SUPERFIELD FORMULATION OF THE LAGRANGIAN BRST QUANTIZATION METHOD

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Lagragian quantization rules for general gauge theories are proposed on a basis of a superfield formulation of the standard BRST symmetry. Independence of the S-matrix on a choice of the gauge is proved. The Ward identities in terms of superfields are derived.

1 Introduction

The majority of field models are based, as a rule, on the fundamental principle of gauge invariance. On the quantum level it leads to the fact that there exists a special type of global supersymmetry, i.e. BRST symmetry,[1,2] underlying the advanced covariant quantization methods for gauge theories.[3–5]

It appears quite natural to refer to papers 4, 5 as the ones to give the most general form of the corresponding quantization rules. The antisymplectic manifold of the BV method[4,5] contains the fields ϕ^A (including the initial classical fields, the ghosts, the antighosts and the Lagrangian multipliers) with assigned to them antifields ϕ^*_A of the opposite Grassmann parity, the usual sources J_A to the fields ϕ^A and finally, the auxiliary fields λ^A, introducing the gauge to the theory.

As shown by Witten,[6] the covariant quantization formalism allows a geometrical interpretation.

In turn, the Yang–Mills type theories permit one to realize the BRST symmetry transformations in superspace.[7–10] This being said, the crucial

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point of the formulations[7−10] is the structure of configuration space of the theories concerned. On the other hand, a closed form of the Lagrangian quantization rules for general gauge theories that would enable one to give the BRST transformations a completely geometrical description has not yet been found.

The purpose of this paper is formulation of the Lagrangian quantization rules on a basis of a superfield approach, revealing the geometrical contents of the BRST symmetry.

We use the condensed notations suggested by De Witt[11]; derivatives with respect to (super)fields are understood as the right-hand and those with respect to (super-)antifields as the left-hand ones. Left derivatives with respect to (super)fields are labelled by the subscript "l". The Grassmann parity of a certain quantity A is denoted $\varepsilon(A)$.

2 Basic Definitions

Let us consider superspace $D+1$, parametrized by coordinates (x^μ, θ); x^μ are the space-time coordinates, $\mu = (0, 1, \ldots, D-1)$; θ is a scalar Grassmann coordinate. Let $\Phi^A(\theta)$ be a set of superfields and $\Phi^*_A(\theta)$ be a set of the corresponding super-antifields

$$\varepsilon(\Phi^A) \equiv \varepsilon_A, \qquad \varepsilon(\Phi^*_A) = \varepsilon_A + 1. \tag{1}$$

In terms of the superfields and super-antifields we define an antibracket by the rule

$$\begin{aligned}(F, G) \;=\; & \int d\theta \Big\{ \frac{\delta F}{\delta \Phi^A(\theta)} \frac{\partial}{\partial \theta} \frac{\delta G}{\delta \Phi^*_A(\theta)} (-1)^{\varepsilon_A + 1} \\ & -(-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)} (F \leftrightarrow G) \Big\} ,\end{aligned} \tag{2}$$

where $F = F[\Phi, \Phi^*]$, $G = G[\Phi, \Phi^*]$ are arbitrary functionals depending on supervariables. From the definition (2) of the antibracket there follow the properties

$$\varepsilon((F, G)) = \varepsilon(F) + \varepsilon(G) + 1 \, ,$$

$$(F,G) = -(G,F)(-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)} ,$$
$$(F,GH) = (F,G)H + (F,H)G(-1)^{\varepsilon(G)\varepsilon(H)} ,$$
$$((F,G),H)(-1)^{(\varepsilon(F)+1)(\varepsilon(H)+1)} + \text{cycl.perm.}(F,G,H) \equiv 0 . \quad (3)$$

The last relation is the generalized Jacobi identity for the antibracket.

Let us also introduce operators Δ, V of the form

$$\Delta = -\int d\theta (-1)^{\varepsilon_A} \frac{\delta_l}{\delta\Phi^A(\theta)} \frac{\partial}{\partial\theta} \frac{\delta}{\delta\Phi^*_A(\theta)} \quad (4)$$

$$V = -\int d\theta \Big\{ \frac{\partial\Phi^*_A(\theta)}{\partial\theta} \frac{\delta}{\delta\Phi^*_A(\theta)} + \frac{\partial\Phi^A(\theta)}{\partial\theta} \frac{\delta_l}{\delta\Phi^A(\theta)} \Big\}, \quad (5)$$

Integration over the coordinate θ is defined by

$$\int d\theta \cdot 1 = 0 , \quad \int d\theta \cdot \theta = 1 ,$$

derivatives with respect to θ are always undestood as the left-hand ones.

The algebra of operators (4),(5) has the form

$$\Delta^2 = 0 , \quad V^2 = 0 , \quad V\Delta + \Delta V = 0 . \quad (6)$$

The action of the operators Δ, V upon the antibracket is given by the following relations

$$\Delta(F,G) = (\Delta F,G) - (-1)^{\varepsilon(F)}(F,\Delta G) ,$$
$$V(F,G) = (VF,G) - (-1)^{\varepsilon(F)}(F,VG) . \quad (7)$$

3 Quantization Rules

Let us now define the generating functional of Green's functions as a functional depending on the super-antifields $Z = Z[\Phi^*]$ in the form

$$Z[\Phi^*] = \int d\Phi^{'} \, d\Phi^{*'} \rho[\Phi^{*'}] \exp\Big\{ \frac{i}{\hbar}\Big(S[\Phi^{'}, \Phi^{*'}] - V^{'}\Psi[\Phi^{'}] - \Phi^*\Phi^{'} \Big)\Big\}, \quad (8)$$

In (8) $S = S[\Phi, \Phi^*]$ is a quantum action satisfying the generating equation

$$\frac{1}{2}(S,S) + VS = i\hbar\Delta S \quad (9)$$

with the boundary condition

$$S|_{\Phi^*=\hbar=0} = \mathcal{S} \ , \tag{10}$$

where $\mathcal{S}$ is a classical gauge invariant action; $\Psi = \Psi[\Phi]$ is a fermion functional introducing the gauge; $\hbar$ is a Plank constant. Besides, the following notations

$$\rho[\Phi^*] = \delta\Big(\int d\theta \Phi^*(\theta)\Big), \qquad \Phi^*\Phi = \int d\theta \Phi^*_A(\theta)\Phi^A(\theta) \tag{11}$$

are used.

An important property of the integrand in (8) for $\Phi^* = 0$ is its invariance under the following global supersymmetry transformations with a constant Grassmann parameter μ

$$\begin{aligned} \delta\Phi^A(\theta) &= \mu\frac{\partial}{\partial\theta}\Phi^A(\theta) \ , \\ \delta\Phi^*_A(\theta) &= \mu\frac{\partial}{\partial\theta}\Phi^*_A(\theta) + \mu\frac{\partial}{\partial\theta}\frac{\delta S}{\delta\Phi^A(\theta)}. \end{aligned} \tag{12}$$

The transformations (12) realize a superfield form of the BRST symmetry transformations and permit one to establish the fact that the vacuum functional $Z_\Psi \equiv Z[0]$ is independent on a choice of the gauge. Indeed, we shall change the gauge by the rule $\Psi \to \Psi + \delta\Psi$. In the functional integral for $Z_{\Psi+\delta\Psi}$ we make the change of variables (12) with the parameter $\mu = \mu[\Phi]$. By virtue of Eq.(9), $Z_{\Psi+\delta\Psi}$ takes on the form

$$\begin{aligned} Z_{\Psi+\delta\Psi} \ &= \ \int d\Phi \ d\Phi^* \rho[\Phi^*] \exp\Big\{\frac{i}{\hbar}\Big(S[\Phi,\Phi^*] - V\Psi[\Phi] \\ &\quad -V\delta\Psi[\Phi] + i\hbar V\mu[\Phi]\Big)\Big\} \ . \end{aligned} \tag{13}$$

Then, choosing for the parameter μ the functional

$$\mu = -\frac{i}{\hbar}\delta\Psi \ , \tag{14}$$

we find that $Z_{\Psi+\delta\Psi} = Z_\Psi$ and conclude that the S-matrix is gauge independent.

Eq.(12) implies that from the geometrical viewpoint the operator V (5) can be considered as a generator of translations in superspace. Given this the transformations (12) take on the form

$$\delta\Phi^A(\theta) = \mu V\Phi^A(\theta)\ ,$$
$$\delta\Phi^*_A(\theta) = \mu V\Phi^*_A(\theta) + \mu\Big(S, \Phi^*_A(\theta)\Big)\ . \tag{15}$$

Another consequence of validity of the transformations (12) are the Ward identities for the generating functional of Green's functions. In fact, making in the functional integral (8) the change of variables (12) and taking the generating equation for $S = S[\Phi, \Phi^*]$ into account, we arrive at the relation

$$\int d\acute{\Phi}\, d\acute{\Phi^*}\rho[\acute{\Phi^*}]\int d\theta\frac{\partial\Phi^*_A(\theta)}{\partial\theta}\acute{\Phi}^A(\theta)\exp\Big\{\frac{i}{\hbar}\Big(S[\acute{\Phi}, \acute{\Phi^*}]$$
$$-\acute{V}\Psi[\acute{\Phi}] - \Phi^*\acute{\Phi}\Big)\Big\} = 0\ , \tag{16}$$

representable, with allowance made for Eq.(8), in the form

$$-\int d\theta\frac{\partial\Phi^*_A(\theta)}{\partial\theta}\frac{\delta}{\delta\Phi^*_A(\theta)}Z[\Phi^*] = VZ[\Phi^*] = 0\ . \tag{17}$$

Now, define the generating functional of vertex functions (effective action) depending on the superfields $\Gamma = \Gamma[\Phi]$ by the Legendre transformation for $\ln Z$ with respect to the super-antifields Φ^*

$$\Gamma[\Phi] = \frac{\hbar}{i}\ln Z[\Phi^*] + \Phi^*\Phi,\ \ \Phi^A(\theta) = -\frac{\hbar}{i}\frac{\delta}{\delta\Phi^*_A(\theta)}\ln Z[\Phi^*]\ , \tag{18}$$

then the identity (17) can be represented in the form

$$-\int d\theta\frac{\partial\Phi^A(\theta)}{\partial\theta}\frac{\delta_l}{\delta\Phi^A(\theta)}\Gamma[\Phi] = V\Gamma[\Phi] = 0\ . \tag{19}$$

Geometrically, the Ward identities (17), (19) imply the fact that the functionals $Z[\Phi^*]$, $\Gamma[\Phi]$ are invariant under supertranslations with respect to the coordinate θ.

4 Relation to the BV Quantization Scheme

It appears very important to establish a relation between the superfield approach in question and the BV quantization rules. To this end, note that the components of superfields $\Phi^A(\theta)$ and super-antifields $\Phi^*_A(\theta)$ are defined by expansions in θ

$$\begin{aligned}
&\Phi^A(\theta) = \phi^A + \lambda^A\theta\ , \ \ \Phi^*_A(\theta) = \phi^*_A - \theta J_A\ ,\\
&\varepsilon(\phi^A) = \varepsilon(J_A) = \varepsilon_A\ , \ \ \varepsilon(\phi^*_A) = \varepsilon(\lambda^A) = \varepsilon_A + 1
\end{aligned} \tag{20}$$

and coincide with the set of variables in the BV quantization scheme (the choice of signs in Eq. (20) is due to considerations of convinience).

Consider by virtue of Eq. (20) the component form of the basic definitions and relations given above.

First, the antibracket (2) is representable in terms of the component fields ϕ^A, ϕ^*_A, λ^A, J_A as follows

$$(F,G) = \frac{\delta F}{\delta\phi^A}\frac{\delta G}{\delta\phi^*_A} - (-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)}(F \leftrightarrow G)\ . \tag{21}$$

Eq.(21) coincides with the usual definition of the antibracket in the framework of BV quantization method.

Second, the corresponding component expressions for the operators Δ, V (4), (5) read

$$\Delta \ = \ (-1)^{\varepsilon_A}\frac{\delta_l}{\delta\phi^A}\frac{\delta}{\delta\phi^*_A}\ , \tag{22}$$

$$V = -J_A\frac{\delta}{\delta\phi^*_A} - (-1)^{\varepsilon_A}\lambda^A\frac{\delta_l}{\delta\phi^A}\ . \tag{23}$$

In virtue of Eqs. (21), (23) we find that the transformations (12) take on the form

$$\begin{aligned}
&\delta\phi^A = \lambda^A\mu, \quad \delta\lambda^A = 0,\\
&\delta\phi^*_A = \mu\Big(\frac{\delta S}{\delta\phi^A} - J_A\Big), \quad \delta J_A = 0.
\end{aligned} \tag{24}$$

Next, making use of Eq. (23), one readily obtains the component form of the Ward identities (17), (19) for the functionals $Z(\phi^*, J) \equiv Z[\phi^*]$, $\Gamma(\phi,\lambda) \equiv \Gamma[\Phi]$

$$J_A\frac{\delta}{\phi^*_A}Z(\phi^*, J) = 0, \quad \lambda^A\frac{\delta}{\delta\phi^A}\Gamma(\phi,\lambda) = 0, \tag{25}$$

Finally, the integration measure in Eq. (8) is understood as follows

$$d\Phi\, d\Phi^*\, \rho(\Phi^*) = d\phi\, d\phi^*\, d\lambda\, dJ\, \delta(J) \tag{26}$$

and the functional $\Phi^*\Phi$ has the form

$$\Phi^*_A\Phi^A = \phi^*_A\lambda^A - J_A\phi^A\, . \tag{27}$$

All things considered, choosing, by virtue of Eqs. (21), (22), (23), for a solution of the generating equation (9) with the boundary condition (10) a functoinal $S = S[\Phi, \Phi^*]$ such that

$$S[\Phi, \Phi^*]|_{J=0} = \overline{S}(\phi, \phi^*) + \phi^*_A\lambda^A, \tag{28}$$

where $\overline{S}$ satisfies the usual master equation of Refs. 4, 5

$$\frac{1}{2}(\overline{S}, \overline{S}) = i\hbar\Delta\overline{S}, \qquad \overline{S}\big|_{\phi^*=\hbar=0} = \mathcal{S}, \tag{29}$$

we arrive, making use of Eqs. (26), (27) at the following representation for the generating functional of Green's functions $Z = Z(J)$ of the fields ϕ^A

$$\begin{aligned} Z(J) \;\; &= \;\; Z[\Phi^*]|_{\phi^*=0} = \int d\phi\, d\phi^*\, d\lambda \exp\Big\{\frac{i}{\hbar}\Big[\overline{S}(\phi, \phi^*) \\ &\quad + \Big(\phi^*_A - \frac{\delta\Psi}{\delta\phi^A}\Big)\lambda^A + J_A\phi^A\Big]\Big\}. \end{aligned} \tag{30}$$

The above relation defines, with allowance made for Eq. (29), the generating functional of Green's functions in the framework of the BV quantization formalism.

5 Conclusion

In this paper the Lagrangian quantization rules for general gauge theories on a basis of a superfield realization of the standard BRST symmetry are presented. The S-matrix is shown to be gauge independent. The Ward identities (17), (19), corresponding to the superfield form (12) (or, equivalently, (15)) of the BRST transformations, imply invariance of the functionals $Z[\Phi^*]$, $\Gamma[\Phi]$ under translations in superspace (x^μ, θ) with respect to the Grassmann coordinate θ. It is shown that the special choice

(28), (29) of a solution to the equation (9) determining the boson functional S leads to the generating functional of Green's functions of the BV quantization scheme.

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