

# N=2 TYPE II – HETEROTIC DUALITY AND HIGHER DERIVATIVE F-TERMS\*

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## Abstract

We test the recently conjectured duality between  $N=2$  supersymmetric type II and heterotic string models by analysing a class of higher dimensional interactions in the respective low-energy Lagrangians. These are  $F$ -terms of the form  $F_g W^{2g}$  where  $W$  is the gravitational superfield. On the type II side these terms are generated at the  $g$ -loop level and in fact are given by topological partition functions of the twisted Calabi-Yau sigma model. We show that on the heterotic side these terms arise at the one-loop level. We study in detail a rank 3 example and show that the corresponding couplings  $F_g$  satisfy the same holomorphic anomaly equations as in the type II case. Moreover we study the leading singularities of  $F_g$ 's on the heterotic side, near the enhanced symmetry point and show that they are universal poles of order  $2g-2$  with coefficients that are given by the Euler number of the moduli space of genus- $g$  Riemann surfaces. This confirms a recent conjecture that the physics near conifold singularity is governed by  $c=1$  string theory at the self-dual point.

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## 1. Introduction

In the last months considerable progress has been achieved in the non-perturbative understanding of string theories with  $N = 2$  space-time supersymmetry. In particular the idea of type II- heterotic string duality has been extended to the  $N = 2$  context and some explicit examples of dual pairs have been proposed by Kachru and Vafa [1]. This gives rise to an exact prepotential for the vector multiplets on the heterotic string side, therefore extending the field theory results of Seiberg and Witten [2] to the string case. An important aspect of this duality is that the dilaton of one model is mapped to an ordinary  $t$ -modulus associated with the compactification of the second model. Moreover, the dilaton in type II belongs to a hypermultiplet while the dilaton in heterotic string belongs to a vector multiplet. Using the fact that vector multiplets and neutral hypermultiplets do not couple to each other, this duality provides a very powerful method for extracting non-perturbative physics of one model from the perturbative computations in the dual model and vice versa.

Let us start by reviewing the main features of  $N = 2$  type II and heterotic strings. The type II string is compactified on a Calabi-Yau threefold<sup>1</sup> which is characterised by the two Betti numbers  $h_{11}$  and  $h_{12}$ . In the type IIA model,  $h_{11}$  gives the number of vector multiplets. Together with the graviphoton the rank of the gauge group is  $r = h_{11} + 1$ . The number of hypermultiplets is  $h_{12} + 1$  where the extra 1 is accounted for by the dilaton. At the perturbative level the gauge group is abelian  $U(1)^r$  and there are no charged matter fields. Since the dilaton belongs to a hypermultiplet, the tree level prepotential is exact at the full quantum level. Moreover this tree level prepotential can be computed exactly, *i.e.* including the world-sheet instanton corrections, by using the mirror symmetry [3, 4, 5]. A generic feature of the prepotential is that it has logarithmic singularities near the conifold locus in the moduli space of the Calabi-Yau threefold [6]. The appearance of this logarithmic singularity at the tree level remained a puzzle for some time. However

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<sup>1</sup>We shall be considering here only (2,2) symmetric compactifications for type II.

recently Strominger [7] proposed a resolution of the singularity as due to the appearance of massless hypermultiplets, corresponding to charged black holes, at the conifold locus. The logarithmic singularity in the prepotential is then understood as a one-loop effect involving this massless black hole in the internal line. For this proposal to work it is crucial that the dilaton, whose expectation value provides the loop expansion parameter, does not couple to the vector multiplets at the two-derivative level.

The  $N = 2$  heterotic string is compactified on  $T^2 \times K_3$  with different possible embeddings of the spin connection in the gauge group, giving rise to different four-dimensional models. The moduli associated with this compactification again split into two classes: the ones in vector multiplets and those in hypermultiplets. Let us denote their numbers by  $n_V$  and  $n_H$  respectively. Contrary to the type II case, the dilaton  $S$  in heterotic string belongs to a vector multiplet. Thus the total rank of the gauge group including the graviphoton becomes  $(n_V + 2)$ . The  $n_V$  moduli in the vector multiplets belong to the coset  $O(2, n_V)/O(2) \times O(n_V)$  modulo discrete identifications that define the duality group. At the classical level this duality group is  $O(2, n_V; \mathbb{Z})$ . At the generic points in the moduli space the gauge group is abelian  $U(1)^{n_V+2}$  and there are no charged massless states. However there are complex codimension 1 surfaces where one of the  $U(1)$ 's is enhanced to  $SU(2)$ , due to the appearance of two extra charged massless vector multiplets. As a result, the perturbative correction to the prepotential, which due to the  $N = 2$  non-renormalization theorem occurs only at the one-loop level, develops a logarithmic singularity near these surfaces [8, 9]. As a result the classical duality group  $O(2, n_V; \mathbb{Z})$  gets modified at the perturbative level [8]. At the full non-perturbative level, from the analysis of Seiberg and Witten [2] in the rigid case, this enhanced symmetry locus is expected to split into several branches where non-perturbative states corresponding to dyonic hypermultiplets become massless. Thus in the full moduli space including the dilaton  $S$ , the singular locus should split into several branches which collapse only in the limit  $S \rightarrow \infty$ .

The candidate dual pairs must of course have the same number of vector and hypermultiplets. This means that  $h_{11} = n_V + 1$  and  $h_{12} = n_H - 1$ . Moreover the singularity structure for the prepotentials discussed above must also be compatible with each other. Since the dilaton  $S$  of heterotic is mapped to an ordinary (1,1) modulus, say  $t$ , of type II, this implies that first of all in the limit  $t \rightarrow \infty$  the type II prepotential must go to the perturbative prepotential of the heterotic theory including the right singularity at the enhanced symmetry locus of the latter. Moreover at finite values of  $t$ , the conifold singularity structure must agree with what one expects from Seiberg-Witten analysis. In other words different branches of conifold singularity must come together as  $t \rightarrow \infty$ .

The examples of dual pairs proposed by Kachru and Vafa satisfy the above qualitative requirements. Moreover for some examples involving rank 3 and 4, more quantitative checks have been made and it has been shown that the prepotential in the type II theory in the limit  $S \rightarrow \infty$  agrees with the perturbative prepotential in the corresponding heterotic model [10, 11, 12]. These checks indicate that at least at the two derivative level, the quantum effective actions of the vector multiplets for these models are equivalent. However to show that the two string theories are equivalent, one must go beyond the two derivative terms. In particular as shown in [13, 14], there is a class of higher derivative  $F$ -terms of the form  $F_g W^{2g}$ , where  $W$  is the  $N = 2$  gravitational superfield, which are generated at genus  $g$  in the type II theory. These terms again should not receive further quantum corrections even at the non-perturbative level due to the fact that the type II dilaton is in the hypermultiplet [14, 15]. The couplings  $F_g$ 's were shown to be proportional genus  $g$  topological partition functions of the twisted Calabi-Yau sigma model, and satisfy certain recursion relations expressing the holomorphic anomaly of genus  $g$  partition function to that of lower genera [13]. If the duality is true at the string level and not just at the level of low energy effective action then such terms must also be present in the heterotic string theory. For genus 1, this coupling corresponds to  $R^2$  term in the effective action,  $R$  being the Riemann tensor, and this has already been investigated previously in the heterotic

string [16] and the anomaly equation satisfied by this coupling has been derived. For rank 3 case it was shown in ref.[10] that  $F_1$ 's of the type II and heterotic models agree in the weak coupling limit.

Our aim here is to analyse the whole sequence of  $F_g$ 's for all  $g$ . It turns out that in heterotic string the terms  $F_g W^{2g}$  already appear at one-loop level. In this paper, we compute these terms at the one-loop level for the rank 3 case, and derive the holomorphic anomaly equations they satisfy. We show that these anomaly equations are the same as the ones in type II theory in the large  $S$  limit. Moreover we analyse the leading singularity structure of  $F_g$ 's near the enhanced symmetry point and show that the order of singularity matches with that in type II. We also compute the coefficient of the leading singularity of  $F_g$  which gets contribution in the degeneration limit of the world-sheet torus. Therefore this coefficient can be obtained by calculating a one-loop diagram in effective field theory with the extra massless states in the internal line. It turns out that this coefficient is the Euler number of the moduli space of genus- $g$  Riemann surfaces, which is exactly the coefficient of the  $\mu^{2-2g}$  term in the expansion of the free energy of  $c = 1$  string theory at the self-dual radius,  $\mu$  being the cosmological constant. Although we do the explicit calculation for the rank 3 case, it will be apparent that this result is in fact universal. One would like to compare these coefficients with the ones in the type II theory, where however one has explicit results only up to  $g = 2$  (and for the quintic) which agree with our results for heterotic string. On the other hand, Ghoshal and Vafa [17] have recently argued that the physics near conifold singularity is described precisely by the  $c = 1$  string theory at the self-dual radius. If this turns out to be confirmed by further calculations on the type II side, then our result would provide a strong evidence in favour of the type II – heterotic duality. Furthermore, the fact that these singularities on heterotic side arise from a one-loop diagram with the would-be massless state in the internal line, provides a strong evidence in favour of Strominger's proposal for the resolution of conifold singularity. In other words in type II the leading singularity in  $F_g$  should arise from a one loop diagram

involving the would-be massless charged black-hole in the internal line.

This paper is organized as follows. In section 2 we discuss the  $N = 2$  effective field theory and show that while the couplings  $F_g W^{2g}$  appear in type II at  $g$ -loop level, in heterotic string they appear at one-loop level. In section 3, we discuss the perturbative prepotential for heterotic string for rank 3 case and compare with the type II. We also discuss the type II holomorphic anomaly equations for  $F_g$ 's in the limit  $S \rightarrow \infty$ . In section 4, we compute the  $F_g$ 's for heterotic string. The generating function for the  $F_g$ 's can be expressed in a compact way as an integral over the moduli space of the world-sheet torus. In section 5, we derive the holomorphic anomaly equations for  $F_g$ ' for heterotic string and show that they are identical to the ones in type II in large  $S$  limit. We also compute the leading singularity near the enhanced symmetry point and show that it gives the Euler number of the moduli space of genus- $g$  Riemann surfaces. Section 6 is devoted to some concluding remarks. Finally in the Appendix we derive a generating function needed for the computations in section 4.

## 2. Effective field theory

The couplings in the effective field theory of type II strings which reproduce the topological partition function  $F_g$  were studied in ref.[14]. It was shown that they correspond to the chiral  $N = 2$  Lagrangian terms

$$I_g = \tilde{F}_g(X) W^{2g}, \quad (2.1)$$

where  $W$  is the (weight 1) Weyl superfield <sup>2</sup>,

$$W_{\mu\nu}^{ij} = F_{\mu\nu}^{ij} - R_{\mu\nu\lambda\rho} \theta^i \sigma_{\lambda\rho} \theta^j + \dots, \quad (2.2)$$

which is anti-self-dual in its Lorentz indices and antisymmetric in the indices  $i, j$  labeling the two supersymmetries;  $W^2 \equiv \epsilon_{ij} \epsilon_{kl} W_{\mu\nu}^{ij} W_{\mu\nu}^{kl}$ .  $R_{\mu\nu\lambda\rho}$  is the anti-self-dual Riemann tensor,

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<sup>2</sup>For a general discussion of  $N = 2$  supergravity see ref.[18].

while  $F_{\mu\nu}^{ij}$  is the (anti-self-dual) graviphoton field strength defined by the supersymmetry transformation of the gravitinos:  $\delta\psi_\mu^i = -\frac{1}{4}\sigma^{\lambda\rho}F_{\lambda\rho}^{ij}\sigma_\mu\bar{\epsilon}_j + \dots$ . Finally  $\tilde{F}_g(X)$  is an analytic function of the  $N = 2$  chiral superfields  $X^I$  (of Weyl weight 1):

$$X^I = X^I + \frac{1}{2}\hat{F}_{\lambda\rho}^I\epsilon_{ij}\theta^i\sigma_{\lambda\rho}\theta^j + \dots, \quad (2.3)$$

where  $\hat{F}_{\lambda\rho}^I$  are the anti-self-dual vector boson field strengths. The scalar component of  $X^0$  corresponds to a constrained field; the unconstrained physical scalars of vector multiplets – the moduli – are parametrized by  $Z^A \equiv X^A/X^0$ . By fixing the superconformal gauge, the scalar component of  $X^0$  can be expressed in terms of the moduli Kähler potential  $K(Z, \bar{Z})$  according to the normalization choice of the coefficient of the Einstein kinetic term  $R$ . In the  $\sigma$ -model representation, this coefficient is set to  $1/g_s^2$ , where  $g_s$  is the four-dimensional string coupling constant, implying that

$$X^0 = \frac{1}{g_s}e^{K/2} \quad (2.4)$$

Since any Lagrangian term in conformal supergravity has Weyl weight 2, it follows that  $\tilde{F}_g(X)$  in eq.(2.1) is a homogenous function of  $X^I$ 's of degree  $2-2g$ . Its lowest component can then be written as

$$\tilde{F}_g(X) = (X^0)^{2-2g}\tilde{F}_g(Z) = (g_s^2)^{g-1}e^{(1-g)K}\tilde{F}_g(Z). \quad (2.5)$$

Note that  $\tilde{F}_0$  coincides with the  $N = 2$  prepotential  $F$  while  $\tilde{F}_1$  gives the gravitational four-derivative  $R^2$ -type couplings. In ref[14] it was shown that  $\tilde{F}_g$  is of the form:

$$\tilde{F}_g = \alpha\beta^{g-1}F_g \quad (2.6)$$

where  $F_g$  is the topological partition function of the twisted Calabi-Yau sigma model and  $\alpha$  and  $\beta$  are some undetermined moduli and  $g$  independent constants. Although  $F_g$ 's are expected to be analytic functions of the moduli, they become non analytic (for  $g \geq 1$ ) due to the holomorphic anomaly in the BRS current of the topological theory [13]. In the context of the effective supergravity, the holomorphic anomaly is a consequence of

propagation of massless particles which lead to non-locality in the gravitational sector of the effective action [14]. This is to be contrasted with the gauge sector which is local at generic values of the moduli space where all non abelian gauge symmetries are broken to  $U(1)$  factors and there are no massless charged hypermultiplets. The moduli dependence of  $F_g(Z, \bar{Z})$  is governed by the following equation (for  $g \geq 2$ ):

$$\bar{\partial}_{\bar{A}} F_g = \frac{1}{2} \bar{C}_{\bar{A}\bar{B}\bar{C}} e^{2K} G^{B\bar{B}} G^{C\bar{C}} \left( D_B D_C F_{g-1} + \sum_r D_B F_r \cdot D_C F_{g-r} \right), \quad (2.7)$$

where  $D$  are Kähler covariant derivatives, and the Yukawa couplings  $C_{ABC} \equiv F_{ABC}$ ,  $F$  being the tree level prepotential. For  $g = 1$ , the equation which governs the moduli dependence of  $F_1$  is:

$$\begin{aligned} \bar{\partial}_{\bar{A}} \partial_A F_1 &= -\frac{\chi}{24} G_{A\bar{A}} + \frac{1}{2} C_{ABC} \bar{C}_{\bar{A}\bar{B}\bar{C}} e^{2K} G^{B\bar{B}} G^{C\bar{C}} \\ &= \frac{1}{2} \left( 3 + h_{11} - \frac{\chi}{12} \right) G_{A\bar{A}} - \frac{1}{2} R_{A\bar{A}} \end{aligned} \quad (2.8)$$

where  $\chi$  is the Euler number of the Calabi-Yau manifold. In the second step we have used the special geometry relation and  $R_{A\bar{A}}$  is the Ricci tensor.

To count string loops, we use the fact that  $N = 2$  conformal supergravity forbids a dependence of  $F_g$ 's on matter hypermultiplets, generalizing the known result on the absence of mixing between vector multiplets and hypermultiplets to the case of higher weight interactions. Next, we note that in type II strings the Kähler potential  $K$  is independent of the dilaton since the latter belongs to a hypermultiplet. Equation (2.5) then implies that the term  $I_g$  in (2.1) can appear only at genus  $g$ . Its highest component contains a genus  $g$  amplitude of two gravitons and  $2g - 2$  graviphotons which was studied in ref.[14] in order to identify  $F_g$  with the topological partition function of Bershadsky et al. [13]. These arguments extend the non-renormalization theorem for the  $N = 2$  prepotential  $F_0$  to all  $F_g$ 's [15]. Therefore, by analogy to the reasoning of ref.[7], we expect that in type II strings,  $F_g$ 's are determined at genus  $g$  and should not receive any further perturbative or non-perturbative corrections.



On the other hand, heterotic – type II string-string duality implies that  $F_g$ 's should also appear in heterotic  $N = 2$  compactifications. In this case, however, the dilaton belongs to a vector multiplet and  $N = 2$  supergravity does not forbid a dependence of  $F_g$ 's on the dilaton. Fortunately, in the weak coupling limit, Peccei-Quinn symmetry of the axion allows at most a linear dilaton dependence for the prepotential  $F_0$  and the  $R^2$  coupling  $F_1$ , for which a constant axion shift gives total divergences. Moreover, the Kähler potential  $K$  contains a  $\ln g_s^2$  term. From eq.(2.4), one now has that  $X^0$  is of order 1 in the string coupling and thus, from eq.(2.5), all  $F_g$ 's should appear at the one loop, with the exception of  $F_0$  and  $F_1$  which have also tree level contributions. In conclusion, for  $N = 2$  compactifications which have dual realizations as type II and heterotic string theories, the one loop corrections to the effective gravitational couplings  $F_g$  (2.1) in the heterotic theory should agree with the corresponding genus  $g$  couplings in the dual type II theory to the order  $(S - \bar{S})^0$ .

### 3. Perturbative prepotential for the rank 3 case

One of the simplest type II – heterotic dual pairs proposed by Kachru and Vafa is for rank 3 case. The type II model is compactified on the Calabi-Yau threefold  $X_{12}(1, 1, 2, 2, 6)$  with Betti numbers  $h_{1,1} = 2$  and  $h_{1,2} = 128$ . Thus the number of vector multiplets is 2 and that of hypermultiplets including the dilaton is  $128 + 1$ . The classical prepotential as a function of the special coordinates  $t_1$  and  $t_2$  of the moduli space of vector multiplets has been studied in ref.[4]. They find the following expression for the Yukawa couplings

$$F_{\alpha\beta\gamma} = F_{\alpha\beta\gamma}^0 + \sum_{0 \leq j, k \in \mathbf{Z}} \frac{c_{\alpha\beta\gamma} n_{jk} q_1^j q_2^k}{1 - q_1^j q_2^k}, \quad (3.1)$$

where  $q_\alpha = \exp(2\pi i t_\alpha)$  and  $F_{\alpha\beta\gamma}^0$  are the interserction numbers with  $F_{111}^0 = 4$ ,  $F_{112}^0 = 2$  and  $F_{122}^0 = F_{222}^0 = 0$ . The coefficients  $c_{111} = j^3$ ,  $c_{112} = j^2 k$ ,  $c_{122} = j k^2$ ,  $c_{222} = k^3$  and  $n_{jk}$  are the instanton numbers, the first few of which have been explicitly given in ref.[4]. Note that  $t_1$  and  $t_2$  are special coordinates so that  $F_{\alpha\beta\gamma}$  are just the appropriate derivatives with

respect to  $t_1$  and  $t_2$  of the prepotential  $F$ . Due to the fact that the dilaton in type II theory belongs to a hypermultiplet, this prepotential is exact and does not receive any quantum correction even at the non-perturbative level.

The dual to the above type II model proposed in ref.[1] is an  $N = 2$  heterotic model with rank 3. The scalar components of the vector multiplets are the dilaton  $S$  and a modulus  $T$  which belongs to the coset  $O(2, 1)/O(2)$ . We use the normalization such that  $\langle S \rangle = \frac{\theta}{\pi} + i\frac{8\pi}{g_s^2}$ . The classical duality symmetry acting on  $T$  is  $O(2, 1; Z) \equiv Sl(2, Z)$ . At generic point in the  $T$ -moduli space, the gauge group is abelian, namely  $U(1)^3$  including the vector partner of dilaton and the graviphoton. However at  $T = i \pmod{Sl(2, Z)}$  two extra vector multiplets become massless, giving rise to an enhanced gauge group  $U(1)^2 \times SU(2)$ . The prepotential for this model is given by

$$F = \frac{1}{2}ST^2 + f(T) + \dots \quad , \quad (3.2)$$

where  $f(T)$  is the one loop correction to the classical prepotential  $\frac{1}{2}ST^2$ , and due to  $N = 2$  non-renormalization theorems is actually the complete perturbative correction. The dots refer to non-perturbative contributions, which are suppressed exponentially as  $e^{2\pi i S}$ . Due to the appearance of extra charged massless states at  $T = i$ , we expect a logarithmic singularity in the one-loop contribution  $f$  to the prepotential. One can construct the Kähler potential starting from the prepotential  $F$  up to order  $1/(S - \bar{S})$  and the result is

$$K = -\ln\left[\frac{i}{2}(S - \bar{S})(T - \bar{T})^2\right] + \frac{2i}{S - \bar{S}}K^{(1)}(T, \bar{T}) \quad (3.3)$$

where

$$K^{(1)} = \frac{i}{T - \bar{T}}\left(\partial_T - \frac{2}{T - \bar{T}}\right)f + c.c \quad (3.4)$$

The requirement that the  $Sl(2, Z)$  transformations of  $T$  should be Kähler transformations implies that  $f(T)$  transforms with weight  $(-4)$  up to additive terms that are at most quartic in  $T$ . Moreover  $S$  also picks up additive terms that depend on  $f(T)$ . Under the transformation  $T \rightarrow \frac{aT+b}{cT+d}$

$$f \rightarrow (cT + d)^{-4}(f + R)$$

$$S \rightarrow S + 2c \frac{f_T + R_T}{cT + d} - 4c^2 \frac{f + R}{(cT + d)^2} - \frac{1}{3}(R_{TT} - R_2) \quad (3.5)$$

where  $R$  is a polynomial with real coefficients which is at most quartic in  $T$  and  $R_2$  is the coefficient of  $T^2$  in  $R$ . The term involving  $R_2$  in the transformation of  $S$  represents a constant axion shift and has been introduced here for convenience. One can construct the one loop correction to the metric by taking derivatives of  $K^{(1)}$  with the result:

$$K_{T\bar{T}}^{(1)} = \frac{i}{(T - \bar{T})^2} (\partial_T - \frac{2}{T - \bar{T}}) (\partial_T - \frac{4}{T - \bar{T}}) f + c.c. \quad (3.6)$$

From the fact that at  $T = i$  extra charged massless states appear it follows that the one-loop metric must have a singularity of the form  $\ln|T - i|$  for  $T$  close to  $i$ . This in turn implies that  $f$  must behave as  $(T - i)^2 \ln(T - i)$ .

Under  $Sl(2, Z)$  duality transformation the one-loop metric must transform covariantly. Using eq.(3.6) one can show the following identity:

$$(\partial_T + \frac{4}{T - \bar{T}}) (\partial_T + \frac{2}{T - \bar{T}}) \partial_T (T - \bar{T})^2 K_{T\bar{T}}^{(1)} = i \partial_T^5 f. \quad (3.7)$$

The left hand side of the above equation transforms with weight  $(6, 0)$  with respect to  $(T, \bar{T})$ . The right hand side is meromorphic in  $T$  with at most a third order pole at  $T = i$ . Moreover as  $T \rightarrow \infty$  we expect the right hand side to vanish as otherwise it would imply that  $K_{T\bar{T}}^{(1)}$  would grow as  $(T - \bar{T})$ . The most general expression compatible with these requirements is

$$\partial_T^5 f = -\frac{1}{18\pi i} \left\{ \frac{j_T}{j - j(i)} \right\}^3 \left\{ \frac{j(i)}{j} \right\}^2 \left\{ 5 + 13 \frac{j}{j(i)} \right\}, \quad (3.8)$$

where  $j \equiv j(T)$  is the unique  $Sl(2, Z)$  invariant meromorphic function with a first order pole at infinity with residue 1 and a third order zero at  $T = \exp(2\pi i/3)$ . The constants appearing inside the bracket on the right hand side are fixed by  $SU(2)$  beta function which determines the singularity in  $K_{T\bar{T}}^{(1)}$  near  $T = i$  and by the requirement that the monodromy of  $f$  as  $T$  goes around  $i$  be imbeddable in the symplectic group  $Sp(6, Z)$  as dictated by  $N = 2$  supergravity.

To check the duality between type II and the heterotic models, we must compare the prepotentials in the two theories in the weak coupling limit. To do that we have to identify the moduli  $(t_1, t_2)$  appearing in the type II theory with  $(S, T)$  appearing in the heterotic theory. The identification proposed in ref.[1] is  $T = t_1$  and  $S = 2t_2$ . One can indeed verify in the large  $S$  limit (*i.e.* ignoring the terms exponentially small in  $t_2$ ) that the two expressions for the prepotentials agree up to the first few terms in  $q_1$  expansion that have been checked so far [10]. Thus the two models seem to agree at least up to the two derivative terms in the effective action in the vector multiplet sector.

As stated in the introduction our aim here is to establish this equivalence for all higher weight F-term couplings of the type  $F_g W^{2g}$  where  $W$  is the  $N = 2$  gravitational Weyl supermultiplet. In the type II theory  $F_g$ 's have already been related to genus  $g$  topological partition function of the twisted version of the Calabi-Yau sigma model. As described in the last section these  $F_g$ 's satisfy the recursion relations (2.7, 2.8). In order to compare these couplings with the similar ones in heterotic string theory we must again consider the large  $S$  limit, as in the heterotic string theory these quantities will be computable only in perturbation theory. In the large  $S$  limit it is easy to see that  $\exp(2K)$  in eqs.(2.7, 2.8) becomes  $-4(S - \bar{S})^{-2}(T - \bar{T})^{-4}$ . In the leading term in  $(S - \bar{S})$ , the only Yukawa coupling that is relevant in the anomaly equation is  $C_{\bar{T}\bar{T}\bar{S}}$ , which is equal to 1. The inverse metrics that enter the equation are in the leading orders

$$\begin{aligned} G^{T\bar{T}} &= -\frac{1}{2}(T - \bar{T})^2 & G^{S\bar{S}} &= -(S - \bar{S})^2 \\ G^{S\bar{T}} &= -i(T - \bar{T})^2 K_T^{(1)} \end{aligned} \tag{3.9}$$

Finally noting that, in the large  $S$  limit,  $F_g$  for  $g \geq 2$  go to constant in  $S$  while  $F_1$  goes to  $-\pi i(S - \bar{S})$  plus zeroeth order in  $S$ , we find that to the leading order in  $S$  the recursion relation (2.7) becomes

$$\partial_{\bar{T}} F_g = \frac{2\pi i}{(T - \bar{T})^2} (D_T F_{g-1} + 2\pi \delta_{g,2} K_T^{(1)}) \quad g \geq 2 \tag{3.10}$$

For  $g = 1$ , to the leading order in  $S$ , the anomaly equation (2.8) becomes

$$\partial_T \partial_{\bar{T}} F_1 = \frac{-25}{(T - \bar{T})^2} \quad (3.11)$$

where we have used the fact that for the present model the number of hypermultiplets is 129.

Note that taking derivatives of  $F_g$ 's with respect to  $\bar{S}$ , one finds contributions which do not vanish exponentially in the large  $S$  limit but they fall off as powers. This implies that in the heterotic theory  $F_g$ 's receive in general higher loop corrections which deserve further study. In this work, we restrict ourselves to the leading weak coupling limit of  $F_g$ 's. However, when making the comparison between type II and heterotic theories, there is a related subtlety which arises from the fact that the natural string basis for the dilaton in heterotic theory corresponds to a linear multiplet  $L$ , while in the dual type II model it is associated to a chiral multiplet  $S$ . The relation between the two fields is [19]

$$\frac{1}{L} = \text{Im}S - K^{(1)} \quad (3.12)$$

Thus, changing variables from  $T$  and  $S$  to  $T$  and  $L$ , one finds that the partial derivatives with respect to  $T$  in the two cases are related as

$$\partial_T|_L = \partial_T|_S + K_T^{(1)} \partial_{\text{Im}S} \quad (3.13)$$

In the following when we derive holomorphic anomaly equation for  $F_g$ 's in heterotic string, the partial derivative with respect to  $T$  will be for fixed  $L$  and therefore to compare with type II equations which are for fixed  $S$  we need to use the above equation.

#### 4. Computation of $W^{2g}$ couplings in Heterotic String

As we argued in section 2, on the basis of non-renormalization properties of type II strings, if type II- heterotic duality is correct, the expression for  $F_g$ , which is purely the

result of a  $g$ -loop computation on the type II side, has to agree, in the weak coupling limit  $S \rightarrow \infty$ , with the result of the perturbative computation on the heterotic side. In particular the zeroth order term in  $(S - \bar{S})$  on the type II side should agree with the one loop results on the heterotic side. As mentioned in section 2, the precise identification of the coupling  $\tilde{F}_g$  and the topological partition function  $F_g$  involved some undetermined constants  $\alpha$  and  $\beta$  in eq.(2.6), therefore in the heterotic string amplitude corresponding to  $F_g$  which we shall compute below, we will not be careful about such constants appearing in various steps. At the end however we will normalize the amplitudes by demanding that the coefficient appearing in the recursion relation on the heterotic side be identical to the one appearing in eqs.(3.10) and (3.11). In the following, therefore, we shall also drop the distinction between  $\tilde{F}_g$  and  $F_g$ .

Consider the amplitude involving two gravitons and  $(2g-2)$  graviphotons. The relevant terms in the action are obtained by expanding  $F_g W^{2g}$  in terms of component fields with the result:

$$S_{eff} = gF_g(R^2)(F^2)^g + 2g(g-1)F_g(RF)^2(F^2)^{g-2} \quad (4.1)$$

where  $R^2 = R_{abcd}R^{abcd}$ ,  $F^2 = F_{ab}F^{ab}$  and  $(RF)^2 = (R_{abcd}F^{cd})(R^{abef}F_{ef})$ . As mentioned previously, it is understood that  $R_{abcd}$  and  $F_{ab}$  represent the anti-self-dual parts of the Riemann tensor and graviphoton field strengths respectively.

The vertices for the anti-self-dual parts of these fields are more easily expressed by going to a complex basis for the four dimensional Euclidean space time. Let us define

$$Z^1 = (X^1 - iX^2)/\sqrt{2} \quad Z^2 = (X^0 - iX^3)/\sqrt{2}, \quad (4.2)$$

and similarly for their left moving fermionic partners

$$\chi^1 = (\psi^1 - i\psi^2)/\sqrt{2} \quad \chi^2 = (\psi^0 - i\psi^3)/\sqrt{2}, \quad (4.3)$$

Then using the results in Appendix of ref.[14] it is easy to see that the following vertices

describe the self-dual parts of Riemann tensor:

$$\begin{aligned} V_h(p_1) &= (\partial Z^2 - ip_1 \chi^1 \chi^2) \bar{\partial} Z^2 e^{ip_1 Z^1} \\ V_h(\bar{p}_2) &= (\partial \bar{Z}^1 - i\bar{p}_2 \bar{\chi}^2 \bar{\chi}^1) \bar{\partial} \bar{Z}^1 e^{i\bar{p}_2 \bar{Z}^2} \end{aligned} \quad (4.4)$$

Here we have chosen convenient kinematics with  $p_1 \neq 0$ ,  $p_2 = \bar{p}_1 = \bar{p}_2 = 0$  for the first vertex and  $\bar{p}_2 \neq 0$ ,  $p_1 = p_2 = \bar{p}_1 = 0$  for the second (as usual we are treating  $p$  and  $\bar{p}$  as independent parameters).

By applying  $N = 2$  space-time supersymmetry transformations twice one can construct the vertices for the graviphotons in the same kinematic configurations. In the zero ghost picture these are

$$\begin{aligned} V_F(p_1) &= (\partial X - ip_1 \chi^1 \Psi) \bar{\partial} Z^2 e^{ip_1 Z^1} \\ V_F(\bar{p}_2) &= (\partial X - i\bar{p}_2 \bar{\chi}^2 \Psi) \bar{\partial} \bar{Z}^1 e^{i\bar{p}_2 \bar{Z}^2} \end{aligned} \quad (4.5)$$

where  $X$  is the complex coordinate of the left-moving torus  $T^2$  and  $\Psi$  is its fermionic partner with  $U(1)$  charge  $+1$ .

Consider now an amplitude  $A_g$  involving one graviton and  $(g - 1)$  graviphotons with  $p_1 \neq 0$ ,  $p_2 = \bar{p}_1 = \bar{p}_2 = 0$  and the remaining graviton and  $(g - 1)$  graviphotons with  $\bar{p}_2 \neq 0$ ,  $p_1 = p_2 = \bar{p}_1 = 0$ . This amplitude gets contribution from both the terms in eq.(4.1) and it is easy to show that

$$\begin{aligned} A_g &= \langle V_h(p_1) V_h(\bar{p}_2) \prod_{i=1}^{g-1} V_F(p_1^{(i)}) \prod_{j=1}^{g-1} V_F(\bar{p}_2^{(j)}) \rangle \\ &= (p_1)^2 (\bar{p}_2)^2 \prod_{i,j=1}^{g-1} p_1^{(i)} \bar{p}_2^{(j)} (g!)^2 F_g. \end{aligned} \quad (4.6)$$

In general this amplitude receives contribution from all the spin structures and one must sum over all the spin structures weighted by a factor half associated to GSO projection. However one can show that the sum over even spin structures gives the same contribution

as the odd one. Thus the full amplitude can be evaluated in the odd spin structure without the factor of half.

In the odd spin structure one of the vertex operators must be inserted in  $(-1)$ -ghost picture due to the presence of a Killing spinor on the world sheet torus, and one must also insert a picture changing operator to take care of the world-sheet gravitino zero mode. It is convenient to take one of the graviphoton vertices in the  $(-1)$ -ghost picture which comes with a fermion  $\Psi$ . Recalling that in the odd-spin structure the space-time fermions  $\chi^i$  and  $\bar{\chi}^i$ , as well as the internal fermions  $\Psi$  and  $\bar{\Psi}$  associated with the left-moving torus  $T^2$  have one zero-mode each, one concludes that the only contribution comes from the term  $e^{\phi\bar{\Psi}}\partial X$  of the picture changing operator. Moreover the space-time fermion zero modes are soaked by the fermionic part of the graviton vertices. From the remaining  $(2g - 3)$  graviphoton vertices in the  $(0)$ -ghost picture only the terms involving  $\partial X$  survive. Together with the  $\partial X$  appearing in the picture changing operator they provide a total of  $(2g - 2)$   $\partial X$ 's which contribute only through their zero modes.

Finally we are left with the correlation functions of space-time bosons. First thing to note is that  $\bar{\partial}Z^2$ 's and  $\bar{\partial}\bar{Z}^1$ 's appearing in the vertex operators cannot contract with each other. The same observation holds for the mutual contractions between  $e^{ip_1Z^1}$ 's and  $e^{i\bar{p}_2\bar{Z}^2}$ 's. Thus  $\bar{\partial}Z^2$ 's and  $\bar{\partial}\bar{Z}^1$ 's must contract with  $e^{i\bar{p}_2\bar{Z}^2}$ 's and  $e^{ip_1Z^1}$ 's, respectively, bringing down the appropriate powers of momenta. Moreover since the correlator  $\langle\bar{\partial}Z\bar{Z}\rangle$  is total derivative, in order to get non-vanishing result each  $e^{i\bar{p}_2\bar{Z}^2}$  must contract with some  $\bar{\partial}Z^2$  and the same holds for  $e^{ip_1Z^1}$ 's. Thus the momentum structure of this amplitude matches with the eq.(4.6) and  $F_g$  is given by the following expression:

$$\begin{aligned}
 F_g &= -\frac{(4\pi i)^{g-1}}{4\pi^2} \frac{1}{(g!)^2} \int \frac{d^2\tau}{\tau_2^3} \frac{1}{\bar{\eta}^3} \langle \prod_{i=1}^g \int d^2x_i Z^1 \bar{\partial}Z^2(x_i) \prod_{j=1}^g \int d^2y_j \bar{Z}^2 \bar{\partial}\bar{Z}^1(y_j) \rangle \\
 &\quad \sum_{\epsilon=0, \frac{1}{2}} C_\epsilon(\bar{\tau}) \sum_{m \in \mathbf{Z} + \epsilon} \sum_{n_1, n_2 \in \mathbf{Z}} \left( \frac{iP_L}{T - \bar{T}} \right)^{2g-2} q^{\frac{1}{2}|P_L|^2} \bar{q}^{\frac{1}{2}P_R^2}
 \end{aligned} \tag{4.7}$$

where  $\tau$  is the Teichmuller parameter of the world-sheet torus,  $q$  is  $e^{2\pi i\tau}$  and  $1/\bar{\eta}^3$  accounts



for the partition function of the two space-time right moving bosons in the light-cone and one free boson corresponding to the right moving  $U(1)$  current. Note that we have normalized the amplitudes by putting in the factor  $-(4\pi i)^{g-1}/4$ , in order to match the coefficients appearing in the recursion relation for  $F_g$ 's to that of eqs.(3.10) and (3.11), as it will be shown in the following. The correlators inside  $\langle \dots \rangle$  are normalized correlators.  $P_L$ ,  $\bar{P}_L$  and  $P_R$  are the left and right moving momenta corresponding to the charges under  $U(1)^3$ . Explicitly they are given in terms of  $n_1$ ,  $n_2$  and  $m$  as:

$$\begin{aligned} P_L &= \frac{i\sqrt{2}}{T - \bar{T}}(n_1 + n_2\bar{T}^2 + 2m\bar{T}) \\ P_R &= \frac{i\sqrt{2}}{T - \bar{T}}(n_1 + n_2T\bar{T} + m(T + \bar{T})) \end{aligned} \quad (4.8)$$

Note that while  $P_L$  is complex as they are the momenta associated with the left-moving two-torus,  $P_R$  is real. This is so because we are considering the two moduli  $(S, T)$  example and hence there is only one  $U(1)$  from the right-movers. The classical duality group that leaves the spectrum invariant is  $O(2, 1)$  and the associated invariant inner product is  $\frac{1}{2}(P_L\bar{P}'_L + \bar{P}_L P'_L) - P_R P'_R = (2mm' - n_1n'_2 - n_2n'_1)$ . The charges for the vector multiplets sit in a lattice  $\Gamma_0$  defined by  $n_1, n_2, m \in \mathbf{Z}$  and is even and integral. However this lattice is not self dual with respect to the above inner product. World-sheet modular invariance requires that the extra vectors contained in the dual lattice  $\Gamma_0^*$  given by  $n_1, n_2 \in \mathbf{Z}$  and  $m \in \mathbf{Z} + \frac{1}{2}$  must also appear in the full string spectrum. In fact  $\Gamma_0^*$  has two classes with respect to  $\Gamma_0$  which we label by  $\Gamma_\epsilon$  for  $\epsilon = 0, \frac{1}{2}$ , with  $\Gamma_{\frac{1}{2}}$  being defined by  $m \in \mathbf{Z} + \frac{1}{2}$ . These two classes couple to different blocks of the remaining conformal field theory whose contribution to the above amplitude is denoted by  $C_\epsilon(\bar{\tau})$ . Actually  $C_\epsilon$  is the trace of  $(-1)^F q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24}$  in the Ramond sectors of the corresponding conformal blocks. They should only depend on  $\bar{\tau}$ . This can be argued as follows. Since we are taking the trace of  $(-1)^F$  in the Ramond sector the non-zero modes of the left moving fermions must cancel exactly with the left moving bosons. The only possible  $\tau$  dependence can come from instanton contributions. However for large Kähler class of the internal  $K_3$  surface these contributions if any would

vanish. Changing the Kähler class amounts to turning on vacuum expectation value for the corresponding moduli which belong to hypermultiplets. Since the couplings  $F_g$  do not depend on hypermultiplets we conclude that these instanton contributions must vanish also for finite Kähler class and hence  $C_\epsilon$  must depend only on  $\bar{\tau}$ . Note that the world sheet modular invariance together with the existence of the tachyon in the right-moving sector implies that

$$\begin{aligned} C_0 &= \bar{q}^{-7/8} \sum_{0 \leq k \in \mathbf{Z}} a_k \bar{q}^k & a_0 = 1, \quad a_1 = -129 \\ C_{\frac{1}{2}} &= \bar{q}^{-\frac{5}{8}} \sum_{0 \leq k \in \mathbf{Z}} b_k \bar{q}^k \end{aligned} \quad (4.9)$$

Here  $a_0 = 1$  accounts for the tachyon and  $a_1 = -n_H$  where  $n_H$  is the number of hypermultiplets. In the case at hand  $n_H = 129$ . This value of  $a_1$  is fixed by the requirement that  $F_1$  reproduces the correct anomaly coefficient.

At the enhanced symmetry point  $T = i$ , one indeed finds two extra massless states given by  $n_1 = n_2 = \pm 1$  and  $m = 0$ , which enhance the gauge symmetry to  $SU(2) \times U(1)^2$ . In the  $\epsilon = 0$  class there are no other points in the fundamental domain of  $T$  where there are extra massless states. However in the  $\epsilon = 1/2$  class, at  $T = \exp(2\pi i/3)$  one could get extra charged massless states corresponding to  $n_1 = n_2 = 2m = \pm 1$  if the coefficient  $b_0$  in eq.(4.9) is not equal to zero. However in this model one knows that there are no extra massless states at  $T = \exp(2\pi i/3)$  and therefore we conclude that  $b_0 = 0$ . Furthermore from the knowledge of the modular transformation properties of the lattice partition functions one can show that  $C_\epsilon$  transform under  $\tau \rightarrow -1/\tau$  as

$$C_\epsilon \rightarrow -\frac{1}{\sqrt{2}}(C_0 + e^{2\pi i \epsilon} C_{\frac{1}{2}}) \quad (4.10)$$

The correlation functions  $\frac{1}{(g!)^2} \langle \prod_{i=1}^g \int d^2 x_i Z^1 \bar{\partial} Z^2(x_i) \prod_{j=1}^g \int d^2 y_j \bar{Z}^2 \bar{\partial} \bar{Z}^1(y_j) \rangle$  appearing in eq.(4.7) are just normalized free field correlators of space-time bosons. In order to evaluate these correlation functions it is convenient to introduce the following generating function:

$$G(\lambda, \tau, \bar{\tau}) = \sum_{g=1}^{\infty} \frac{1}{(g!)^2} \left(\frac{\lambda}{\tau_2}\right)^{2g} \langle \prod_{i=1}^g \int d^2 x_i Z^1 \bar{\partial} Z^2(x_i) \prod_{j=1}^g \int d^2 y_j \bar{Z}^2 \bar{\partial} \bar{Z}^1(y_j) \rangle$$

$$\equiv \sum_{g=1}^{\infty} \lambda^{2g} G_g(\tau, \bar{\tau}) \quad (4.11)$$

The coefficient  $G_g$  times  $\tau_2^{2g}$  is what appears in the expression for  $F_g$  (4.7). Note that under the world-sheet modular transformation  $\tau \rightarrow \frac{a\tau+b}{c\tau+d}$  the  $G_g$  transforms with weight  $2n$  in  $\bar{\tau}$ . Thus by assigning the transformation  $\lambda \rightarrow \frac{\lambda}{c\bar{\tau}+d}$ ,  $G$  becomes invariant. In other words

$$G\left(\frac{\lambda}{c\bar{\tau}+d}, \frac{a\tau+b}{c\tau+d}, \frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) = G(\lambda, \tau, \bar{\tau}) \quad (4.12)$$

Now  $G$  can be expressed as the following normalized functional integral of four bosonic fields:

$$G(\lambda, \tau, \bar{\tau}) = \frac{\int \prod_{i=1,2} DZ^i D\bar{Z}^i \exp(-S + \int \frac{\lambda}{\tau_2} (Z^1 \bar{\partial} Z^2 + \bar{Z}^2 \partial \bar{Z}^1) d^2x)}{\int \prod_{i=1,2} DZ^i D\bar{Z}^i \exp(-S)} \quad (4.13)$$

The action  $S$  is the free field action  $S = \sum_{i=1,2} \frac{1}{\pi} \int d^2x (\partial Z^i \bar{\partial} \bar{Z}^i + \partial \bar{Z}^i \bar{\partial} Z^i)$ . Note that  $\frac{1}{(g!)^2}$  appearing in eq.(4.11) is exactly taken care of in eq.(4.13). The right hand side of this equation can be easily evaluated since the functional integrals are gaussian. One can use  $\zeta$ -function regularization as in ref.[20] to evaluate these functional integrals. In Appendix A we show that the result of these functional integral gives the following simple formula for  $G$ :

$$G(\lambda, \tau, \bar{\tau}) = \left(\frac{2\pi i \lambda \bar{\eta}^3}{\Theta_1(\lambda, \bar{\tau})}\right)^2 e^{-\frac{\pi \lambda^2}{\tau_2}} \quad (4.14)$$

where  $\Theta_1(z, \tau)$  is the odd theta-function. This formula for  $G$  can be understood as follows. The term involving  $\lambda$  in eq.(4.13) is just the right moving part of the space-time Lorentz current. This term therefore effectively twists the boundary conditions of the bosons by  $e^{\pm 2\pi i \lambda}$ . This explains the appearance of  $\bar{\Theta}_1(\lambda, \bar{\tau})^{-2}$  in eq.(4.14) since we have four bosons.  $\bar{\eta}^6$  appears because we are considering normalized correlators.  $\exp(-\frac{\pi \lambda^2}{\tau_2})$  is just due to the shift in the zero point energy of the twisted bosons. In fact this can also be deduced from the modular transformation of  $\Theta$ -function and the modular invariance of  $G$  following from eq.(4.12). The appearance of  $\lambda^2$  also follows from modular invariance of  $G$ . The fact that the right-moving Lorentz current is not a dimension (1,1) conformal operator does not create any problem, since we are integrating over flat world sheet torus. Indeed we have

explicitly verified eq.(4.14) for  $g = 1, 2, 3, 4$  by direct evaluation of the leading behaviour in  $1/\tau_2$  of the correlation functions (4.11). These leading terms also turn out to be crucial in studying the leading singularities of  $F_g$  near  $T = i$ , as we shall see below.

The generating function  $G$  satisfies the following differential equation which can be easily seen from eq.(4.14)

$$\partial_\tau G(\lambda, \tau, \bar{\tau}) = -\frac{i\pi \lambda^2}{2 \tau_2^2} G(\lambda, \tau, \bar{\tau}) \quad (4.15)$$

This equation turns out to be important in evaluating the holomorphic anomaly of  $F_g$ 's. In terms of the coefficients  $G_g$  defined in eq.(4.11), and which appear in the definition of  $F_g$  (4.7), this equation reads as:

$$\partial_\tau G_g = -\frac{i\pi}{2} \frac{1}{\tau_2^2} G_{g-1} \quad (4.16)$$

Finally we close this section by giving an expression for the generating function of  $F_g$ 's in terms of world-sheet integral. Define the following generating function

$$F(\lambda, T, \bar{T}) = \sum_{g=1}^{\infty} \lambda^{2g} F_g \quad (4.17)$$

Then using the fact that  $\lambda^{2g}$  appears with  $\frac{1}{\tau_2} (i\tau_2 P_L / (T - \bar{T}))^{2g-2}$  as follows from eqs.(4.7) and (4.11), and using the explicit formula for  $G$  (4.14), we can write the following expression for  $F(\lambda, T, \bar{T})$ :

$$F(\lambda, T, \bar{T}) = -\frac{1}{4\pi^2} \int \frac{d^2\tau}{\tau_2} \frac{1}{\bar{\eta}^3} \sum_{\epsilon=0, \frac{1}{2}} C_\epsilon(\bar{\tau}) \sum_{m \in \mathbf{Z} + \epsilon} \sum_{n_1, n_2 \in \mathbf{Z}} \left( \frac{2\pi i \lambda \bar{\eta}^3}{\bar{\Theta}_1(\tilde{\lambda}, \bar{\tau})} \right)^2 e^{-\frac{\pi}{2} \tilde{\lambda}^2 \tau_2} q^{\frac{1}{2}|P_L|^2} \bar{q}^{\frac{1}{2}P_R^2} \quad (4.18)$$

where  $\tilde{\lambda} = \sqrt{-4\pi i \lambda P_L / (T - \bar{T})}$ .

## 5. Holomorphic anomaly and the leading singularity of $F_g$

In this section we are going to use the results of section 4 to perform the tests of heterotic-type II duality we have promised in the introduction. First, we are going to

compare the recursion relations obeyed by the  $F_g$ 's, as computed in the previous section, to those of the type II side in the  $S \rightarrow \infty$  limit. Second, we will compute the leading infrared singularity in the  $F_g$ 's near the enhanced symmetric point  $T = i$ , and compare it with what one expects from the type II side.

Recall that in terms of the functions  $G_g$ , the couplings  $F_g$  are expressed as:

$$F_g = -\frac{(4\pi i)^{g-1}}{4\pi^2} \int \frac{d^2\tau}{\tau_2^3} \frac{\tau_2^{2g}}{\bar{\eta}^3} G_g(\tau, \bar{\tau}) \sum_{\epsilon=0, \frac{1}{2}} C_\epsilon(\bar{\tau}) \sum_{m \in \mathbf{Z} + \epsilon} \sum_{n_1, n_2 \in \mathbf{Z}} \left( \frac{iP_L}{T - \bar{T}} \right)^{2g-2} q^{\frac{1}{2}|P_L|^2} \bar{q}^{\frac{1}{2}P_R^2} \quad (5.1)$$

We now wish to find the holomorphic anomaly equation satisfied by  $F_g$  and relate it to the corresponding equations in type II theory. Let us take the derivative of  $F_g$  with respect to  $\bar{T}$ . One can prove the following identities which follow from the definitions of  $P_L$  and  $P_R$  in eq.(4.8):

$$\begin{aligned} \partial_{\bar{T}} \left( \frac{P_L}{T - \bar{T}} \right) &= 2 \frac{P_R}{(T - \bar{T})^2} \\ \partial_{\bar{T}} (\bar{P}_L (T - \bar{T})) &= 0 \\ \partial_{\bar{T}} P_R &= \frac{\bar{P}_L}{T - \bar{T}} \end{aligned} \quad (5.2)$$

There are of course similar identities for derivatives with respect to  $T$  which are just the complex conjugate of the above. Using these identities one can easily show that for  $g \geq 2$ ,

$$\partial_{\bar{T}} F_g = \frac{i}{4\pi^3} \frac{(4\pi i)^{g-1}}{(T - \bar{T})^2} \int d^2\tau G_g(\tau, \bar{\tau}) \times \quad (5.3)$$

$$\partial_\tau \left[ \frac{\tau_2^{2g-3}}{\bar{\eta}^3} \sum_{\epsilon=0, \frac{1}{2}} C_\epsilon(\bar{\tau}) \sum_{m \in \mathbf{Z} + \epsilon} \sum_{n_1, n_2 \in \mathbf{Z}} \left( \frac{iP_L}{T - \bar{T}} \right)^{2g-4} \partial_T (q^{\frac{1}{2}|P_L|^2} \bar{q}^{\frac{1}{2}P_R^2}) \right]$$

Now we can perform a partial integration with respect to  $\tau$ . The boundary term vanishes for generic values of  $T$  away from the singularity  $T = i$ . The only nonvanishing contribution then appears when  $\partial_\tau$  acts on  $G_g$ . Using now eq.(4.16) one obtains

$$\partial_{\bar{T}} F_g = \frac{2\pi i}{(T - \bar{T})^2} D_T F_{g-1} \quad (5.4)$$

where  $D_T$  is the Kähler covariant derivative. Recalling that  $F_g$  has Kähler weight  $(g - 1)$ , which implies that it transforms as weight  $(2g - 2)$  with respect to  $T$ , one has the following action of  $D_T$

$$D_T F_g = (\partial_T - (g - 1)(\partial_T K)) F_g = (\partial_T + \frac{2g - 2}{T - \bar{T}}) F_g \quad (5.5)$$

For  $g = 1$ , the anomaly equation has been derived before in ref.[16], or one can alternatively derive it using eqs.(5.1) and (5.2), and the result is

$$\begin{aligned} \partial_{\bar{T}} \partial_T F_1 &= \frac{25}{2} K_{T\bar{T}}^{(0)} + \frac{2i}{\pi(T - \bar{T})^2} \int \frac{d^2\tau}{\tau_2^{3/2} \bar{\eta}^3} \frac{1}{\epsilon=0, \frac{1}{2}} \sum C_\epsilon(\bar{\tau}) \sum_{m \in \mathbf{Z} + \epsilon} \sum_{n_1, n_2 \in \mathbf{Z}} \partial_{\bar{\tau}} (\tau_2^{1/2} q^{\frac{1}{2}|P_L|^2} \bar{q}^{\frac{1}{2}P_R^2}) \\ &= \frac{25}{2} K_{T\bar{T}}^{(0)} + 2\pi K_{T\bar{T}}^{(1)} \end{aligned} \quad (5.6)$$

where  $K_{T\bar{T}}^{(0)}$  and  $K_{T\bar{T}}^{(1)}$  are the tree level metric and one loop correction to the metric respectively. The first term on the right hand side comes from the boundary term as  $\tau_2 \rightarrow \infty$ , and the second term appears as a result of partial integration and the fact that  $\partial_{\bar{\tau}} G_1 = \frac{-i\pi}{2\tau_2^2}$ . In the second step, we have used the fact that the second term on the right hand side is in fact just the Green-Schwarz term, which is proportional to the one loop correction to the Kähler metric namely  $K_{T\bar{T}}^{(1)}$

Now we wish to compare these results with the anomaly equations for type II couplings  $F_g$ , in the leading  $S$  limit eq.(3.11). Note that the coefficient of tree level metric  $K_{T\bar{T}}^{(0)} = -2/(T - \bar{T})^2$  agrees in the two equations (3.11) and (5.6). The appearance of the extra term  $K_{T\bar{T}}^{(1)}$  in (5.6) can be understood as follows. As discussed in section 2, in the type II case the anomaly equation is derived treating  $T$  and  $S$  as independent variables, while in the heterotic string the independent variables are  $T$  and the linear dilaton multiplet  $L$ . Thus, in order to compare the two equations, one must change the variables. As mentioned in section 3, the linear dilaton  $L$  is related to  $S$  via a duality transformation which gives eq.(3.12). As a result, the partial derivatives with respect to  $T$  for fixed  $L$  and  $S$  respectively are related by eq.(3.13). As noted above only  $F_1$ , goes linearly as  $\text{Im}(S)$  with constant coefficient, while the remaining  $F_g$ 's for  $g \geq 2$  have no linear dependence on

$S$ . This means that the second term on the right hand side of eq.(3.13) is non-trivial only when it acts on  $F_1$ . It is then easy to see that

$$\partial_T \partial_{\bar{T}} F_1|_L = \partial_T \partial_{\bar{T}} F_1|_S + 2\pi K_{T\bar{T}}^{(1)} \quad (5.7)$$

Comparing now the equations (5.6), (3.11) and (5.7), we find that the two anomaly equations do agree.

The anomaly equation for  $F_g$ 's (5.4), is identical to the one for the type II case (3.10) for  $g \geq 3$ . The reason is of course that the second term on the right hand side of (3.13) vanishes when acting on  $F_g$  for  $g \geq 2$ . The only exception is for  $g = 2$ , in which case taking into account eq.(3.13), one finds that the equation (5.4) becomes

$$\partial_{\bar{T}} F_2|_S = \frac{2\pi i}{(T - \bar{T})^2} (\partial_T F_1|_S + 2\pi K_T^{(1)}) \quad (5.8)$$

This equation again agrees with (3.10). Thus we conclude that the anomaly equations for type II and heterotic strings agree at the perturbative level.

We now turn to the question of holomorphic ambiguities in  $F_g$ 's. In heterotic string, we have a closed form expression for  $F_g$  as integral over the moduli of the world-sheet torus. For type II on the other hand the  $F_g$ 's involve integration of the topological partition functions of the twisted Calabi-Yau sigma models over the moduli space of genus- $g$  Riemann surfaces, and therefore the determination of holomorphic ambiguities in this case is extremely difficult. However, one can try to compare the leading singularities in  $F_g$  near  $T = i$ . In the type II case, as noted in ref.[13], the leading singularity in  $F_g$  for  $g \geq 2$  near the conifold locus is  $\mu^{2-2g}$ , while for  $F_1$  it is  $\ln|\mu|$ , where  $\mu$  is the local coordinate which goes to zero at the conifold. These leading singularities are meromorphic and therefore are not captured in the holomorphic anomaly equations. As  $S \rightarrow \infty$ , the two branches of the conifold meet at  $T = i$ . Thus in this limit we can identify  $\mu$  with  $(T - i)$ , up to a constant multiplicative factor. The coefficient of this leading singularity is expected to be universal. This follows from Strominger's interpretation of the conifold singularity as due

to the appearance of massless charged black holes [7]. The singularity in  $F_g$  then would be due to a one-loop diagram involving this massless black hole as the internal line. The universality follows from the fact that the graviton and graviphoton couple universally to massless hypermultiplets. It has also been argued by Ghoshal and Vafa [17], that the leading singularity structure of  $F_g$ 's is described by the free energy of the  $c = 1$  string theory at the self dual radius:

$$Z_{c=1} = \frac{1}{2}\mu^2 \ln \mu - \frac{1}{12} \ln \mu + \sum_{g \geq 2} \chi(g)\mu^{2-2g} \quad (5.9)$$

where  $\chi(g)$  is the Euler number of the moduli space of genus- $g$  Riemann surfaces and  $\mu$  is the cosmological constant. The identification with the singularities of  $F_g$  follows from identifying  $\mu$  with a local coordinate near conifold which vanishes at the conifold. The normalization of the coordinate is fixed by comparing the tree level singularity with that in eq.(5.9). In ref[17], it was shown that with this normalization, the coefficient of the singularity for genus  $g = 2$  is exactly  $\chi(2)$  for the type II string compactified on the quintic threefold.

In heterotic case also we expect the same singularity structure near  $T = i$ . Here however, two elementary string states corresponding to charged vector multiplets become massless and as a result a string one-loop computation should exhibit this singularity structure. The universality of the coefficient of the leading singularity follows from the same argument as in the type II case. We now evaluate these coefficients from the explicit expression eq.(5.1) for  $F_g$ 's in the heterotic string. Near  $T = i$ , the extra massless states correspond to the lattice states with  $n_1 = n_2 = \pm 1$  and  $m = 0$  in eq.(4.8). In this limit the left and the right moving momenta  $P_L$  and  $P_R$  behave as  $P_L = -i\sqrt{2}(\bar{T} + i)$  and  $P_R^2 = 2 + 2|T - i|^2$ . The singularity appears from the region of integration corresponding to large  $\tau_2$ . To perform this integration and isolate the singularity it is convenient to rescale  $\tau_2$  as  $4\pi\tau_2|T - i|^2$ . Taking into account the powers of  $\tau_2$  and  $P_L$  appearing in eq.(5.1), one finds that this change of variable brings about a factor of  $(4\pi\sqrt{2}(T - i))^{2-2g}$  which is exactly the expected



leading singular behaviour. Furthermore only the constant term in  $(\bar{q}\bar{\eta}^{-3}C_0G_g)$  as  $\tau_2 \rightarrow \infty$  contribute to the coefficient of singularity. In this limit,  $\bar{q}\bar{\eta}^{-3}C_0 = a_0 = 1$ , as follows from eq.(4.9). Finally the leading term in  $G_g$  is simply the coefficient of  $\lambda^{2g}$  in the expansion of

$$\int d\tau_2 \tau_2^{2g-3} \left(\frac{\pi\lambda}{\sin \pi\lambda}\right)^2 e^{-\tau_2} \quad (5.10)$$

We can compute this as follows:

$$\begin{aligned} \left(\frac{\pi\lambda}{\sin \pi\lambda}\right)^2 &= -2\pi i \lambda^2 \partial_\lambda \frac{1}{e^{2\pi i\lambda} - 1} \\ &= -\frac{(2g-1)(-1)^g}{(2g)!} B_{2g} (2\pi\lambda)^{2g} \end{aligned} \quad (5.11)$$

where  $B_{2g}$  is the  $2g$ -th Bernoulli number, and in the second step we have used the definition of the generating function for Bernoulli numbers. For  $g = 1$ , the integral over  $\tau_2$  has a logarithmic divergence near  $T = i$ , and taking into account the value of the Bernoulli number  $B_2 = 1/6$ , one finds that  $F_1$  behaves as  $\frac{1}{6} \ln(T - i)$ . Note that the coefficient of the logarithmic singularity is  $(-2)$  times the one appearing near the conifold singularity in eq.(5.9), namely  $(-1/12)$ . Finally for  $g \geq 2$ , the integral  $\int d\tau_2 \tau_2^{2g-3} \exp(-\tau_2)$  provides an extra factor of  $(2g-3)!$ . Thus altogether for  $g \geq 2$ , the coefficient of the leading singularity  $(\sqrt{\frac{2}{\pi i}}(T - i))^{2-2g}$  is  $-2B_{2g}/2g(2g-2)$  which is just  $(-2)$  times the Euler number  $\chi(g)$  of the moduli space of genus- $g$  Riemann surfaces. Thus by identifying  $\mu$  with  $\sqrt{\frac{2}{\pi i}}(T - i)$ , we find that the singularity structure for  $F_g$ 's near  $T = i$  is described by  $(-2)$  times the free energy of  $c = 1$  string theory at the self-dual point. In fact, with this identification, the tree level term  $-2(\frac{1}{2}\mu^2 \ln \mu) = \frac{2i}{\pi}(T - i)^2 \ln(T - i)$ , exactly reproduces the singularity of the prepotential  $f$  discussed in section 3. The relative factor of  $(-2)$  can be understood from the fact, that while near  $T = i$  one has two extra massless vector multiplets, near the conifold only one hypermultiplet corresponding to a black hole becomes massless. The ratio of their contributions to the trace anomaly in the two cases respectively is exactly  $(-2)$ . Moreover this also supports the argument of ref.[17] that the physics near conifold singularity is described by  $c = 1$  string theory at the self-dual radius. A more invariant identification of

$\mu$ , which applies generically to the singularities associated with the appearance of massless states, is given by  $\mu = \sqrt{i/\pi} \exp(-K^{(0)}/2)Z$ , where  $Z$  is the central charge of the  $N = 2$  supersymmetry algebra [21] and  $K^{(0)}$  is the tree level Kähler potential corresponding to the  $T$  moduli. Note that  $\mu$  transforms with weight 1 under Kähler transformations, however recalling that  $\mu^{2-2g}$  appears together with  $(2g - 2)$  graviphotons, the corresponding term in the effective action is Kähler invariant, due to the transformation properties of the graviphotons discussed in section 2.

It is interesting to note that a one loop computation in heterotic string reproduces the Euler number of moduli space of genus- $g$  Riemann surfaces. In fact since the leading singularity appears from  $\tau_2 \rightarrow \infty$  limit, it should be possible to understand it purely at the effective field theory level and this in the type II context would then be consistent with Strominger’s interpretation of the conifold singularity. In fact the effective action for QED in the presence of constant electric and magnetic fields has been computed long ago by Schwinger [22]. If one considers self-dual background, then Schwinger’s formula exactly coincides with eq.(5.10) with the identification  $\lambda^2 = F_{\mu\nu}^2$ . It is interesting to note that QED already computes the Euler number of the moduli space of genus- $g$  Riemann surfaces! However, we are concerned here with  $N = 2$  supergravity sector and Schwinger’s formalism needs to be extended to this case.

It should be pointed out that from the heterotic side, using the general formula (5.1), one can also compute quite easily the subleading singularities near  $T = i$ . This involves expanding  $P_L$  near  $T = i$  as well as the generating function  $G_g$  whose relevant part is obtained by expanding  $(\frac{\pi\lambda}{\sin \pi\lambda})^2 \exp(-\frac{\pi\lambda^2}{\tau_2})$ . It would be interesting to compare also these subleading singularities with the ones in type II, where however we do not have similar results at present. On the other hand assuming duality, the fact that we can compute the coefficients of all the poles in  $F_g$ , can help at least partially in fixing the holomorphic ambiguity on the type II side.

## 6. Concluding remarks

In this paper we examined the proposed  $N = 2$  type II-heterotic duality in a class of higher derivative F-terms of the form  $F_g W^{2g}$ ,  $W$  being the gravitational multiplet. While in type II side, they appear at  $g$ -loop level and are exact at the quantum level, on the heterotic side, to the leading order in the string coupling, they appear at the one-loop level. We analysed in detail the rank 3 example and showed that to this order, the holomorphic anomaly equations for  $F_g$ 's are identical for the two models. We also computed the leading singularity near the enhanced symmetry point in the heterotic string and showed that the corresponding coefficient is universal and is given by the Euler number of the moduli space of genus- $g$  Riemann surfaces. Therefore if the conjecture of Ghoshal and Vafa, relating the conifold singularity to the  $c = 1$  string theory at self-dual radius, would receive more evidence for  $g \geq 3$ , then our result would represent a very strong argument in favour of the type II-heterotic duality. Although we have focussed here on the rank 3 example, one can easily extend the above analysis to the rank 4 case.

There are several questions which need further investigation. One of the issues is regarding corrections to  $F_g$ 's that are higher order in  $1/(S - \bar{S})$ . This involves going beyond one loop on the heterotic side. On the type II side, the open issue is the structure of leading and subleading singularities near the conifold. Another issue which has not been investigated so far is the comparison of the hypermultiplet sectors of the two theories.

In conclusion, our analysis of higher-dimensional effective Lagrangian interactions provides a very strong quantitative evidence supporting the duality conjecture for certain pairs of type II and heterotic superstring models in four dimensions. The most intriguing aspect of duality which emerges very clearly from this work is the apparent equivalence of physical effects which occur at different loop orders, or even non-perturbatively, as viewed from dual descriptions. This goes very far beyond our experience with low-energy quantum field theory, and our intuition what is classical and what is quantum. Uncovering the origin

of duality may indeed provide a clue to understanding the physical content of superstring theory.

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## Appendix

This appendix is devoted to the derivation of eq.(4.14). We will adopt the  $\zeta$ -function regularization used in ref.[20] to evaluate the determinant of a scalar field on a world sheet torus. The functional integral in eq.(4.13) is quadratic in the scalar fields, so we can evaluate it by expanding the  $Z$ 's into an orthonormal basis of eigenfunctions of the scalar laplacian. Let us choose torus coordinates  $\sigma^1, \sigma^2$ , with  $0 \leq \sigma^1, \sigma^2 \leq 1$ , and the corresponding metric to be given by  $ds^2 = |d\sigma^1 + \tau d\sigma^2|^2$ . The orthonormal basis is given by  $\phi_{n,m} = \frac{1}{\sqrt{\tau_2}} \exp(2\pi i(n\sigma^1 + m\sigma^2))$  where  $n, m \in \mathbf{Z}$ . It is then easy to see that as a result of the functional integration in the numerator of eq.(4.13) we will get the following determinant:

$$\det' \Delta = \prod_{(n,m) \neq (0,0)} \left(\frac{2\pi}{\tau_2}\right)^2 [|n - m\tau|^4 - \lambda^2(n - m\tau)^2] \quad (\text{A.1})$$

which defines the operator  $\Delta$  in terms of its eigenvalues. To evaluate (A.1) it is useful to split  $\Delta$  as  $\Delta = \Delta^+ \Delta^-$ , where  $\Delta^\pm$  have eigenvalues

$$\lambda_{n,m}^\pm = \frac{2\pi^2}{\tau_2} [|n - m\tau|^2 \pm \lambda(n - m\tau)] \quad (\text{A.2})$$

We can then evaluate  $\ln \det' \Delta^\pm$  following ref.[20] by using  $\zeta$ -function regularization and

converting the sum over  $n$  into an integral using the Sommerfeld-Watson transformation.

The result is:

$$\begin{aligned}
\ln \det' \Delta^\pm &= \lim_{s, \mu^2 \rightarrow 0} \\
&- 2 \frac{d}{ds} \left[ \left( \frac{2\pi}{\tau_2^2} \right)^{-s} \int_c dz \sum_m \frac{e^{i\pi z}}{2i \sin \pi z} [(z - m\tau_1)^2 + m\tau_2^2 \pm \lambda(z - m\tau) + \mu^2]^{-s} + \text{h.c.} \right] \\
&+ \frac{d}{ds} \left[ \left( \frac{2\pi}{\tau_2^2} \right)^{-s} \int_c dz \sum_m [(z - m\tau_1)^2 + m\tau_2^2 \pm \lambda(z - m\tau) + \mu^2]^{-s} + \text{h.c.} \right] \\
&- \left( \frac{2\pi\mu^2}{\tau_2^2} \right)^{-s} \ln \left( \frac{2\pi\mu^2}{\tau_2^2} \right). \tag{A.3}
\end{aligned}$$

The contour passes above the real axis from  $+\infty + i\epsilon$  to  $-\infty + i\epsilon$  and we have introduced a mass  $\mu$  as an infrared regulator. Notice that the sum over  $m$  includes  $m = 0$ . Let us first do the computation for  $\Delta^+$ . The first term in the bracket converges at  $s = 0$  and gives

$$2 \sum_{m=1}^{\infty} \ln(1 - q^m) + \sum_{m=0}^{\infty} \ln(1 - \bar{q}^m e^{-2\pi i \lambda}) + \sum_{m=1}^{\infty} \ln(1 - \bar{q}^m e^{2\pi i \lambda}) + \ln(2\pi\mu^2) - \ln(\lambda) \tag{A.4}$$

Note that  $\ln(2\pi\mu^2)$  term cancels between eq.(A.4) and the last term in eq.(A.3). The second integral can be expanded in powers of  $\lambda$ . After performing the sum over  $m \neq 0$ , it turns out that only up to quadratic terms in  $\lambda$  survive in the limit  $s \rightarrow 0$  with the result:

$$-\frac{\pi}{3} \tau_2 + \frac{\pi}{2} \frac{\lambda^2}{\tau_2} + i\pi\lambda. \tag{A.5}$$

For  $m = 0$  the result vanishes in the limit  $s, \mu \rightarrow 0$ . Combining now the contribution  $\Delta^-$  and taking into account the normalization i.e. the partition function of four scalars, we get the desired result eq.(4.14).

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