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NON-PERTURBATIVE MONODROMIES IN $N = 2$ HETEROTIC STRING VACUA

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ABSTRACT

We address non-perturbative effects and duality symmetries in $N = 2$ heterotic string theories in four dimensions. Specifically, we consider how each of the four lines of enhanced gauge symmetries in the perturbative moduli space of $N = 2$ T_2 compactifications is split into 2 lines where monopoles and dyons become massless. This amounts to considering non-perturbative effects originating from enhanced gauge symmetries at the microscopic string level. We show that the perturbative and non-perturbative monodromies consistently lead to the results of Seiberg-Witten upon identication of a consistent truncation procedure from local to rigid $N = 2$ supersymmetry.

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1 Introduction

Recently some major progress has been obtained in the understanding of the non-perturbative dynamics of $N=2$ Yang-Mills theories as well as $N=2$ superstrings in four dimensions. The quantum moduli space of the $N=2$ $SU(r+1)$ Yang-Mills theories is described [1, 2, 3, 4] by *rigid special geometry*, which is based on an (auxiliary) Riemann surface of genus r with the $2r$ periods (a_i, a_{Di}) $(r=1,\dots r)$ being holomorphic sections of the $Sp(2r,\mathbf{Z})$ bundle defined by the first homology group of the Riemann surface. The periods a_i are associated to r $N=2$ vector superfields in the Cartan subalgebra of $SU(r+1)$; hence non-zero vacuum expectation values break $SU(r+1)$ down to $U(1)^{r+1}$. The quantum moduli space possesses singular points with non-trivial monodromies around these points. The semiclassical monodromies are due to the one-loop contributions to the holomorphic prepotential, and the corresponding logarithmic singularities are the left-over signal of the additional non-Abelian massless vector fields at $a_i=0$. However in the full quantum moduli space there are no points of enhanced non-Abelian gauge symmetries, and the semiclassical monodromies are split into non-perturbative monodromies, where the monodromy group Γ_M is a subgroup of $Sp(2r,\mathbf{Z})$. The corresponding singular points in the quantum moduli space are due to magnetic monopoles and dyons becoming massless at these points.

During the past years duality symmetries in string theory gained a lot of attention. First, the socalled target space duality symmetry (T-duality) (for a review see [5]) is known to be a true symmetry in every order of string perturbation theory. Second, S-duality [6] was proposed to be a non-perturbative string symmetry, and evidence for S-duality in $N=4$ heterotic strings is now accumulating. Moreover, string-string dualities [7, 8] between type II,I and heterotic theories in various different dimensions play an important role in the understanding of string dynamics.

In this paper we will address non-perturbative effects and duality symmetries in $N=2$ heterotic string theories in four dimensions. When coupling the $N=2$ Yang-Mills gauge theory to supergravity, as it is necessary in the context of superstrings, different additional effects play an important role. First, there is always one additional $U(1)$ vector field, the socalled graviphoton. As a result of the gravitational interactions the couplings of n $N=2$ vector multiplets plus the graviphoton are now described by *local special geometry* with the $2n+2$ holomorphic periods (X^I, iF_I) $(I=0,\dots,n)$ being holomorphic sections of an $Sp(2n+2)$ bundle [9]. Due to the absence of a propagating scalar partner of the graviphoton, the associated special Kähler space can be parametrized by projective, *special* coordinates $z^A = X^A/X^0$ $(A=1,\dots,n)$. In the context of four-dimensional

$N = 2$ heterotic string vacua another $U(1)$ gauge boson plus a corresponding (complex) scalar field exists besides the vector fields of the rigid theory, namely the dilaton-axion field S. In the rigid theories, S appears merely as a parameter for the classical gauge coupling plus theta angle, but in string theories S becomes a dynamical field. It can be either described by a vector-tensor multiplet [10, 11] or, as we will keep it in the following, by an $N = 2$ vector multiplet. Thus the total number of physical vector multiplets is given by $n = r + 1$, where r is the number of moduli fields in vector multiplets which are in one to one correspondence with the Higgs fields of the rigid theory. In a recent very interesting developement, some convincing evidence accumulated [8, 12] that the periods (X^I, iF_I) of the full heterotic $N = 2$ quantum theory are given by a suitably chosen Calabi-Yau threefold with dimension of the third cohomology group $b_3 = 4 + 2r$. Moreover, based on the ideas of heterotic versus type II string duality, this Calabi-Yau space does not just serve as an auxiliary construction, but there exists a dual type II, $N = 2$ string compactified on this Calabi-Yau space. This observation opens the exiting possibility to obtain non-perturbative quantum effects on the heterotic side by computing the classical vector couplings on the type II side, since in the type II theories the dilaton as the loop counting parameter sits in a hyper multiplet [13] and, at lowest order, does not couple to the type II vector fields. If this picture turns out to be true, it consequently implies that the Riemann surface of the rigid theory is embedded into the six-dimensional Calabi-Yau space.

In this paper we investigate (however without reference to an underlying Calabi-Yau space) how the semiclassical singular lines of enhanced gauge symmetries in $N = 2$ heterotic strings can be split non-perturbatively each into two singular loci, namely each into two lines where magnetic monopoles or dyons respectively beome massless. Specifically, under the (reasonable) assumption that the non-perturbative dynamics well below the Planck scale is governed by the Yang-Mills gauge interactions a la Seiberg and Witten [1], we are able to construct the associated non-perturbative monopol and dyon monodromy matrices. In addition, we address the question, how the already known perturbative as well as the here newly derived non-perturbative monodromies of the $N = 2$ heterotic moduli space lead to the rigid monodromies of [1, 3, 4]. This embedding of the rigid monodromies into the local ones implies a very well defined truncation procedure. As we will show this does not agree with the naive limit of $M_{\text{Planck}} \to \infty$, because one also has to take into account the fact that in the string case the dilaton as well as the graviphoton are in general not invariant under the Weyl transformation. Thus, in other words, the dilaton and graviphoton fields have to be frozen, before one can perform the limit $M_{\text{Planck}} \to \infty$.

The classical as well as the perturbative (one-loop) holomorphic prepotentials for $N = 2$ heterotic strings were derived in [11, 14, 15]. In particular, [11, 15] focused on heterotic string vacua which are given as a compactification of the six-dimensional $N = 1$ heterotic string on a two-dimensional torus T_2. These types of backgrounds always lead to two moduli fields, T and U, of the underlying T_2; the underlying holomorphic prepotential is then a function of S, T and U. At special lines (points) in the perturbative (T, U) moduli space, part of the Abelian gauge group is enhanced to $SU(2)$, $SU(2)^2$ or $SU(3)$ respectively. This is the stringy version of the Higgs effect with Higgs fields given as certain combinations of the moduli T and U. Thus this situation is completely analogeous to the rigid case discussed in [1, 2, 3, 4]; in the string case, however, the Weyl group, acting on the Higgs fields a_1 and a_2, of $SU(2)$, $SU(2)^2$ or $SU(3)$ as the classical symmetry group of the effective action is extended to be the full target space duality group acting on the moduli fields T and U [16]. In the next chapter we will recall the classical prepotential and the classical duality symmetries; in particular we will work out the precise relation between the field theory Higgs fields and the string moduli, and the relation of the four indepedent Weyl transformations to specific elements of the target space duality group. At the one loop level, the holomorphic prepotential exhibits logarithmic singularities precisely at the critical lines of enhanced gauge symmetries; moving in moduli space around the critical lines one obtaines the semiclassical monodromies. In the third chapter we will determine the $Sp(8, \mathbf{Z})$ one-loop monodromy matrices corresponding to the four independent Weyl transformations of the enhanced gauge groups. We will discuss the consistent truncation to the rigid case and show that the truncated one-loop monodomies agree with the semi-classical monodromies obtained in [1, 3, 4]. Finally, in chapter four, we derive, under a few physical assumptions, the splitting of the one-loop (i.e. semi-classical) monodromies into a pair of non-perturbative monopole and dyon monodromies. We show that with the same truncation procedure as in the one loop case we arrive at the non-perturbative monodromies of the rigid cases. This analyis adresses the non-perturbative effects in the gauge sectors far below the Planck scale. Thus the dilaton field is kept large at the points where monopoles or dyons become massless. In addition one expects [8] non-perturbative, genuine stringy monodromies at finite values of the dilaton, e.g. when gravitational instantons, black holes etc. become massless. It would be of course very interesting to compare our results with some computations of monodromies in appropriately choosen (such as $X_{24}(1, 1, 2, 8, 12)$ [8]) Calabi-Yau moduli spaces.

2 Classical results, enhanced gauge symmetries and Weyl reflections

In this section we collect some results about $N = 2$ heterotic strings and the related classical prepotential; we will in particular work out the relation between the enhanced gauge symmtries, the duality symmetries and Weyl transformations. We will consider four-dimensional heterotic vacua which are based on compactifications of six-dimensional vacua on a two-torus T_2. The moduli of T_2 are commonly denoted by T and U where U describes the deformations of the complex structure, $U = (\sqrt{G} - iG_{12})/G_{11}$ (G_{ij} is the metric of T_2), while T parametrizes the deformations of the area and of the antisymmetric tensor, $T = 2(\sqrt{G} + iB)$. (Possibly other existing vector fields will not play any role in our discussion.) The scalar fields T and U are the spin-zero components of two $U(1)$ $N = 2$ vector supermultiplets. All physical properties of the two-torus compactifications are invariant under the group $SO(2, 2, \mathbf{Z})$ of discrete target space duality transformations. It contains the $T \leftrightarrow U$ exchange, with group element denoted by σ and the $PSL(2, \mathbf{Z})_T \times PSL(2, \mathbf{Z})_U$ dualities, which act on T and U as

$$(T, U) \longrightarrow \left(\frac{aT - ib}{icT + d}, \frac{a'U - ib'}{ic'U + d'}\right), \quad a, b, c, d, a', b', c', d' \in \mathbf{Z}, \quad ad - bc = a'd' - b'c' = 1. \tag{2.1}$$

The classical monodromy group, which is a true symmetry of the classical effective Lagrangian, is generated by the elements σ, g_1, g_1: $T \to 1/T$ and g_2, g_2: $T \to 1/(T - i)$. The transformation t: $T \to T + i$, which is of infinite order, corresponds to $t = g_2^{-1} g_1$. Whereas $PSL(2, \mathbf{Z})_T$ is generated by g_1 and g_2, the corresponding elements in $PSL(2, \mathbf{Z})_U$ are obtained by conjugation with σ, i.e. $g_i' = \sigma^{-1} g_i \sigma$.

As mentioned already in the introduction, the $N = 2$ heterotic string vacua contain two further $U(1)$ vector fields, namely the graviphoton field, which has no physical scalar partner, and the dilaton-axion field, denoted by S. Thus the full Abelian gauge symmetry we consider is given by $U(1)_L^2 \times U(1)_R^2$. At special lines in the (T, U) moduli space, additional vector fields become massless and the $U(1)_L^2$ becomes enlarged to a non-Abelian gauge symmetry. Specifically, there are four inequivalent lines in the moduli space where two charged gauge bosons become massless. The quantum numbers of the states that become massless can be easily read of from the holomorphic mass formula [17, 18, 19]

$$\mathcal{M} = m_2 - i m_1 U + i n_1 T - n_2 T U, \tag{2.2}$$

where n_i, m_i are the winding and momentum quantum numbers associated with the i-th direction of the target space T_2. Let us now collect the fixed lines, the quantum numbers of the states which become massless at the fixed lines; we already include in the following table the Weyl transformations under which the corresponding lines are fixed:

FP Transformations	Fixed Points	Quantum Numbers
w_1	$U = T$	$m_1 = n_1 = \pm 1, \quad m_2 = n_2 = 0$
$w_2 = w_1'$	$U = \frac{1}{T}$	$m_2 = n_2 = \pm 1, \quad m_1 = n_1 = 0$
w_2'	$U = T - i$	$m_1 = m_2 = n_1 = \pm 1, \quad n_2 = 0$
w_0'	$U = \frac{T}{iT+1}$	$m_1 = n_1 = -n_2 = \pm 1, \quad m_2 = 0$

(2.3)

At each of the four critical lines the $U(1)_L^2$ is extended to $SU(2)_L \times U(1)_L$. Moreover, these lines intersect one another in two inequivalent critical points (for a detailed discussion see ([19])). At $(T, U) = (1, 1)$ the first two lines intersect. The four extra massless states extend the gauge group to $SU(2)_L^2$. At $(T, U) = (\rho, \bar{\rho})$ $(\rho = e^{i\pi/6})$ the last three lines intersect. The six additional states extend the gauge group to $SU(3)$. (In addition, the first and the third line intersect at $(T, U) = (\infty, \infty)$, whereas the first and the last line intersect at $(T, U) = (0, 0)$.)

The Weyl groups of the enhanced gauge groups $SU(2)^2$ and $SU(3)$, realized at $(T, U) = (1, 1), (\rho, \bar{\rho})$ respectively, have the following action on T and U:

Weyl Reflections	$T \to T'$	$U \to U'$
w_1	$T \to U$	$U \to T$
w_2	$T \to \frac{1}{U}$	$U \to \frac{1}{T}$
w_1'	$T \to \frac{1}{U}$	$U \to \frac{1}{T}$
w_2'	$T \to U + i$	$U \to T - i$
w_0'	$T \to \frac{U}{-iU+1}$	$U \to \frac{T}{iT+1}$

(2.4)

w_1, w_2 are the Weyl reflections of $SU(2)_{(1)} \times SU(2)_{(2)}$, whereas w_1' and w_2' are the fundamental Weyl reflections of the enhanced $SU(3)$. For later reference we have also listed the $SU(3)$ Weyl reflection $w_0' = w_2'^{-1} w_1' w_2'$ at the hyperplane perpendicular to the highest root of $SU(3)$. Note that $w_2 = w_1'$. All these Weyl transformations are target space modular transformations and therefore elements of the monodromy group. All Weyl reflections can be expressed in terms of the generators g_1, g_2, σ and, moreover, all Weyl reflections are conjugated to the mirror symmetry σ by some group element:

$$w_1 = \sigma, \quad w_2 = w_1' = g_1 \sigma g_1 = g_1^{-1} \sigma g_1 \tag{2.5}$$

$$w_2' = t\sigma t^{-1} = (g_1^{-1}g_2)^{-1}\sigma(g_1^{-1}g_2), \quad w_0' = w_2'^{-1}w_1'w_2' \tag{2.6}$$

As already mentioned the four critical lines are fixed under the corresponding Weyl transformation. Thus it immediately follows that the numbers of additional massless states agrees with the order of the fixed point transformation at the critical line, points respectively [19].

Let us now express the moduli fields T and U in terms of the field theory Higgs fields whose non-vanishing vacuum expectation values spontaneously break the enlarged gauge symmetries $SU(2)^2$, $SU(3)$ down to $U(1)^2$. First, the Higgs field[3] of $SU(2)_{(1)}$ is given by $a_1 \propto (T-U)$. Taking the rigid field theory limit $\kappa^2 = \frac{8\pi}{M^2_{\text{Planck}}} \to 0$ we will expand $T = T_0 + \kappa\delta T$, $U = T_0 + \kappa\delta U$. Then, at the linearized level, the $SU(2)_{(1)}$ Higgs field is given as $a_1 \propto (\delta T - \delta U)$. Analogously, for the enhanced $SU(2)_{(2)}$ the Higgs field is $a_2 \propto (T - 1/U)$. Again, we expand as $T = T_0(1+\kappa\delta T)$, $U = \frac{1}{T_0}(1+\delta U)$ which leads to $a_2 \propto \delta T + \delta U$. Finally, for the enhanced $SU(3)$ we obtain as Higgs fields $a_1' \propto \delta T + \delta U$, $a_2' \propto \rho^2\delta T + \rho^{-2}\delta U$, where we have expanded as $T = \rho + \delta T$, $U = \rho^{-1} + \delta U$ (see section 3 for details).

The classical vector couplings are determined by the holomorphic prepotential which is a homogeneous function of degree two of the fields X^I ($I = 1, \ldots, 3$). It is given by [14, 11, 15]

$$F = i\frac{X^1X^2X^3}{X^0} = -STU, \tag{2.7}$$

where the physical vector fields are defined as $S = i\frac{X^1}{X^0}$, $T = -i\frac{X^2}{X^0}$, $U = -i\frac{X^3}{X^0}$ and the graviphoton corresponds to X^0. As explained in [14, 11], the period vector (X^I, iF_I) ($F_I = \frac{\partial F}{X^I}$), that follows from the prepotential (2.7), does not lead to classical gauge couplings which all become small in the limit of large S. Specifically, the gauge couplings which involve the $U(1)_S$ gauge group are constant or even grow in the string weak coupling limit $S \to \infty$. In order to choose a 'physical' period vector one has to replace $F^S_{\mu\nu}$ by its dual which is weakly coupled in the large S limit. This is achieved by the following symplectic transformation $(X^I, iF_I) \to (P^I, iQ_I)$ where[4]

$$P^1 = iF_1,\ Q_1 = iX^1, \text{ and } P^i = X^i,\ Q_i = F_i \quad \text{for} \quad i = 0, 2, 3. \tag{2.8}$$

In this new basis the classical period vector takes the form

$$\Omega^T = (1, TU, iT, iU, iSTU, iS, -SU, -ST), \tag{2.9}$$

[3]Note that the Higgs fields just correspond to the uniformizing variables of modular functions at the critical points, lines respectively.

[4]Note however that the new coordinates P^I are not independent and hence there is no prepotential $Q(P^I)$ with the property $Q_I = \frac{\partial Q}{\partial P^I}$.

where $X^0 = 1$. One sees that after the transformation (2.8) all electric vector fields P^I depend only on T and U, whereas the magnetic fields Q_I are all proportional to S.

The basis Ω is also well adapted to discuss the action of the target space duality transformations and, as particular elements of the target space duality group, of the four inequivalent Weyl reflections given in (2.4). In general, the field equations of the $N = 2$ supergravity action are invariant under the following symplectic $Sp(8, \mathbf{Z})$ transformations, which act on the period vector Ω as

$$\begin{pmatrix} P^I \\ iQ_I \end{pmatrix} \to \Gamma \begin{pmatrix} P^I \\ iQ_I \end{pmatrix} = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} P^I \\ iQ_I \end{pmatrix}, \tag{2.10}$$

where the 4×4 sub-matrices U, V, W, Z have to satisfy the symplectic constraints

$$U^T V - W^T Z = V^T U - Z^T W = 1, \quad U^T W = W^T U, \quad Z^T V = V^T Z. \tag{2.11}$$

Invariance of the lagrangian implies that $W = Z = 0$, $VU^T = 1$. In case that $Z = 0$, $W \neq 0$ and hence $VU^T = 1$ the action is invariant up to shifts in the θ-angles; this is just the situation one encounters at the one-loop level. The non-vanishing matrix W corresponds to non-trivial one-loop monodromy due to logarithmic singularities in the prepotential. (This will be the subject of section 3.) Finally, if $Z \neq 0$ then the electric fields transform into magnetic fields; these transformations are the non-perturbative monodromies due to logarithmic singularities induced by monopoles, dyons or other non-perturbative excitations (see section 4).

The classical action is completely invariant under the target space duality transformations. Thus the classical monodromies have $W, Z = 0$. The matrices U (and hence $V = U^{T,-1} = U^*$) are given by

$$U_\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad U_{g_1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad U_{g_2} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \tag{2.12}$$

At the classical level the S-field is invariant under these transformations. The corresponding symplectic matrices for the four inequivalent Weyl reflections then immediately follow as

$$U_{w_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \quad U_{w_2} = U_{w_1'} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$U_{w_2'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} , \quad U_{w_0'} = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \tag{2.13}$$

Let us now discuss the masses of those states which saturate the socalled BPS bound. These masses are dermined by the complex central charge of the $N=2$ supersymmetry algebra. In general the mass formula is given by the following expression [18, 14]

$$M^2 = e^K |M_I P^I + i N^I Q_I|^2 = e^K |\mathcal{M}|^2. \tag{2.14}$$

Here K is the Kähler potential, the M_I are the electric quantum numbers of $U(1)^4$ and the N^I are the magnetic quantum numbers. It follows that the classical spectrum of electric states, i.e. $N^I = 0$, agrees with the string momentum and winding spectrum of eq.(2.2), upon identification $M_I = (m_2, -n_2, n_1, -m_1)$. Moreover, if one chooses linearly dependent electric and magnetic charges, i.e. $M_I/p = (m_2, -n_2, n_1, -m_1)$, $N^I/q = (-n_2, m_2, -m_1, n_1)$, then the classical mass formula factorizes as[5]

$$M^2 = \frac{|(p+iqS)(m_2 - i m_1 U + i n_1 T - n_2 UT)|^2}{(S+\bar{S})(T+\bar{T})(U+\bar{U})}. \tag{2.15}$$

The moduli dependent part again agrees with the classical string momentum and winding spectrum; p and q are the electric and magnetic S-quantum numbers. Note that the factorized classical mass formula can be obtained by truncating the BPS mass formula of the $N=4$ heterotic string to the S, T, U subspace.

Finally, requiring the holomorphic mass $\mathcal{M}$ in (2.14) to be invariant under the symplectic transformations (2.10) yields that the quantum numbers M_I and N^I have to transform as

$$(N, -M)^T \to (N, -M)^T \, \Gamma^T \tag{2.16}$$

[5] We will, however, in the following not rely on this factorisation, but rather use equation (2.14).

under (2.10).

3 Perturbative results

Let us first review the main results about the one-loop perturbative holomorphic prepotential which were derived in [11, 15]. Using simple power counting arguments it is clear that the one-loop prepotential must be independent of the dilaton field S. The same kind of arguments actually imply that there are no higher loop corrections to the prepotential in perturbation theory. Thus the perturbative, i.e. one loop prepotential takes the form

$$F = F^{(Tree)}(X) + F^{(1-loop)}(X) = i\frac{X^1X^2X^3}{X^0} + (X^0)^2 f(T,U) = -STU + f(T,U) \quad (3.1)$$

where

$$S = i\frac{X^1}{X^0}, \quad T = -i\frac{X^2}{X^0}, \quad U = -i\frac{X^3}{X^0} \quad (3.2)$$

Taking derivatives of the 1-loop corrected prepotential gives that

$$\begin{aligned} F_0 &= -i\frac{X^1X^2X^3}{(X^0)^2} + 2X^0 f(T,U) + iX^2 f_T + iX^3 f_U\ ,\ F_1 = i\frac{X^2X^3}{X^0}, \\ F_2 &= i\frac{X^1X^3}{X^0} - iX^0 f_T, \quad F_3 = i\frac{X^1X^2}{X^0} - iX^0 f_U \end{aligned} \quad (3.3)$$

which, in special coordinates, turns into

$$F_0 = STU + 2f(T,U) - Tf_T - Uf_U, \quad F_1 = -iTU, \quad F_2 = iSU - if_T, \quad F_3 = iST - if_U \quad (3.4)$$

The 1-loop corrected Kähler potential is, in special coordinates, given by

$$\begin{aligned} K(S,T,U) = -\log Y, \quad Y = Y_{tree} + Y_{pert}, \quad Y_{tree} = (S+\overline{S})(T+\overline{T})(U+\overline{U}) \\ Y_{pert} = 2(f+\overline{f}) - (T+\overline{T})(f_T + \overline{f_T}) - (U+\overline{U})(f_U + \overline{f_U}) \end{aligned} \quad (3.5)$$

The 1-loop corrected section $\Omega^T = (P, iQ)^T = (P^0, P^1, P^2, P^3, iQ_0, iQ_1, iQ_2, iQ_3)$ is given by

$$\begin{aligned} \Omega^T &= (X^0, iF_1, X^2, X^3, iF_0, -X^1, iF_2, iF_3) \\ &= (1, TU, iT, iU, iSTU + 2if - iTf_T - iUf_U, iS, -SU + f_T, -ST + f_U) \end{aligned} \quad (3.6)$$

Since the target space duality transformations are known to be a symmetry in each order of perturbation theory, the tree level plus one-loop effective action must be invariant under these transformations, where however one has to allow for discrete shifts in the various θ angles due to monodromies around semi-classical singularities in the moduli space where massive string modes become massless. Instead of the classical transformation rules, in the quantum theory, (P^I, iQ_I) transform according to

$$P^I \rightarrow U^I_{\ J}\, P^J, \qquad iQ_I \rightarrow V_I^{\ J}\, iQ_J + W_{IJ}\, P^J \ , \tag{3.7}$$

where

$$V = (U^{\rm T})^{-1}, \quad W = V\Lambda\,, \quad \Lambda = \Lambda^{\rm T} \tag{3.8}$$

and U belongs to $SO(2,2,\mathbf{Z})$. Classically, $\Lambda = 0$, but in the quantum theory, Λ is a real symmetric matrix, which should be integer valued in some basis.

Besides the target space duality symmetries, the effective action is also invariant, up to discrete shifts in the θ-angles, under discrete shifts in the S-field, D: $S \rightarrow S + i$. Thus the full perturbative monodromies contain the following $Sp(8,\mathbf{Z})$ transformation:

$$V_S = U_S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad W_S = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad Z_S = 0. \tag{3.9}$$

Invariance of the one-loop action up to discrete θ-shifts then implies that

$$F^{(1-loop)}(X) \longrightarrow F^{(1-loop)}(X) - \frac{i}{2}\Lambda_{IJ}P^I P^J \tag{3.10}$$

This reads in special coordinates like

$$f(T,U) \rightarrow (icT + d)^{-2}(f(T,U) + \Psi(T,U)) \tag{3.11}$$

for an arbitrary $PSL(2,\mathbf{Z})_T$ transformation. $\Psi(T,U)$ is a quadratic polynomial in T and U.

As explained in [11, 15] the dilaton is not any longer invariant under the target space duality transformations at the one-loop level. Indeed, the relations (3.2) and (3.7) imply[6]

$$S \longrightarrow S + \frac{V_1^{\ J}(F_J^{(1-loop)} - i\Lambda_{JK}P^K)}{U^0_{\ I}P^I} \tag{3.12}$$

Near the singular lines the one-loop prepotential exhibits logarithmic singularities and is therefore not a singlevalued function when transporting the moduli fields around the

[6] It is still possible to define an invariant dilaton field which is however not an $N = 2$ special coordinate [11].

singular lines. For example around the singular $SU(2)_{(1)}$ line $T=U\neq 1,\rho$ the function f must have the following form

$$f(T,U)=\frac{1}{\pi}(T-U)^2\log(T-U)+\Delta(T,U), \tag{3.13}$$

where $\Delta(T,U)$ is finite and single valued at $T=U\neq 1,\rho$. At the remaining three critical lines $f(T,U)$ takes an analogous form. Moreover at the intersection points the residue of the singularity must change in agreement with the number of states which become massless at these critical points (These residues are of course just given by the $N=2$ pure Yang-Mills β-functions for $SU(2)$, $SU(2)^2$ and $SU3$) (there are no massless additional flavors at the points of enhanced symmetries).) Specifically at the point $(T,U)=(1,1)$ the prepotential takes the form

$$f(T,U=1)=\frac{1}{\pi}(T-1)\log(T-1)^2+\Delta'(T) \tag{3.14}$$

and around $(T,U)=(\rho,\bar{\rho})$

$$f(T,U=\bar{\rho})=\frac{1}{\pi}(T-\rho)\log(T-\rho)^3+\Delta''(T), \tag{3.15}$$

where $\Delta'(T)$, $\Delta''(T)$ are finite at $T=1$, $T=\rho$ respectively. Since $f(T,U)$ is not a true modular form, but has non-trivial monodromy properties, it is not possible to determine the exact analytic form of $f(T,U)$. However the third derivative transforms nicely under target space duality transformtions, and using the informations about the order of poles and zeroes one can uniquely determine

$$\begin{aligned}\partial_T^3 f(T,U) &\propto \frac{+1}{2\pi}\frac{E_4(iT)E_4(iU)E_6(iU)\eta^{-24}(iU)}{j(iT)-j(iU)}\\ \partial_U^3 f(T,U) &\propto \frac{-1}{2\pi}\frac{E_4(iT)E_6(iU)\eta^{-24}(iT)E_4(iU)}{j(iT)-j(iU)}\end{aligned} \tag{3.16}$$

This result has recently prooved to be important to support the hypotheses [8, 12] that the quantum vector moduli space of the $N=2$ heterotic string is given by the tree level vector moduli space of an dual type II, $N=2$ string, compactified on a suitably choosen Calabi-Yau space. In addition to eq.(3.16) one can also deduce that [19, 11]

$$\partial_T\partial_U f=-\frac{2}{\pi}\log(j(iT)-j(iU))+\text{finite} \tag{3.17}$$

which has precisely the right property that the coefficient of the logarthmic singularity is proportional to the number of generically massive states that become massless.

Using eqs. (2.14) and (3.6) we can also determine the loop corrected mass formula for the $N=2$ BPS states:

$$\begin{aligned}\mathcal{M} &= M_0+M_1TU+iM_2T+iM_3U+iN^0(STU+2f-Tf_T-Uf_U)\\ &+ iN^1S+N^2(f_T-SU)+N^3(f_U-ST)\end{aligned} \tag{3.18}$$

We recognize that electric states with $N^I = 0$ do not get a mass shift at the perturbative level. It follows that the positions of the singular loci of enhanced gauge symmetries are unchanged in perturbation theory. However the masses of states with magnetic charges $N^I \neq 0$ are already shifted at the perturbative level.

3.1 Perturbative $SU(2)_{(1)}$ monodromies

Let us now consider the element σ which corresponds to the Weyl reflection in the first enhanced $SU(2)_{(1)}$.

Under the mirror transformation σ, $T \leftrightarrow U, T-U \rightarrow e^{-i\pi}(T-U)$, and the P transform classically and perturbatively as

$$P^0 \rightarrow P^0, \quad P^1 \rightarrow P^1, \quad P^2 \rightarrow P^3, \quad P^3 \rightarrow P^2 \tag{3.19}$$

The one–loop correction $f(T,U)$ transforms as[7]

$$\begin{aligned} f(T,U) &\rightarrow f(U,T) = f(T,U) - i(T-U)^2 \\ f_T(U,T) &= f_U(T,U) + 2i(T-U), \quad f_U(U,T) = f_T(T,U) - 2i(T-U) \end{aligned} \tag{3.20}$$

The f function must then have the following form for $T \rightarrow U$

$$f(T,U) = \frac{1}{\pi}(T-U)^2 \log(T-U) + \Delta(T,U) \tag{3.21}$$

with derivatives

$$\begin{aligned} f_T(T,U) &= \frac{2}{\pi}(T-U)\log(T-U) + \frac{1}{\pi}(T-U) + \Delta_T \\ f_U(T,U) &= -\frac{2}{\pi}(T-U)\log(T-U) - \frac{1}{\pi}(T-U) + \Delta_U \end{aligned} \tag{3.22}$$

$\Delta(T,U)$ has the property that it is finite as $T \rightarrow U \neq 1, \rho$ and that, under mirror symmetry $T \leftrightarrow U$, $\Delta_T \leftrightarrow \Delta_U$. The 1-loop corrected Q_2 and Q_3 are thus given by

$$\begin{aligned} Q_2 &= iSU - \frac{2i}{\pi}(T-U)\log(T-U) - \frac{i}{\pi}(T-U) - i\Delta_T \\ Q_3 &= iST + \frac{2i}{\pi}(T-U)\log(T-U) + \frac{i}{\pi}(T-U) - i\Delta_U \end{aligned} \tag{3.23}$$

It follows from (3.12) that, under mirror symmetry $T \leftrightarrow U$, the dilaton S transforms as

$$S \rightarrow S + i \tag{3.24}$$

[7]Note that one can always add polynomials of quadratic order in the moduli to a given $f(T,U)$ [15]. This results in the conjugation of the monodromy matrices. Hence, all the monodromy matrices given in the following are unique up to conjugation.

Then, it follows that perturbatively

$$\begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix} \to \begin{pmatrix} Q_3 \\ Q_2 \end{pmatrix} + \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} T \\ U \end{pmatrix} \tag{3.25}$$

Thus, the section Ω transforms perturbatively as $\Omega \to \Gamma_\infty^{w_1}\Omega$, where

$$\begin{aligned} \Gamma_\infty^{w_1} &= \begin{pmatrix} \mathcal{U} & 0 \\ \mathcal{U}\Lambda & \mathcal{U} \end{pmatrix} \quad , \quad \mathcal{U} = \begin{pmatrix} I & 0 \\ 0 & \eta \end{pmatrix} \quad , \quad \Lambda = -\begin{pmatrix} \eta & 0 \\ 0 & \mathcal{C} \end{pmatrix} \\ \eta &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \mathcal{C} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \end{aligned} \tag{3.26}$$

3.2 Truncation to the rigid case of Seiberg/Witten

In order to truncate the perturbative $SU(2)_{(1)}$ monodromy $\Gamma_\infty^{w_1}$ to the rigid one of Seiberg/Witten [1], we will take the limit $\kappa^2 = \frac{8\pi}{M_{pl}^2} \to 0$ as well as expand

$$\begin{aligned} T &= T_0 + \kappa\delta T \\ U &= T_0 + \kappa\delta U \end{aligned} \tag{3.27}$$

Here we have expanded the moduli fields T and U around the same vev $T_0 \neq 1, \rho$. Both δT and δU denote fluctuating fields of mass dimension one. We will also freeze in the dilaton field to a large vev, that is we will set $S = \langle S\rangle \to \infty$. Then, the Q_2 and Q_3 given in (3.23) can be expanded as

$$\begin{aligned} Q_2 &= i\langle S\rangle T_0 + \kappa\tilde{Q}_2 \quad , \quad Q_3 = i\langle S\rangle T_0 + \kappa\tilde{Q}_3 \\ \tilde{Q}_2 &= i\langle S\rangle\delta U - \frac{2i}{\pi}(\delta T - \delta U)\log\kappa^2(\delta T - \delta U) - \frac{i}{\pi}(\delta T - \delta U) - i\Delta_T(\delta T, \delta U) \\ \tilde{Q}_3 &= i\langle S\rangle\delta T + \frac{2i}{\pi}(\delta T - \delta U)\log\kappa^2(\delta T - \delta U) + \frac{i}{\pi}(\delta T - \delta U) - i\Delta_U(\delta T, \delta U) \end{aligned} \tag{3.28}$$

Next, one has to specify how mirror symmetry is to act on the vev's T_0 and $\langle S\rangle$ as well as on δT and δU. We will take that under mirror symmetry

$$T_0 \to T_0 \quad , \quad \delta T \leftrightarrow \delta U \quad , \quad \langle S\rangle \to \langle S\rangle \tag{3.29}$$

Note that we have taken $\langle S \rangle$ to be invariant under mirror symmetry. This is an important difference to (3.24). Using (3.29) and that $\delta T - \delta U \rightarrow e^{-i\pi}(\delta T - \delta U)$, it follows that the truncated quantitities $\tilde{Q}_2$ and $\tilde{Q}_3$ transform as follows under mirror symmetry

$$\begin{pmatrix} \tilde{Q}_2 \\ \tilde{Q}_3 \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{Q}_3 \\ \tilde{Q}_2 \end{pmatrix} + \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} \delta T \\ \delta U \end{pmatrix} \tag{3.30}$$

Defing a truncated section $\tilde{\Omega}^T = (\tilde{P}^2, \tilde{P}^3, i\tilde{Q}_2, i\tilde{Q}_3) = (i\delta T, i\delta U, i\tilde{Q}_2, i\tilde{Q}_3)$, it follows that $\tilde{\Omega}$ transforms as $\tilde{\Omega} \rightarrow \tilde{\Gamma}_\infty^{w_1} \tilde{\Omega}$ under mirror symmetry (3.29) where

$$\tilde{\Gamma}_\infty^{w_1} = \begin{pmatrix} \tilde{U} & 0 \\ \tilde{U}\tilde{\Lambda} & \tilde{U} \end{pmatrix} \ , \ \tilde{U} = \eta \ , \ \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \tilde{\Lambda} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \tag{3.31}$$

Note that, because of the invariance of $\langle S \rangle$ under mirror symmetry, $\tilde{\Lambda} \neq -\mathcal{C}$, contrary to what one would have gotten by performing a naive truncation of (3.26) consisting in keeping only rows and columns associated with (P^2, P^3, iQ_2, iQ_3).

Finally, in order to compare the truncated $SU(2)$ monodromy (3.31) with the perturbative $SU(2)$ monodromy of Seiberg/Witten [1], one has to perform a change of basis from moduli fields to Higgs fields, as follows

$$\begin{pmatrix} a \\ a_D \end{pmatrix} = M\tilde{\Omega} \ , \ M = \begin{pmatrix} m & \\ & m^* \end{pmatrix} \ , \ m = \frac{\gamma}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{3.32}$$

where γ denotes a constant to be fixed below. Then, the perturbative $SU(2)$ monodromy in the Higgs basis is given by

$$\tilde{\Gamma}_\infty^{Higgs} = M\tilde{\Gamma}_\infty^{w_1} M^{-1} = \begin{pmatrix} m\tilde{U}m^{-1} & 0 \\ m^*\tilde{U}\tilde{\Lambda}m^{-1} & m^*\tilde{U}m^T \end{pmatrix} \tag{3.33}$$

which is computed to be

$$\tilde{\Gamma}_\infty^{Higgs,w_1} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ \frac{4}{\gamma^2} & 0 & -1 & 0 \\ & & & 1 \end{pmatrix} \tag{3.34}$$

Note that (3.34) indeed correctly shows that, under the Weyl reflection in the first $SU(2)$, the second $SU(2)$ is left untouched. The fact that (3.34) reproduces this behaviour can be easily traced back to the fact that we have assumed that $\langle S\rangle$ stays invariant under the mirror transformation $\delta T \leftrightarrow \delta U$. Finally, comparing with the perturbative $SU(2)$ monodromy of Seiberg/Witten [1] yields that $\gamma^2 = 2$, whereas comparision with the perturbative $SU(2)$ monodromy of Klemm et al [4] gives that $\gamma^2 = 1$.

3.3 Relating Λ to the dilaton vev $\langle S\rangle$

In the following we will consider the rigid limit and relate the dynamically generated scale Λ of Seiberg/Witten [1] to the frozen dilaton vev $\langle S\rangle$.

We took the f function to be of the following form for $T \to U$

$$f(T,U) = \frac{1}{\pi}(T-U)^2 \log(T-U) + \Delta(T,U) \tag{3.35}$$

$\Delta(T,U)$ denotes a 1-loop contribution coming from additional heavy modes associated with $SU(2)_{(2)}$. For energies E^2 in a regime where $|\delta T + \delta U|^2 \gg |\delta T - \delta U|^2 \gg E^2 \gg \Lambda^2$, these heavier modes decouple from the low energy effective action and the 1-loop correction is due to the light modes associated with the first $SU(2)_{(1)}$, only. Then, in this regime the 1-loop contribution $\Delta(T,U)$ can be safely ignored.

The Higgs section $(a, a_D)^T = (a_1, a_2, a_{D1}, a_{D2})$ is obtained from the truncated section $\tilde{\Omega}^T = (\tilde{P}^2, \tilde{P}^3, i\tilde{Q}_2, i\tilde{Q}_3) = (i\delta T, i\delta U, i\tilde{Q}_2, i\tilde{Q}_3)$ via

$$\begin{pmatrix} a \\ \\ a_D \end{pmatrix} = M\tilde{\Omega} \ , \ \ M = \begin{pmatrix} m & \\ & m^* \end{pmatrix} \ , \ \ m = \frac{\gamma}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ \\ 1 & 1 \end{pmatrix} \tag{3.36}$$

Then

$$\begin{aligned}
a_1 &= \frac{i\gamma}{\sqrt{2}}(\delta T - \delta U) \ , \ \ a_2 = \frac{i\gamma}{\sqrt{2}}(\delta T + \delta U) \\
a_{D1} &= \frac{i}{\sqrt{2}\gamma}\left(\tilde{Q}_2 - \tilde{Q}_3\right) \\
&= \frac{1}{\sqrt{2}\gamma}\left[\langle S\rangle(\delta T - \delta U) + \frac{4}{\pi}(\delta T - \delta U)\log(\delta T - \delta U) + \frac{2}{\pi}(\delta T - \delta U)\right] \\
a_{D2} &= \frac{i}{\sqrt{2}\gamma}\left(\tilde{Q}_2 + \tilde{Q}_3\right) = -\frac{1}{\sqrt{2}\gamma}\langle S\rangle(\delta T + \delta U)
\end{aligned} \tag{3.37}$$

and consequently

$$a_{D1} = -\frac{i}{\gamma^2}\langle S\rangle a_1 - \frac{4i}{\pi\gamma^2}a_1 \log\left(\frac{\sqrt{2}}{\gamma}a_1\right) - \frac{2i}{\pi\gamma^2}a_1 - \frac{2}{\gamma^2}a_1$$

$$= \frac{i}{\gamma^2}a_1\left(-\langle S\rangle - \frac{4}{\pi}\log\left(\frac{\sqrt{2}}{\gamma}a_1\right) - \frac{2}{\pi} + 2i\right)$$

$$a_{D2} = \frac{i}{\gamma^2}\langle S\rangle a_2 \tag{3.38}$$

Setting[8]

$$a_{D1} = -\frac{4i}{\pi\gamma^2}a_1\log\left(\frac{a_1}{\Lambda}\right) - \frac{2i}{\pi\gamma^2}a_1 \tag{3.39}$$

it follows that

$$\Lambda = e^{-\frac{\pi}{4}\langle S\rangle - \log\frac{\sqrt{2}}{\gamma} + \frac{i\pi}{2}} \tag{3.40}$$

in the rigid case.

In the local case, on the other hand, the dynamically generated scale

$$\Lambda = e^{-\frac{\pi}{4}S - \log\frac{\sqrt{2}}{\gamma} + \frac{i\pi}{2}} \tag{3.41}$$

is in general not invariant under modular transformations due to an associated transformation of the dilaton S.

3.4 Perturbative $SU(2)_{(2)}$ monodromies

Under the Weyl twist w_2 in the second $SU(2)_{(2)}$, the moduli T and U transform as $T \to \frac{1}{U}, U \to \frac{1}{T}$. The section Ω transforms perturbatively as $\Omega \to \Gamma_\infty^{w_2}\Omega$. $\Gamma_\infty^{w_2}$ is conjugated to $\Gamma_\infty^{w_1}$ by $\Gamma^{(g_1)}$. Since $\Gamma^{(g_1)}$ can be taken to have no perturbative corrections [15], we get that

$$\Gamma_\infty^{w_2} = \begin{pmatrix} \mathcal{U} & 0 \\ \mathcal{U}\Lambda & \mathcal{U} \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} \eta & 0 \\ 0 & I \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -\mathcal{C} & 0 \\ 0 & -\eta \end{pmatrix}$$

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{3.42}$$

It then follows that perturbatively

$$\begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix} \to \begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix} - \begin{pmatrix} U \\ T \end{pmatrix} \tag{3.43}$$

[8]Seiberg/Witten corresponds to $\gamma^2 = 2$. Taking into account that their looping around singular points is opposite to ours gives total agreement between our and their results.

Next, let us construct 1-loop corrected Q_2 and Q_3 which have the above monodromy properties. We will show that the 1-loop correction $f(T,U)$ reproducing the perturbative monodromy (3.43) is, in the vicinity of $T = \frac{1}{U}$, given by

$$f(T,U) = -\frac{1}{\pi}(\delta T + \delta U)^2 \log(\delta T + \delta U) + \Xi(T,U) \tag{3.44}$$

where we have expanded $T = T_0(1+\delta T), U = \frac{1}{T_0}(1+\delta U)$. $\Xi(T,U)$ and its derivatives $\Xi_{T,U}$ have the property that $\Xi \to \Xi, \Xi_{T,U} \to \Xi_{T,U}$ under the linearised transformation laws $\delta T \to -\delta U, \delta U \to -\delta T, \delta T + \delta U \to e^{-i\pi}(\delta T + \delta U)$. An example of a $\Xi(T,U)$ meeting these requirements is given by $\Xi = \frac{1}{\pi}(\delta T - \delta U)^2 \log(\delta T - \delta U)$. Using (3.44), it follows that Q_2 and Q_3 are at the linearised level given by

$$\begin{aligned}
Q_2 &= iSUP^0 - iP^0 f_T \\
&= iS\frac{(1+\delta U)}{T_0} - i\left(-\frac{2}{\pi}\frac{(\delta T+\delta U)}{T_0}\log(\delta T + \delta U) - \frac{1}{\pi}\frac{(\delta T+\delta U)}{T_0} + \Xi_T\right) \\
Q_3 &= iSTP^0 - iP^0 f_U \\
&= iST_0(1+\delta T) - i\left(-\frac{2}{\pi}T_0(\delta T+\delta U)\log(\delta T + \delta U) - \frac{1}{\pi}T_0(\delta T+\delta U) + \Xi_U\right)
\end{aligned} \tag{3.45}$$

Now, under $T \to \frac{1}{U}, U \to \frac{1}{T}$, the dilaton transforms as $S \to S - i + \frac{2i}{TU} + \frac{1}{TU}(2f - Tf_T - Uf_U)$, whereas the graviphoton transforms as $P^0 \to P^1$. Linearising these transformation laws, using the properties of $\Xi_{T,U}$ given above as well as

$$2f - Tf_T - Uf_U = \frac{2}{\pi}\left(\delta T + \delta U + 2(\delta T + \delta U)\log(\delta T + \delta U)\right) \tag{3.46}$$

gives that

$$\begin{aligned}
Q_2 &\to iS\frac{(1+\delta U)}{T_0} - i\left(-\frac{2}{\pi}\frac{(\delta T+\delta U)}{T_0}\log(\delta T + \delta U) - \frac{1}{\pi}\frac{(\delta T+\delta U)}{T_0} + \Xi_T\right) - \frac{(1+\delta U)}{T_0} \\
&= Q_2 - \frac{(1+\delta U)}{T_0}
\end{aligned} \tag{3.47}$$

and similarly that

$$Q_3 \to Q_3 - T_0(1+\delta T) \tag{3.48}$$

Thus, the 1-loop correction f given in (3.44) correctly reproduces the perturbative monodromy (3.43). Note that (3.46) implies that $2\Xi - T\Xi_T - U\Xi_U = 0$ which is an independent constraint on Ξ. Again this requirement is satisfied by

$$\Xi = \frac{1}{\pi}(\delta T - \delta U)^2 \log(\delta T - \delta U) \tag{3.49}$$

at the linearized level.

3.5 Truncation to the rigid case

Next, consider truncating (3.43) to the rigid case. In the rigid case one expects to recover a second copy of the $SU(2)$-case discussed by Seiberg/Witten [1]. In order to do so, we will freeze in both the graviphoton $\langle P^0\rangle = 1$ and the dilaton $\langle S\rangle = \infty$. That is, both P^0 and S will be taking to be invariant under $\delta T \to -\delta U, \delta U \to -\delta T$. Note that, in particular, $\langle P^0\rangle = 1$ is a fixed point of $P^0 \to P^1 = TU = (1+\delta T+\delta U)$ in the local case.

Then, (3.45) can be written as

$$\begin{aligned}
Q_2 &= i\langle S\rangle\frac{1}{T_0} + \frac{1}{T_0}\tilde{Q}_2 \ , \quad Q_3 = i\langle S\rangle T_0 + T_0\tilde{Q}_3 \\
\tilde{Q}_2 &= i\langle S\rangle\delta U - i\left(-\frac{2}{\pi}(\delta T+\delta U)\log(\delta T+\delta U) - \frac{1}{\pi}(\delta T+\delta U) + \Xi_{\delta T}\right) \\
\tilde{Q}_3 &= i\langle S\rangle\delta T - i\left(-\frac{2}{\pi}(\delta T+\delta U)\log(\delta T+\delta U) - \frac{1}{\pi}(\delta T+\delta U) + \Xi_{\delta U}\right) \qquad (3.50)
\end{aligned}$$

Let as impose yet another condition on $\Xi_{T,U}$, namely that $\Xi_{\delta T} = -\Xi_{\delta U}$ at the linearised level. Note that $\Xi = \frac{1}{\pi}(\delta T - \delta U)^2\log(\delta T - \delta U)$ is an example of a Ξ meeting this additional requirement. Then, it follows that under $\delta T \to -\delta U, \delta T \to -\delta U, \delta T+\delta U \to e^{-i\pi}(\delta T+\delta U)$

$$\begin{pmatrix} \tilde{Q}_2 \\ \tilde{Q}_3 \end{pmatrix} \to \begin{pmatrix} -\tilde{Q}_3 \\ -\tilde{Q}_2 \end{pmatrix} - 2\begin{pmatrix} \delta T+\delta U \\ \delta T+\delta U \end{pmatrix} \qquad (3.51)$$

Thus, the truncated section $\tilde{\Omega}^T = (i\delta T, i\delta U, i\tilde{Q}_2, i\tilde{Q}_3)$ transforms as $\tilde{\Omega} \to \tilde{\Gamma}^{w_2}_\infty\tilde{\Omega}$ where

$$\tilde{\Gamma}^{w_2}_\infty = \begin{pmatrix} -\eta & 0 \\ -2I-2\eta & -\eta \end{pmatrix} \ , \ \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (3.52)$$

The two critical lines $T = U$ and $TU = 1$ are in the local case related by the group element g_1 which acts by $T \to 1/T$, $U \to U$. The monodromy matrices associated with the two lines are related through conjugation by $\Gamma^{(g_1)}$. This transformation permutes the two $SU(2)$ factors (outer automorphism of $SU(2)^2$) and therefore is also present in the rigid theory. Consequently we expect that the two truncated monodromies $\tilde{\Gamma}^{(w_i)}_\infty$ are also conjugated. The conjugation matrix is then the truncated version of $\Gamma^{(g_1)}$. Now the linearized section transforms classically under g_1 by

$$(i\delta T, i\delta U, i\tilde{Q}_2, i\tilde{Q}_3) \to (-i\delta T, i\delta U, i\tilde{Q}_2, -i\tilde{Q}_3) \qquad (3.53)$$

Since $\Gamma^{(g_1)}$ has been taken to have no perturbative corrections [15], we expect that this transformation law is likewise not modified in the rigid case. Then

$$\tilde{\Gamma}^{(g_1)} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = (\tilde{\Gamma}^{(g_1)})^{-1} \tag{3.54}$$

should be the truncated version of the permutation of the two $SU(2)$s. And indeed one easily verifies that $\tilde{\Gamma}_\infty^{(w_2)} = \tilde{\Gamma}^{(g_1)}\tilde{\Gamma}_\infty^{(w_1)}\tilde{\Gamma}^{(g_1)}$.

3.6 Perturbative $SU(2)^2$ monodromies

Under the combined Weyl twists $w_1 w_2$ of $SU(2)^2$, the moduli T and U transform as $T \to \frac{1}{T}, U \to \frac{1}{U}$. The section Ω transforms perturbatively as $\Omega \to \Gamma_\infty^{SU(2)^2}\Omega$, where

$$\Gamma_\infty^{SU(2)^2} = \Gamma_\infty^{w_1}\Gamma_\infty^{w_2} = \begin{pmatrix} \mathcal{U} & 0 \\ X & \mathcal{U} \end{pmatrix}, \mathcal{U} = \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix}, X = -2\begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix}, \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.55}$$

It then follows that perturbatively

$$\begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix} \to \begin{pmatrix} Q_3 \\ Q_2 \end{pmatrix} - 2\begin{pmatrix} U \\ T \end{pmatrix} \tag{3.56}$$

Inspection of (3.21) and of (3.44) shows that the 1-loop correction $f(T,U)$ reproducing the perturbative monodromy (3.56) should, in the vicinity of $T = U = 1$, be given by

$$f(T,U) = \frac{1}{\pi}\left((\delta T - \delta U)^2 \log(\delta T - \delta U) - (\delta T + \delta U)^2 \log(\delta T + \delta U)\right) \tag{3.57}$$

Note that (3.57) satisfies all the requirements imposed on $\Delta(T,U)$ and on $\Xi(T,U)$ in the previous sections. Using (3.57), it follows that Q_2 and Q_3 are at the linearised level given by

$$\begin{aligned} Q_2 &= iSUP^0 - iP^0 f_T \\ &= iS(1+\delta U) - \frac{2i}{\pi}\left((\delta T - \delta U)\log(\delta T - \delta U) - (\delta T + \delta U)\log(\delta T + \delta U) - \delta U\right) \\ Q_3 &= iSTP^0 - iP^0 f_U \\ &= iS(1+\delta T) - \frac{2i}{\pi}\left(-(\delta T - \delta U)\log(\delta T - \delta U) - (\delta T + \delta U)\log(\delta T + \delta U) - \delta T\right) \end{aligned} \tag{3.58}$$

Under $w_1 w_2$, $T \to \frac{1}{T}, U \to \frac{1}{U}$, and the dilaton transforms as $S \to S + \frac{2i}{TU} + \frac{1}{TU}(2f - Tf_T - Uf_U)$, whereas the graviphoton transforms as $P^0 \to P^1$. Linearising these transformation laws und using that

$$2f - Tf_T - Uf_U = \frac{2}{\pi}\left(\delta T + \delta U + 2(\delta T + \delta U)\log(\delta T + \delta U)\right) \tag{3.59}$$

it follows that under $\delta T \to -\delta T = e^{-i\pi}\delta T, \delta U \to -\delta U = e^{-i\pi}\delta U$

$$\begin{aligned} Q_2 &\to iS(1+\delta T) - \frac{2i}{\pi}\left(-(\delta T - \delta U)\log(\delta T - \delta U) - (\delta T + \delta U)\log(\delta T + \delta U) - \delta T\right) \\ &- 2(1+\delta U) = Q_3 - 2(1+\delta U) \end{aligned} \tag{3.60}$$

and similarly that

$$Q_3 \to Q_2 - 2(1+\delta T) \tag{3.61}$$

Thus, the 1-loop correction f given in (3.57) indeed correctly reproduces the perturbative monodromy (3.56).

3.7 Truncation to the rigid case

Next, consider truncating the above $SU(2)^2$ monodromies to the rigid case. In the rigid case one expects to recover 2 copies of the $SU(2)$-case discussed by Seiberg/Witten. As before, we will freeze in both the graviphoton and the dilaton to its fixed point values, i.e. $\langle P^0 \rangle = 1$, $\langle S \rangle = \infty$.

Then, (3.58) can be written as

$$\begin{aligned} Q_2 &= i\langle S\rangle + \tilde{Q}_2 \quad , \quad Q_3 = i\langle S\rangle + \tilde{Q}_3 \\ \tilde{Q}_2 &= i\langle S\rangle\delta U - \frac{2i}{\pi}\left((\delta T - \delta U)\log(\delta T - \delta U) - (\delta T + \delta U)\log(\delta T + \delta U) - \delta U\right) \\ \tilde{Q}_3 &= i\langle S\rangle\delta T - \frac{2i}{\pi}\left(-(\delta T - \delta U)\log(\delta T - \delta U) - (\delta T + \delta U)\log(\delta T + \delta U) - \delta T\right) \end{aligned} \tag{3.62}$$

Then, it follows that under $\delta T \to e^{-i\pi}\delta T = -\delta T, \delta U \to e^{-i\pi}\delta U = -\delta U$

$$\begin{pmatrix} \tilde{Q}_2 \\ \tilde{Q}_3 \end{pmatrix} \to \begin{pmatrix} -\tilde{Q}_2 \\ -\tilde{Q}_3 \end{pmatrix} - 4\begin{pmatrix} \delta U \\ \delta T \end{pmatrix} \tag{3.63}$$

Consequently, the truncated section $\tilde{\Omega}$ transforms as $\tilde{\Omega} \to \tilde{\Gamma}_\infty^{SU(2)^2}\tilde{\Omega}$ with

$$\tilde{\Gamma}_\infty^{SU(2)^2} = \begin{pmatrix} -I & 0 \\ -4\eta & -I \end{pmatrix} \quad , \quad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.64}$$

It can be checked that

$$\tilde{\Gamma}_\infty^{SU(2)^2} = \tilde{\Gamma}_\infty^{w_1} \tilde{\Gamma}_\infty^{w_2} \tag{3.65}$$

as it must for consistency. Finally, rotating to the Higgs basis gives that

$$\tilde{\Gamma}_\infty^{Higgs,SU(2)^2} = M \tilde{\Gamma}_\infty^{SU(2)^2} M^{-1} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ \frac{4}{\gamma^2} & 0 & -1 & 0 \\ & -\frac{4}{\gamma^2} & & -1 \end{pmatrix} \tag{3.66}$$

3.8 The first Weyl twist w_1' of $SU(3)$

Under the first Weyl twist w_1' of $SU(3)$, the moduli T and U transform as $T \to \frac{1}{U}, U \to \frac{1}{T}$. The section Ω transforms perturbatively as $\Omega \to \Gamma_\infty^{w_1'} \Omega$, where

$$\begin{aligned} \Gamma_\infty^{w_1'} &= \Gamma_\infty^{w_2} = \begin{pmatrix} \mathcal{U} & 0 \\ \mathcal{U}\Lambda & \mathcal{U} \end{pmatrix} , \; \mathcal{U} = \begin{pmatrix} \eta & 0 \\ 0 & I \end{pmatrix} , \; \Lambda = \begin{pmatrix} -\mathcal{C} & 0 \\ 0 & -\eta \end{pmatrix} \\ \eta &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \; \mathcal{C} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \end{aligned} \tag{3.67}$$

It then follows that perturbatively

$$\begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix} \to \begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix} - \begin{pmatrix} U \\ T \end{pmatrix} \tag{3.68}$$

Next, let us construct 1-loop corrected Q_2 and Q_3 which have the above monodromy properties. We will show that the 1-loop correction $f(T,U)$ reproducing the perturbative monodromy (3.68) is, in the vicinity of $T = \rho = \frac{1}{2}\sqrt{3} + \frac{i}{2}, U = \rho^{-1}$ given by

$$f(T,U) = -\frac{1}{2\pi} \left(\sum_i Z_i^2 \log Z_i - \frac{1}{2} \sum_i Z_i^2 \right) \quad , \quad i = 1,2,3 \tag{3.69}$$

where

$$\begin{aligned} Z_1 &= c\big((2-\rho^2)\delta T + (2-\rho^{-2})\delta U\big) \\ Z_2 &= c\big((2\rho^2-1)\delta T + (2\rho^{-2}-1)\delta U\big) \\ Z_3 &= c\big((\rho^2+1)\delta T + (\rho^{-2}+1)\delta U\big) \end{aligned} \tag{3.70}$$

and where we have expanded $T=\rho+\delta T, U=\rho^{-1}+\delta U$. c denotes a constant which can be determined as follows. Differentiation of (3.69) gives that

$$f_{TU}(\delta T,\delta U)=-\frac{3c^2}{\pi}\left(\log Z_1+\log Z_2+\log Z_3\right)+finite \tag{3.71}$$

It follows that

$$f_{TU}(\delta T,\delta U=0)=-\frac{9c^2}{\pi}\log\delta T+finite \tag{3.72}$$

The logarithmic singularity (3.72) should be 3 times as strong as the logarithmic singularity of the $SU(2)_1$ case given by $f_{TU}(T=U+\delta T,U)=-\frac{2}{\pi}\log\delta T$ as computed from (3.22). Thus it follows that $c^2=\frac{2}{3}$. Using (3.69), it follows that Q_2 and Q_3 are at the linearised level given by

$$\begin{aligned}
Q_2 &= iSUP^0-iP^0f_T\\
&= iS(\rho^{-1}+\delta U)+\frac{ci}{\pi}\Big((2-\rho^2)Z_1\log Z_1+(2\rho^2-1)Z_2\log Z_2+(\rho^2+1)Z_3\log Z_3\Big)\\
Q_3 &= iSTP^0-iP^0f_U\\
&= iS(\rho+\delta T)+\frac{ci}{\pi}\Big((2-\rho^{-2})Z_1\log Z_1+(2\rho^{-2}-1)Z_2\log Z_2+(\rho^{-2}+1)Z_3\log Z_3\Big)
\end{aligned} \tag{3.73}$$

Now, under $T\rightarrow\frac{1}{U}, U\rightarrow\frac{1}{T}$, the dilaton transforms as $S\rightarrow S-i+\frac{2i}{TU}+\frac{1}{TU}(2f-Tf_T-Uf_U)$, whereas the graviphoton transforms as $P^0\rightarrow P^1$. Also, at the linearised level, $\delta T\rightarrow-\rho^2\delta U, \delta U\rightarrow-\rho^{-2}\delta T$ and, consequently, $Z_1\rightarrow e^{-i\pi}Z_1=-Z_1, Z_2\leftrightarrow Z_3$. Using that $c^2=\frac{2}{3}$ and that

$$2f-Tf_T-Uf_U=\frac{c\sqrt{3}}{\pi}\left(2Z_1\log Z_1-Z_2\log Z_2+Z_3\log Z_3\right)+\mathcal{O}(\delta T\delta U) \tag{3.74}$$

it follows that at the linearised level Q_2 transforms into

$$\begin{aligned}
Q_2 &\rightarrow iS(\rho^{-1}+\delta U)-\frac{ci}{\pi}\Big((\frac{3}{2}-\frac{i}{2}\sqrt{3})Z_1\log Z_1+i\sqrt{3}Z_2\log Z_2+(\frac{3}{2}+\frac{i}{2}\sqrt{3})Z_3\log Z_3\Big)\\
&- (\rho^{-1}+\delta U)=Q_2-U
\end{aligned} \tag{3.75}$$

and similarly that

$$Q_3\rightarrow Q_3-T \tag{3.76}$$

Thus, the 1-loop correction f given in (3.69) correctly reproduces the perturbative monodromy (3.68). Note that, although the 1-loop correction f given in (3.69) differs radically from the f function for the $SU(2)_2$ case given in (3.44), both 1-loop f functions nevertheless give rise to the same perturbative monodromy matrix (3.67). This is a consequence of the nontrivial transformation law of the dilaton.

3.9 Truncation to the rigid case

Next, consider truncating the above to the rigid case. In order to do so, we will freeze in both the graviphoton and the dilaton to its fixed point values, i.e. $\langle P^0\rangle = 1$ and $\langle S\rangle = \infty$. In order to compare the truncated $SU(3)$ monodromies with the rigid monodromies of Klemm et al [4], one has to perform a change of basis from the moduli fields to the Higgs fields. The Higgs section $(a, a_D)^T = (a_1, a_2, a_{D1}, a_{D2})$ is obtained from the truncated section $\tilde{\Omega}^T = (i\delta T, i\delta U, i\tilde{Q}_2, i\tilde{Q}_3)$ via

$$\begin{pmatrix} a \\ \\ a_D \end{pmatrix} = \begin{pmatrix} m & \\ & m^* \end{pmatrix} \begin{pmatrix} \tilde{P} \\ \\ i\tilde{Q} \end{pmatrix} \quad , \quad m = -ic \begin{pmatrix} 1 & 1 \\ \\ \rho^2 & \rho^{-2} \end{pmatrix} \quad , \quad c^2 = \frac{2}{3} \tag{3.77}$$

Then, indeed,

$$\begin{aligned} Z_1 &= c\Big((2-\rho^2)\delta T + (2-\rho^{-2})\delta U\Big) = 2a_1 - a_2 \\ Z_2 &= c\Big((2\rho^2-1)\delta T + (2\rho^{-2}-1)\delta U\Big) = 2a_2 - a_1 \\ Z_3 &= c\Big((\rho^2+1)\delta T + (\rho^{-2}+1)\delta U\Big) = a_1 + a_2 \end{aligned} \tag{3.78}$$

thus precisely reproducing equation (3.9) of Klemm et al [4].

Equation (3.73) can now be written as

$$Q_2 = i\langle S\rangle\rho^{-1} + \tilde{Q}_2 \quad , \quad Q_3 = i\langle S\rangle\rho + \tilde{Q}_3 \tag{3.79}$$

with

$$\begin{aligned} \tilde{Q}_2 &= \frac{1}{\sqrt{3}c}\langle S\rangle(-a_2+\rho^2 a_1) \\ &+ \frac{ci}{\pi}\Big((2-\rho^2)Z_1\log Z_1 + (2\rho^2-1)Z_2\log Z_2 + (\rho^2+1)Z_3\log Z_3\Big) \\ \tilde{Q}_3 &= \frac{1}{\sqrt{3}c}\langle S\rangle(a_2-\rho^{-2}a_1) \\ &+ \frac{ci}{\pi}\Big((2-\rho^{-2})Z_1\log Z_1 + (2\rho^{-2}-1)Z_2\log Z_2 + (\rho^{-2}+1)Z_3\log Z_3\Big) \end{aligned} \tag{3.80}$$

Then, using (3.77) it follows that

$$\begin{aligned} a_{D1} &= \frac{i}{2}\langle S\rangle(a_2-2a_1) - \frac{i}{\pi}(2Z_1\log Z_1 - Z_2\log Z_2 + Z_3\log Z_3) \\ a_{D2} &= \frac{i}{2}\langle S\rangle(a_1-2a_2) - \frac{i}{\pi}(-Z_1\log Z_1 + 2Z_2\log Z_2 + Z_3\log Z_3) \end{aligned} \tag{3.81}$$

Writing

$$
\begin{aligned}
a_{D1} &= -\frac{i}{\pi}\left(2Z_1\log\frac{Z_1}{\Lambda} - Z_2\log\frac{Z_2}{\Lambda} + Z_3\log\frac{Z_3}{\Lambda}\right) \\
a_{D2} &= -\frac{i}{\pi}\left(-Z_1\log\frac{Z_1}{\Lambda} + 2Z_2\log\frac{Z_2}{\Lambda} + Z_3\log\frac{Z_3}{\Lambda}\right)
\end{aligned} \tag{3.82}
$$

yields that

$$
\Lambda = e^{-\frac{\langle S\rangle \pi}{6}} \tag{3.83}
$$

(3.81), on the other hand, reproduces, up to an overall minus sign, equation (3.13) of Klemm et al [4]. The Higgs fields a_1 and a_2 transform as $a_1 \rightarrow a_2 - a_1, a_2 \rightarrow a_2$ under $\delta T \rightarrow -\rho^2 \delta U, \delta U \rightarrow -\rho^{-2}\delta T$. It follows that the Higgs section transforms perturbatively as

$$
\begin{pmatrix} a \\ a_D \end{pmatrix} \rightarrow \tilde{\Gamma}_{\infty}^{Higgs,w_1'} \begin{pmatrix} a \\ a_D \end{pmatrix} \quad , \quad \tilde{\Gamma}_{\infty}^{Higgs,w_1'} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & -2 & -1 & 0 \\ -2 & 1 & 1 & 1 \end{pmatrix} \tag{3.84}
$$

which reproduces equation (3.20) of Klemm et al [4]. Note that in Klemm et al [4] one loops around singular points in the opposite way we do. Since the function we chose, equation (3.69), has an opposite overall sign as compared to their function (3.16), it follows that our and their perturbative monodromies should coincide, as they indeed do.

Finally note that, although $\Gamma_{\infty}^{w_2} = \Gamma_{\infty}^{w_1'}$ in the local case, the truncated monodromies $\tilde{\Gamma}_{\infty}^{Higgs,w_2}$ and $\tilde{\Gamma}_{\infty}^{Higgs,w_1'}$ are very different from each other. This is due to the fact that the associated 1-loop f functions are very different and that the dilaton has been frozen to $\langle S\rangle = \infty$ in the rigid case.

3.10 The second Weyl twist w_2' and the third Weyl twist w_0' of $SU(3)$

Under the second Weyl twist w_2' of $SU(3)$, the moduli T and U transform as $T \rightarrow U + i, U \rightarrow T - i$. Taking as the 1-loop corrected function $f(T,U)$ the one given in (3.69), it can be checked using (3.12) that $S \rightarrow S + i$. Then, indeed, the 1-loop corrected Kähler potential $K = -\log(Y_{tree} + Y_{pert})$ is invariant under w_2'. The resulting perturbative monodromy $\Gamma_{\infty}^{w_2'}$ is then given by

$$
\Gamma_{\infty}^{w_2'} = \begin{pmatrix} U_{w_2'} & 0 \\ U_{w_2'}^* \Lambda_{w_2'} & U_{w_2'}^* \end{pmatrix} \tag{3.85}
$$

where $U_{w_2'}$ is given in (2.13) and where

$$\Lambda_{w_2'} = \begin{pmatrix} -2 & -1 & -2 & 2 \\ -1 & 0 & 0 & 0 \\ -2 & 0 & -2 & 1 \\ 2 & 0 & 1 & -2 \end{pmatrix} \tag{3.86}$$

The perturbative monodromy $\Gamma_\infty^{w_0'}$ associated with the third Weyl twist is obtained from $\Gamma_\infty^{w_1'}$ by conjugation as

$$\Gamma_\infty^{w_0'} = \left(\Gamma_\infty^{w_2'}\right)^{-1} \Gamma_\infty^{w_1'} \Gamma_\infty^{w_2'} = \begin{pmatrix} U_{w_0'} & 0 \\ U^*_{w_0'} \Lambda_{w_0'} & U^*_{w_0'} \end{pmatrix} \tag{3.87}$$

where $U_{w_0'}$ is given in (2.13) and where

$$\Lambda_{w_0'} = \begin{pmatrix} 0 & -3 & -2 & 2 \\ -3 & 0 & -2 & 2 \\ -2 & -2 & -4 & 3 \\ 2 & 2 & 3 & -4 \end{pmatrix} \tag{3.88}$$

Truncation to the rigid case is again achieved by freezing in both the graviphoton and the dilaton, i.e. $\langle P^0 \rangle = 1, \langle S \rangle = \infty$. Due to the choice (3.69) of the 1-loop correction $f(T, U)$, the resulting rigid monodromy matrices for the second and the third Weyl twists are again the ones given in equation (3.20) of [4].

3.11 Summary

In summary, the complete semiclassical monodromy is given by the product of the four Weyl-reflection monodromies times the monodromy matrix eq. (3.9) which corresponds to the discrete shifts in the dilaton field. In the following we will show how the four perturbative monodromies associated with the enhancement of gauge symmetries are to be decomposed into non-perturbative monodromies due to monopoles and dyons becoming massless at points in the interior of moduli space.

4 Non perturbative monodromies

4.1 General remarks

In order to obtain some information about non-perturbative monodromies in $N = 2$ heterotic string compactifications, we will follow Seiberg/Witten's strategy in the rigid case [1] and try to decompose the perturbative monodromy matrices Γ_∞ into $\Gamma_\infty = \Gamma_m \Gamma_d$ with Γ_m (Γ_d) possessing monopole like (dyonic) fixed points. Thus each semi-classical singular line will split into two non-perturbative singular lines where magnetic monopoles or dyons respectively become massless. In doing so we will work in the limit of large dilaton field S assuming that in this limit the non-perturbative dynamics is dominated by the Yang-Mills gauge forces. Nevertheless, the monodromy matrices we will obtain are not an approximation in any sense, since the monodromy matrices are of course field independent. They are just part of the full quantum monodromy of the four-dimensional heterotic string.

Let us now precisely list the assumptions we will impose when performing the split of any of the semiclassical monodromies into the non-perturbative ones:

1. Γ_∞ must be decomposed into precisely two factors.

$$\Gamma_\infty = \Gamma_M \Gamma_D \tag{4.1}$$

2. Γ_M and therefore Γ_D must be symplectic.

3. Γ_M must have a unique monopole like fixed point. For the case of w_1, for instance, it must be of the form

$$(N, -M) = \left(0, 0, N^2, -N^2, 0, 0, 0, 0\right) \tag{4.2}$$

4. Γ_D must have a unique dyonic fixed point. For the case of w_1, for instance, it must be of the form

$$(N, -M) = \left(0, 0, N^2, -N^2, 0, 0, -M_2, M_2\right) \tag{4.3}$$

   where N^2 and M_2 are proportional.

5. Γ_M and Γ_D should be conjugated, that is, they must be related by a change of basis, as it is the case in the rigid theory.

6. The limit of large S should be respected. This means that S should only transform into a function of T and U (for at least one of the four $SU(2)$ lines, as will be discussed in the following).

In the following we will show that under these assumptions the splitting can be performed in a consistent way. We will discuss the non perturbative monodromies for the $SU(2)_{(1)}$ case in big detail. Unlike the rigid case, however, where the decomposition of the perturbative monodromy into a monopole like monodromy and a dyonic monodromy is unique (up to conjugation), it will turn out that there are several distinct decompositions, depending on four (discrete) parameters. Only a subset of these distinct decompositions should be, however, the physically correct one. One way of deciding which one is the physically correct one is to demand that, when truncating this decomposition to the rigid case, one recovers the rigid non perturbative monodromies of Seiberg/Witten. This, however, requires one to have a reasonable prescription of taking the flat limit, and one such prescription was given in section (3.2).

The non-perturbative part $f^{\rm NP}$ of the prepotential will depend on the S-field. We will make the following ansatz for the prepotential

$$F = i\frac{X^1X^2X^3}{X^0} + (X^0)^2\left(f(T,U) + f^{\rm NP}(S,T,U)\right) \tag{4.4}$$

Then the non-perturbative period vector $\Omega^T = (P, iQ)^T$ takes the form

$$\begin{aligned}\Omega^T &= (1, TU - f_S^{\rm NP}, iT, iU, iSTU + 2i(f + f^{\rm NP}) - iT(f_T + f_T^{\rm NP}) - iU(f_U + f_U^{\rm NP}) \\ &- iSf_S^{\rm NP}, iS, -SU + f_T + f_T^{\rm NP}, -ST + f_U + f_U^{\rm NP})\end{aligned} \tag{4.5}$$

This leads to the following non-perturbative mass formula for the BPS states

$$\begin{aligned}\mathcal{M} &= M_IP^I + iN^IQ_I = M_0 + M_1(TU - f_S^{\rm NP}) + iM_2T + iM_3U + iN^0(STU \\ &+ 2(f + f^{\rm NP}) - T(f_T + f_T^{\rm NP}) - U(f_U + f_U^{\rm NP}) - Sf_S^{\rm NP}) + iN^1S \\ &+ iN^2(iSU - if_T - if_T^{\rm NP}) + iN^3(iST - if_U - if_U^{\rm NP})\end{aligned} \tag{4.6}$$

Then we see that all states with $M_1 \neq 0$ or $N^I \neq 0$ undergo a non-perturbative mass shift. In the following we will use this formula to determine (as a function of $f_{\rm NP}$ and its derivatives) the singular loci where monopoles or dyons become massless. This will, for concreteness, be done for the case of $SU(2)_{(1)}$.

4.2 Non perturbative monodromies for $SU(2)_{(1)}$

In order to find a decomposition of $\Gamma_\infty^{w_1}$, $\Gamma_\infty^{w_1} = \Gamma_M^{w_1}\Gamma_D^{w_1}$, we will now make the following ansatz:

1. $\Gamma_\infty^{w_1}$ has a peculiar block structure in that the indices $j = 0, 1$ of the section (P_j, iQ_j) are never mixed with the indices $j = 2, 3$. We will assume that $\Gamma_M^{w_1}$ and $\Gamma_D^{w_1}$ also have

this structure. This implies that the problem can be reduced to two problems for 4×4 matrices. The existence of a basis where the non–perturbative monodromies have this special form will be aposteriori justified by the fact that it leads to a consistent truncation to the rigid case.

2. We will take $\Gamma^{w_1}_M$ to have two $-U_\sigma$ blocks on its diagonal, which is necessary in order to insure that the eigenvectors have the correct form.

Then, let us first consider the submatrix of $\Gamma^{w_1}_\infty$ which acts on $(P^2, P^3, iQ_2, iQ_3)^T$. We will show that its decomposition into non-perturbative pieces is almost unique. More precisely, there will be a one parameter family of decompositions, as follows. The submatrix of $\Gamma^{w_1}_\infty$ acting on $(P^2, P^3, iQ_2, iQ_3)^T$ is given by

$$\Gamma_{\infty,23} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 1 \\ -2 & 1 & 1 & 0 \end{bmatrix} \tag{4.7}$$

It will be decomposed into $\Gamma_{\infty,23} = \Gamma_{M,23}\Gamma_{D,23}$. As stated above, we will make the following ansatz for the monopole monodromy matrix $\Gamma_{M,23}$

$$\Gamma_{M,23} = \begin{bmatrix} 0 & -1 & a & b \\ -1 & 0 & c & d \\ p & q & 0 & -1 \\ r & s & -1 & 0 \end{bmatrix} \tag{4.8}$$

The existence of an eigenvector of the form $(1, -1, 0, 0)$ implies that $p = q, r = s$, whereas symplecticity implies $r = p, a = -b = -c = d$. Thus

$$\Gamma_{M,23} = \begin{bmatrix} 0 & -1 & a & -a \\ -1 & 0 & -a & a \\ p & p & 0 & -1 \\ p & p & -1 & 0 \end{bmatrix} \tag{4.9}$$

Computing the eigenvectors we find that the monopole fixed point is unique (though the eigenvalue 1 has multiplicity 2). Thus, $\Gamma_{M,23}$ appears to be reasonable. Computing $\Gamma_{D,23}$ we find

$$\Gamma_{D,23} = \begin{bmatrix} -1-3\,a & 3\,a & a & -a \\ 3\,a & -1-3\,a & -a & a \\ -p+2 & -p-1 & -1 & 0 \\ -p-1 & -p+2 & 0 & -1 \end{bmatrix} \tag{4.10}$$

Requiring the existence of a dyonic fixed point of $\Gamma_{D,23}$ fixes $a = -\frac{2}{3}$. Moreover one automatically gets that $-M_2 = \frac{3}{2}N^2$. Hence

$$\Gamma_{M,23} = \begin{bmatrix} 0 & -1 & -2/3 & 2/3 \\ -1 & 0 & 2/3 & -2/3 \\ p & p & 0 & -1 \\ p & p & -1 & 0 \end{bmatrix}, \quad \Gamma_{D,23} = \begin{bmatrix} 1 & -2 & -2/3 & 2/3 \\ -2 & 1 & 2/3 & -2/3 \\ -p+2 & -p-1 & -1 & 0 \\ -p-1 & -p+2 & 0 & -1 \end{bmatrix} \tag{4.11}$$

For $p \neq 0$ these matrices are conjugated, because they have the same Jordan normal form. This is, however, not the case if $p = 0$. Naively one might have expected this to be the natural choice because it makes $\Gamma_{M,23}$ block triangular. But in the case of $p = 0$, $\Gamma_{M,23}$ has an additional eigenvector, whereas $\Gamma_{D,23}$ doesn't have one, and hence the matrices are not conjugated.

Next, consider the submatrix of $\Gamma_\infty^{w_1}$ which acts on $(P^0, P^1, iQ_0, iQ_1)^T$. Its decomposition is less constrained. One has to decompose this perturbative submatrix such that there are no fixed points with non–vanishing N^i, M_i, $i = 0, 1$ quantum numbers and such that the decomposition is symplectic. Since we are in the perturbative regime with respect to S, namely at $S = \infty$, we are not looking for non–perturbative effects in the graviton/dilaton sector, but only for non–perturbative effects in the gauge sector. Thus, the decomposition of $\Gamma_{\infty,01}$ should be of the perturbative type.

This, on the other hand, gives a three parameter family of decompositions of the perturbative monodromy $\Gamma_{\infty,01}$, namely

$$\Gamma_{\infty,01} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ x & y & -1 & 0 \\ y & v & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -x & -y+1 & -1 & 0 \\ -y+1 & -v & 0 & -1 \end{bmatrix} \tag{4.12}$$

where both parts have no fixed point and are conjugated to each other, because they have the same Jordan normal form.

Putting all these things together yields the following 8×8 non–perturbative monodromy matrices that consistently describe the splitting of the $T = U$ line

$$\Gamma_M^{w_1} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -2/3 & 2/3 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2/3 & -2/3 \\ x & y & 0 & 0 & -1 & 0 & 0 & 0 \\ y & v & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & p & p & 0 & 0 & 0 & -1 \\ 0 & 0 & p & p & 0 & 0 & -1 & 0 \end{bmatrix} \tag{4.13}$$

$$
\Gamma_D^{w_1} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & -2/3 & 2/3 \\ 0 & 0 & -2 & 1 & 0 & 0 & 2/3 & -2/3 \\ -x & -y+1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -y+1 & -v & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -p+2 & -p-1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -p-1 & -p+2 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{4.14}
$$

The associated fixed points have the form

$$
(N, -M) = \Big(0, 0, N^2, -N^2, 0, 0, 0, 0\Big) \tag{4.15}
$$

for the monopole and

$$
(N, -M) = \left(0, 0, N^2, -N^2, 0, 0, \frac{3}{2}N^2, -\frac{3}{2}N^2\right) \tag{4.16}
$$

for the dyon.

4.3 Truncating the $SU(2)_{(1)}$ monopole monodromy to the rigid case

The monopole monodromy matrix for the first $SU(2)$, given in equation (4.13), depends on 4 undetermined parameters, namely x, y, v and $p \neq 0$. Note that demanding the monopole monodromy matrix to be conjugated to the dyonic monodromy matrix led to the requirement $p \neq 0$.

On the other hand, it follows from (4.13) that

$$
S \to S + i\Big(y + v(TU - f_S^{\mathrm{NP}})\Big) \tag{4.17}
$$

Consider now the 4×4 monopole subblock given in (4.11)

$$\Gamma^{w_1}_{M23} = \begin{bmatrix} 0 & -1 & -2\alpha & 2\alpha \\ -1 & 0 & 2\alpha & -2\alpha \\ p & p & 0 & -1 \\ p & p & -1 & 0 \end{bmatrix} \quad , \quad \alpha = \frac{1}{3} \quad , \quad p \neq 0 \tag{4.18}$$

Rotating it into the Higgs basis gives that

$$\Gamma^{Higgs,w_1}_{M} = M\Gamma^{w_1}_{M23}M^{-1} = \begin{bmatrix} 1 & 0 & -4\alpha\gamma^2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{2p}{\gamma^2} & 0 & -1 \end{bmatrix} \quad , \quad \alpha = \frac{1}{3} \quad , \quad p \neq 0 \tag{4.19}$$

where M is given in equation (3.32). In the rigid case, on the other hand, one expects to find for the rigid monopole monodromy matrix in the Higgs basis that

$$\tilde{\Gamma}^{Higgs,w_1}_{M} = \begin{bmatrix} 1 & 0 & -4\tilde{\alpha}\gamma^2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{2\tilde{p}}{\gamma^2} & 0 & -1 \end{bmatrix} \quad , \quad \tilde{\alpha} = \frac{1}{4} \quad , \quad \tilde{p} = 0 \tag{4.20}$$

The first and third lines of (4.20) are, for $\tilde{\alpha} = \frac{1}{4}$, nothing but the monodromy matrix for one $SU(2)$ monopole ($\gamma^2 = 2$ in the conventions of Seiberg/Witten [1] , and $\gamma^2 = 1$ in the conventions of Klemm et al [4]). Line 2 shows that $a_2 \to -a_2$, which is necessary in order to insure that the monopoles associated with $SU(2)_{(1)}$ and $SU(2)_{(2)}$ do not simultaneously become massless.

Thus, truncating the monopole monodromy matrix (4.13) to the rigid case appears to produce jumps in the parameters $p \to \tilde{p} = 0$ and $\alpha \to \tilde{\alpha}$ as given above. In the following we would like to present a field theoretical explanation for the jumps occuring in the parameters p and α when taking the rigid limit.

In the perturbative regime, that is at energies E^2 satisfying $|\delta T+\delta U|^2 \gg |\delta T-\delta U|^2 \gg E^2 \gg \Lambda^2$, we saw in subsection (3.3) that a_{D1} and a_{D2} were given by

$$\begin{aligned} a_{D1} &= -\frac{i}{\gamma^2}Sa_1 - \frac{4i}{\pi\gamma^2}a_1\log\left(\frac{\sqrt{2}}{\gamma}a_1\right) - \frac{2i}{\pi\gamma^2}a_1 - \frac{2}{\gamma^2}a_1 \\ a_{D2} &= \frac{i}{\gamma^2}Sa_2 \end{aligned} \tag{4.21}$$

Note that a_{D2} didn't get any 1-loop correction in this regime. On the other hand, as $E^2 \to \Lambda^2$, non-perturbative corrections become important. For a_{D2} one expects these non-perturbative corrections to be given by [1]

$$a_{D2} = \frac{i}{\gamma^2}Sa_2 + \sum_{k\geq 1}\mathcal{F}_k\left(\frac{\Lambda}{a_2}\right)^{4k}a_2^2 \tag{4.22}$$

However, since $|a_2 \propto \delta T+\delta U| \gg \Lambda$, it follows that the non-perturbative corrections to a_{D2} can here also be ignored, that is $a_{D2} = \frac{i}{\gamma^2}Sa_2$ in the regime under consideration. For a_{D1}, on the other hand, the non-perturbative corrections become important when $E^2 \to \Lambda^2$.

Now, under the monopole monodromy (4.13) the dilaton shifts as in (4.17), whereas $a_2 \to -a_2$ as can be seen from (4.19). Then it follows that

$$\begin{aligned} a_{D2} &= \frac{i}{\gamma^2}Sa_2 \to -\frac{i}{\gamma^2}\left(S+i\left[y+v(TU-f_S^{\mathrm{NP}})\right]\right)a_2 \\ &= -a_{D2} + \frac{1}{\gamma^2}\left[y+v(TU-f_S^{\mathrm{NP}})\right]a_2 \end{aligned} \tag{4.23}$$

Comparing with (4.19) shows that $v=0, 2p=y$ for consistency. Next, consider taking the rigid limit by freezing in the dilaton to $\langle S\rangle$. Then, under $a_2 \to -a_2$ it follows that

$$a_{D2} = \frac{i}{\gamma^2}\langle S\rangle a_2 \to -\frac{i}{\gamma^2}\langle S\rangle a_2 = -a_{D2} \tag{4.24}$$

Thus, due to the freezing in of the dilaton field, one finds that $p \neq 0 \to \tilde{p} = 0$!

Next, consider the dynamically generated scale Λ which, in the local case, is given by

$$\Lambda = e^{-\frac{\pi}{4}S - \log\frac{\sqrt{2}}{\gamma} + \frac{i\pi}{2}} \tag{4.25}$$

Under (4.17), it follows that $\log\Lambda$ transforms into

$$\log\Lambda \to \log\Lambda - \frac{i\pi}{4}\left(y+v(TU-f_S^{\mathrm{NP}})\right) \tag{4.26}$$

which for $v=0$ turns into

$$\log\Lambda \to \log\Lambda - \frac{i\pi}{4}y \tag{4.27}$$

In the rigid case, as $E^2 \to \Lambda^2$, a_1 was determined by Seiberg/Witten to be given by

$$\begin{aligned} a_1 &= constant - \frac{2i\tilde{\alpha}\gamma^2}{\pi} a_{D1} \log \frac{a_{D1}}{\Lambda} \quad , \quad \tilde{\alpha} = \frac{1}{4} \\ a_{D1} &= c_0(u - \Lambda) \end{aligned} \tag{4.28}$$

Note that $\frac{a_{D1}}{\Lambda}$ is dimensionless. Indeed, as $u - \Lambda \to e^{-2i\pi}(u - \Lambda)$, $a_{D1} \to e^{-2i\pi} a_{D1}$, it follows that

$$\begin{aligned} a_1 &\to a_1 - 4\tilde{\alpha}\gamma^2 \, a_{D1} \\ a_{D1} &\to a_{D1} \end{aligned} \tag{4.29}$$

which is consistent with (4.20). The 1-loop contribution to a_1 can also be understood as arising from a Feynman graph in the dual theory with 2 external magnetic photon lines and a light monopole hypermultiplet of mass $m \propto a_{D1}$ running in the loop. The 1-loop beta function coefficient is proportional to $\tilde{\alpha}$.

In the local case, on the other hand, nothing changes in the computation of this magnetic Feynman graph. Thus, in the local case one has again that

$$\begin{aligned} a_1 &= constant - \frac{2i\tilde{\alpha}\gamma^2}{\pi} a_{D1} \log \frac{a_{D1}}{\Lambda} \quad , \quad \tilde{\alpha} = \frac{1}{4} \\ a_{D1} &= c_0(u - \Lambda) \end{aligned} \tag{4.30}$$

A crucial difference, however, arises in that the dynamically generated scale Λ now transforms as well under modular transformations, namely as given in (4.27). Then, it follows that

$$\begin{aligned} a_1 &\to a_1 - \frac{2i\tilde{\alpha}\gamma^2}{\pi}(-2i\pi + \frac{i\pi}{4} y) a_{D1} = a_1 - 4\alpha\gamma^2 \, a_{D1} \\ a_{D1} &\to a_{D1} \end{aligned} \tag{4.31}$$

where $\alpha = \tilde{\alpha}(1 - \frac{y}{8})$. Thus, one sees that the jump in $\alpha \to \tilde{\alpha}$ when taking the rigid limit is a direct consequence of the freezing in of the dilaton. Finally, with $\tilde{\alpha} = \frac{1}{4}$ and $\alpha = \frac{1}{3}$ it follows that $y = -\frac{8}{3}$ and that $p = -\frac{4}{3}$.

Thus, we have given a field theoretical explanation for the jumping occuring in certain parameters when taking the rigid limit. As a bonus we have also been able to determine the value of the parameters v, y and p. Moreover, one can show that, in order to decouple the four $U(1)$'s at the non-perturbative level, one has to have $x = v$ and consequently $x = 0$. Note that $v = 0$ ensures that $S \to S + iy$ under the $SU(2)_{(1)}$ monopole monodromy.

4.4 Singular loci for $SU(2)_{(1)}$

Let us consider the Weyl twist w_1 in the first $SU(2)$. The associated monopole eigenvector has non vanishing quantum numbers $N^3 = -N^2$. Then, it follows from (4.6) that its mass vanishes for $Q_2 = Q_3$, which gives that

$$iS(T-U) - i(f_T - f_U) - i(f_T^{\mathrm{NP}} - f_U^{\mathrm{NP}}) = 0 \tag{4.32}$$

Under the monopole monodromy (4.13), it follows that

$$\begin{aligned} T &\to U + \frac{2}{3}(Q_2 - Q_3) \\ U &\to T - \frac{2}{3}(Q_2 - Q_3) \end{aligned} \tag{4.33}$$

Then, on the locus of vanishing monopole masses (4.32), one has that $T \leftrightarrow U$.

The associated dyon eigenvector, on the other hand, has non vanishing quantum numbers $M_3 = -M_2 = \frac{3}{2}N^2, N^3 = -N^2$. Then, it follows from (4.6) that its mass vanishes for

$$T - U = \frac{2}{3}(Q_2 - Q_3) \tag{4.34}$$

Under the dyon monodromy (4.14), it follows that

$$\begin{aligned} T &\to -T + 2U + \frac{2}{3}(Q_2 - Q_3) \\ U &\to -U + 2T - \frac{2}{3}(Q_2 - Q_3) \end{aligned} \tag{4.35}$$

On the locus of vanishing dyon masses (4.34) one then has again that $T \leftrightarrow U$.

Thus, on the vanishing locus of the monopoles and dyons associated with $SU(2)_{(1)}$ one recovers the classical modular transformation law $\frac{P^2}{P^0} \leftrightarrow \frac{P^3}{P^0}$ or, equivalently, $T \leftrightarrow U$.

Similar considerations can be made for any of the other 3 $SU(2)$ lines.

4.5 Non perturbative decomposition of the other 3 $SU(2)$ lines

As discussed in section 2, the perturbative monodromy matrices associated with the 4 $SU(2)$ lines are conjugated to each other by the generators σ, g_1 and g_2. Then, it follows that the non-perturbative decomposition of any of the perturbative monodromies associated with w_2, w_1' and w_2' is conjugated to the non-perturbative decomposition given above for $\Gamma_\infty^{w_1}$. For concreteness, we will below show how the non-perturbative monodromies of $SU(2)_{(2)}$ can be obtained from the ones of $SU(2)_{(1)}$ by conjugation with the generator g_1. We will find one additional monopole and one additional dyon eigenvector for $SU(2)_{(2)}$. An analogous decomposition of the remaining perturbative matrices associated with w_2' and w_0' leads to 1 additional monopole and to 3 additional dyons. Thus, similarly to what one has in the rigid case, one finds 2 monopoles and 2 dyons for the case of $SU(2)_{(1)} \times SU(2)_{(2)}$, whereas for the $SU(3)$ case one finds 2 monopole and 4 dyon eigenvectors, which are conjugated to each other [3, 4].

The explicit matrix representation of the generator g_1 is

$$\Gamma^{(g_1)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \qquad (4.36)$$

where $\left(\Gamma^{(g_1)}\right)^2 = -I$. The perturbative and non-perturbative monodromies for $SU(2)_{(2)}$ are obtained from the monodromies of $SU(2)_{(1)}$ by conjugation with $\Gamma^{(g_1)}$, $\Gamma^{(w_2)}_{\infty,M,D} =$

$(\Gamma^{(g_1)})^{-1}\Gamma^{(w_1)}_{\infty,M,D}\Gamma^{(g_1)}$. They are computed to be

$$\Gamma^{(w_2)}_{\infty} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.37}$$

$$\Gamma^{(w_2)}_{M} = \begin{bmatrix} 0 & -1 & 0 & 0 & -2/3 & 2/3 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2/3 & -2/3 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ p & p & 0 & 0 & 0 & -1 & 0 & 0 \\ p & p & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & x & y & 0 & 0 & -1 & 0 \\ 0 & 0 & y & v & 0 & 0 & 0 & -1 \end{bmatrix} \tag{4.38}$$

$$
\Gamma_D^{(w_2)} = \begin{bmatrix} 1 & -2 & 0 & 0 & -2/3 & 2/3 & 0 & 0 \\ -2 & 1 & 0 & 0 & 2/3 & -2/3 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -p+2 & -p-1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -p-1 & -p+2 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -x & -y+1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -y+1 & -v & 0 & 0 & 0 & -1 \end{bmatrix} \tag{4.39}
$$

First note that now P_0 transforms into some Q_i, and therefore the constraint $S = \infty$ seems to be violated. However, since now something non–trivial has to happen with the quantum numbers N_0, N_1 which are related to the magnetic quantum numbers of $SU(2)_{(2)}$, it is inevitable, that some non–vanishing entries appear at that place. Moreover, the physics should be the same as on the line $T = U$ because both sets of matrices are conjugated by a perturbative monodromy transformation.

The associated fixed points have the expected form, namely

$$
(N, -M) = \left(-N^2, N^2, 0, 0, 0, 0, 0, 0\right) \tag{4.40}
$$

for the monopole and

$$
(N, -M) = \left(-N^2, N^2, 0, 0, -\frac{3}{2}N^2, \frac{3}{2}N^2, 0, 0\right) \tag{4.41}
$$

for the dyon.

5 Conclusions

We have shown in the context of four-dimensional heterotic strings that the semiclassical monodromies associated with lines of enhanced gauge symmetries can be consistently split into pairs of non-perturbative lines of massless monopoles and dyons. Furthermore, all monodromies obtained in the context allow for a consistent truncation to the rigid monodromies of [1, 3, 4]. It would be very interesting to compare the monodromies obtained on the heterotic side with computations on the type II side of monodromies in appropriately chosen Calabi-Yau spaces.

In this paper we have not addressed the splitting of the semiclassical monodromy (3.9), associated with discrete shifts in the S field, into non-perturbative monodromies. If indeed such a splitting occurs, then it should be due to new gravitational stringy non-perturbative effects occuring at finite S, i.e. $S \approx 1$.

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