta_j^l \frac{1}{R^3} X_k X_l f + q^{-1} \epsilon_a{}^{kl} \epsilon_i{}^{ja} X_k X_l f \frac{1}{R^3} \\ &\quad - q^{-1} (q - 1) \gamma^{kl} \gamma_{ij} \frac{1}{R^3} X_k X_l f \\ &= q^{-1} \gamma_{ij} f \frac{1}{R} + \frac{1}{R} \tilde{R}^{-1}{}_{ij}{}^{kl} X_k \partial_l f \end{aligned}$$

where we used the fact that $\epsilon_a{}^{kl}X_kX_l = 0$. At the same time

$$\begin{aligned}
\partial_i \frac{1}{R} X_j f &= \left(q^{-1} \frac{1}{R} \partial_i - q^{-2} \frac{1}{R^3} X_i \right) X_j f \\
&= q^{-1} \frac{1}{R} [\gamma_{ij} + q \tilde{R}^{-1}{}^{kl} X_k \partial_l] f - q^{-2} \frac{1}{R^3} X_i X_j f \\
&= \partial_i X_j f \frac{1}{R}
\end{aligned}$$

We must also show that $\partial_i \frac{1}{R} f \frac{1}{R} = \partial_i f \frac{1}{R} \frac{1}{R}$ in order to complete this inductive proof. However this is trivial having assumed that $\partial_i f \frac{1}{R} = \partial_i \frac{1}{R} f$. Thus $\hat{\mathcal{A}}_{R^3}^X$ and $\hat{\mathcal{D}}_{R^3}^X$ are consistently defined.

If $\frac{1}{R}$ is truly a scalar, it should commute with the L_i and W . This is indeed true:

$$\begin{aligned}
W &= \mu[1 + q^{-2}(1 - q)X \cdot \bar{\partial}] \\
X \cdot \bar{\partial} &= \gamma^{ij} X_i \bar{\partial}_j \frac{1}{R} \\
&= \gamma^{ij} X_i \left[q \frac{1}{R} \bar{\partial}_j - q^2 \frac{1}{R^3} X_j \right] \\
&= q \frac{1}{R} X \cdot \bar{\partial} - q^2 \frac{1}{R} \\
W \frac{1}{R} &= \mu \left[\frac{1}{R} + (q^{-2} - q^{-1}) \left(q \frac{1}{R} X \cdot \bar{\partial} - q^2 \frac{1}{R} \right) \right] \\
&= \mu \left[q \frac{1}{R} + q \frac{1}{R} (q^{-2} - q^{-1}) X \cdot \bar{\partial} \right] \\
&= \frac{1}{R} \mu [1 + (q^{-2} - q^{-1}) X \cdot \bar{\partial}] \\
&= \frac{1}{R} W
\end{aligned}$$

$$L_i = \mu q^{-1} [X \times \bar{\partial}]_i$$

$$\begin{aligned}
[X \times \bar{\partial}]_i &= \epsilon_i{}^{jk} X_j \bar{\partial}_k \\
&= \epsilon_i{}^{jk} X_j \left[q \frac{1}{R} \bar{\partial}_k - q^2 \frac{1}{R^3} X_k \right] \\
&= q \frac{1}{R} \epsilon_i{}^{jk} X_j \bar{\partial}_k \\
&= q \frac{1}{R} [X \times \bar{\partial}]_i \\
L_i \frac{1}{R} &= \mu q^{-1} [X \times \bar{\partial}]_i \frac{1}{R} \\
&= \mu q \frac{1}{R} q^{-1} [X \times \bar{\partial}]_i
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{R} \mu q^{-1} [X \times \bar{\partial}]_i \\
&= \frac{1}{R} L_i
\end{aligned}$$

B q -deformed Rayleigh formulas

Let D be the symmetric q -derivative:

$$Df(x) \equiv \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x} \quad (31)$$

Acting on monomials,

$$Dx^n = [n]_q x^{n-1} \quad (32)$$

Let us define the following q -generalizations of some common functions by replacing factorials with q -deformed factorials in their Taylor expansions:

$$\exp_q(x) \equiv \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} \quad (33)$$

$$\cos_q(x) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{[2n]_q!} \quad (34)$$

$$\sin_q(x) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{[2n+1]_q!}. \quad (35)$$

Then $\exp_q(ix) = \cos_q(x) + i \sin_q(x)$

The q -deformed Rayleigh formulas,

$$j_{[q]l}(x) = (-x)^l \left(\frac{1}{x} D \right)^l \left(\frac{\sin_q(x)}{x} \right) \quad (36)$$

$$n_{[q]l}(x) = (-x)^l \left(\frac{1}{x} D \right)^l \left(\frac{-\cos_q(x)}{x} \right), \quad (37)$$

can then be proved by induction after noting that

$$j_{[q]0}(x) = \frac{\sin_q(x)}{x} \quad (38)$$

and

$$n_{[q]0}(x) = -\frac{\cos_q(x)}{x}. \quad (39)$$

The q -deformed spherical Hankel functions are

$$h_{[q]l}^{(1)}(x) \equiv j_{[q]l}(x) + in_{[q]l}(x) \quad (40)$$

and $h_{[q]l}^{(2)}(x)$ is just the complex conjugate of $h_{[q]l}^{(1)}(x)$. Since $(-x)^l (\frac{1}{x}D)^l$ is a linear operator, it follows that $h_{[q]l}^{(1)}$ should satisfy

$$h_{[q]l}^{(1)}(x) = (-x)^l \left(\frac{1}{x}D\right)^l \left(\frac{-i \exp_q(ix)}{x}\right).$$

One can show by induction that

$$h_{[q]l}^{(1)}(x) = \sum_{n=0}^l q^{\frac{1}{2}[l(l+1)-n(n+1)]} \frac{i^{n-l-1} [l+n]_q! \exp_q(iq^n x)}{[2n]_q! [l-n]_q! x^{n+1}} \quad (41)$$

satisfies this Rayleigh formula for all l . Thus these functions must equal the Hankel functions.

The powers of q that appear both inside and outside the q -exponential arise because

$$\begin{aligned} D(x^k \exp_q(\alpha x)) &= [k]_q x^{k-1} \exp_q(\alpha q x) + \alpha q^{-k} x^k \exp_q(\alpha x) \\ &= [-k]_q x^{k-1} \exp_q(\alpha \frac{1}{q} x) + \alpha q^k x^k \exp_q(\alpha x). \end{aligned}$$

This results from the q -deformed arithmetic in which

$$[n+k]_q = q^n [k]_q + q^{-k} [n]_q = q^{-n} [k]_q + q^k [n]_q \quad (42)$$

It appears to be a common trend of solutions to self-adjoint q -deformed Hamiltonians that they can be expressed as series of the form $\sum_n A_n \exp_q(q^n x)$.

C q -Coulomb Hamiltonian Eigenvalue Problem

Let $\beta = \sqrt{-E}$ and $[\gamma]_q = \frac{\alpha}{\beta}$. In order to obtain a difference equation for the A_p , we need to express $(H + \beta^2)\psi$ as a series of the form $\sum_p B_p \exp_q(q^p x)$ where B_p is a function of the A_p .

If we simply apply $(H + \beta^2)$ to ψ using Equation 13, we get

$$\begin{aligned} (H + \beta^2) R^{l+p} \exp_q(-q^s \beta R) Y_q^l{}_m = \\ - \sum_{n=0}^{\infty} \frac{[n+p+2l+1]_q [n+p]_q}{[n]_q!} (-q^s \beta)^n R^{n+p+l-2} Y_q^l{}_m \\ - \alpha \sum_{n=0}^{\infty} \frac{(q^{n+l+p-1} + q^{-n-l-p-1})}{[n]_q!} (-q^s \beta)^n R^{n+p+l-1} Y_q^l{}_m \\ + \beta^2 R^{l+p} \exp_q(-q^s \beta R) Y_q^l{}_m. \end{aligned}$$

If we decompose $[n+p+2l+1]_q [n+p]_q$, we get terms proportional to $[n]_q [n-1]_q$, $q^{-n} [n]_q$, and q^{-2n} . By resumming these terms, we can rewrite

them as powers of R times exponentials. We wish to rewrite the entire equation in terms of functions of the form $R^{l+p-m} \exp_q(-q^{s+m}\beta R) Y_q^l$ where m is an integer and s is a function of p . Then we can obtain a difference equation for the coefficients of these functions. In the following, we assume that $s = -l-1-p+\gamma$. The only terms which need to be rewritten are then

$$\begin{aligned}
& (K^2 P^2 - \alpha \frac{1}{R} q^{-1} \mu^{-1}) R^{l+p} \exp_q(-q^{-l-1-p+\gamma} \beta R) Y_q^l = \\
& - \sum_{n=0}^{\infty} \frac{[n+2l+p+1]_q [n+p]_q}{[n]_q!} (q^{l+1+p-\gamma} \beta)^n R^{n+l+p-2} Y_q^l \\
& \quad - [\gamma]_q \beta \sum_{n=0}^{\infty} \frac{q^{n+l+p+1}}{[n]_q!} (-q^{l+1+p-\gamma} \beta)^n R^{n+l+p-1} Y_q^l \\
& = - \sum_{n=0}^{\infty} ([n+2l+p+1]_q [n+p]_q - q^{-1+n+\gamma} [n]_q [\gamma]_q) \\
& \quad \times \frac{(-q^{l+1+p-\gamma} \beta)^n}{[n]_q!} R^{n+l+p-2} Y_q^l \\
& = - \sum_{n=0}^{\infty} \{ (q^{2p+2l+2} + 1 - q^{2\gamma}) [n]_q [n-1]_q + q^{-2n} [p+2l+1]_q [p]_q \\
& \quad + q^{-n} [n]_q (-q^{1+\gamma} [\gamma]_q q^{p+2l+1} [p]_q + q^{1+p} [2l+2+p]_q) + q^{-2n} [p+2l+1]_q [p]_q \} \\
& \quad \times \frac{(-q^{l+1+p-\gamma} \beta)^n}{[n]_q!} R^{n+l+p-2} Y_q^l \\
& = \beta^2 (q^{2l+2+2p} - q^{2l+2+2p-2\gamma} - q^{4l+4p+4-2\gamma}) R^{l+p} \exp_q(-q^{-l-1-p+\gamma} \beta R) Y_q^l \\
& \quad - \beta (q^{l+p+1} [\gamma]_q - q^{2p+3l+1-\gamma} [p]_q - q^{l+2p+1-\gamma} [2l+p+2]_q) \\
& \quad \times R^{l+p-1} \exp_q(-q^{-l-p+\gamma} \beta R) Y_q^l \\
& \quad - [p+2l+1]_q [p]_q R^{l+p-2} \exp_q(-q^{-l-p+1+\gamma} \beta R) Y_q^l
\end{aligned}$$

Thus

$$\begin{aligned}
& (H + \beta^2) R^{l+p} \exp_q(-q^{-l-p-1+\gamma} \beta R) Y_q^l \\
& = q^{2l+2p+2-\gamma} [2]_{q^{-l-p-1}} [l+p+1-\gamma]_q (q^{-1} - q) \\
& \quad \times \beta^2 R^{l+p} \exp_q(-q^{-l-1-p+\gamma} \beta R) Y_q^l \\
& + (q^{2p+3l+1-\gamma} [p]_q + q^{l+2p+1-\gamma} [2l+p+2]_q - [2]_{q^{l+p+1}} [\gamma]_q) \\
& \quad \times \beta R^{l+p-1} \exp_q(-q^{-l-p+\gamma} \beta R) Y_q^l \\
& - [p+2l+1]_q [p]_q R^{l+p-2} \exp_q(-q^{-l-p+1+\gamma} \beta R) Y_q^l
\end{aligned}$$

If we sum over p from 0 to ∞ , we can obtain the desired difference equation. We normalize the wave functions so that $A_0 = 1$. Then

$$A_1 = q^{l+1} \beta \frac{[2]_{q^{l+1}} [l+1-\gamma]_q}{[2l+2]_q}$$

and for $p > 0$,

$$\begin{aligned} & \{[p+2l+3]_q [p+2]_q\} A_{p+2} \\ &= q^{2l+2p+2-\gamma} [2]_{q^{l+p+1}} [l+p+1-\gamma]_q (q^{-1} - q) \beta^2 A_p \\ &+ (q^{2p+3l+3-\gamma} [p+1]_q + q^{l+2p+3-\gamma} [2l+p+3]_q - [2]_{q^{-l-p-2}} [\gamma]_q) \beta A_{p+1} \end{aligned}$$

Equation 24 is then a solution to this difference equation. Thus the theorem is proved.

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