A Topological Bound for Electroweak Vortices from Supersymmetry

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Abstract

We study the connection between $N = 2$ supersymmetry and a topological bound in a two-Higgs-doublet system having an $SU(2) \times U(1)_Y \times U(1)_{Y'}$ gauge group. We derive Bogomol'nyi equations from supersymmetry considerations showing that they hold provided certain conditions on the coupling constants, which are a consequence of the huge symmetry of the theory, are satisfied.

Supersymmetric Grand Unified Theories (SUSY GUTs) have attracted much attention in connection with the hierarchy problem in possible unified theories of strong and electroweak interactions [1, 2]. In view of the requirement of electroweak symmetry breaking, these models necessitate an enrichment of the Higgs sector [3], posing many interesting questions both from the classical and quantum point of view. In fact, many authors have explored the existence of stable vortex solutions in a variety of multi-Higgs systems [4, 5] which mimic the bosonic sector of SUSY GUTs, as it happens in the abelian Higgs model [6].

Vortices emerging as finite energy solutions of gauge theories can be usually shown to satisfy a topological bound for the energy, the so-called Bogomol'nyi bound [7]. Bogomol'nyi bounds were shown to reflect the presence of an extended supersymmetric structure [8]-[11] - this requiring certain conditions on coupling constants - where the central charge coincides with the topological charge. Being originated in the supercharge algebra, the bound is expected to be exact quantum mechanically.

Since multi-Higgs models can be understood to be motivated by SUSY GUTs, Supersymmetry is a natural framework to investigate Bogomol'nyi bounds. We shall study, then, the supersymmetric generalization of the $SU(2) \times U(1)_Y \times U(1)_{Y'}$ model with two-Higgs first introduced in Ref.[5]. The theory has the same gauge group structure as that of supersymmetric extensions of the Weinberg-Salam Model that arise as low energy limits of E_6 based Grand Unified or superstring theories. In spite of being a simplified model (in the sense that its Higgs structure is not so rich as that of Grand Unified theories), it can be seen as the minimal extension of the Standard Model necessary for having Bogomol'nyi equations. We show that the Bogomol'nyi bound of the model, as well as the Bogomol'nyi equations, are straight consequences of the requirement of $N = 2$ supersymmetry imposed on the theory. We also show explicitely that a necessary condition to achieve the $N = 2$ model implies certain relations between coupling constants that equal those found in [5] for the existence of a Bogomol'nyi bound.

The $SU(2) \times U(1)_Y \times U(1)_{Y'}$ gauge theory in $2 + 1$, introduced in Ref.[5], is described by the action

$$S = \int d^3x \left[-\frac{1}{4}\vec{W}_{\mu\nu} \cdot \vec{W}^{\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}\sum_{q=1}^{2} |\mathcal{D}_\mu^{(q)}\Phi_{(q)}|^2 - V(\Phi_{(1)}, \Phi_{(2)}) \right] \tag{1}$$

where $\Phi_{(1)}$ and $\Phi_{(2)}$ are a couple of Higgs doublets under the $SU(2)$ factor of the gauge group, A and B are real scalar fields and $\vec{W} = W^a\tau^a$ is a real scalar in the adjoint representation of $SU(2)$. The specific

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form of the potential will be determined below. The strength fields can be written in terms of the gauge fields A_μ, B_μ and $\vec{W}_\mu$. The covariant derivative is defined as:

$$\mathcal{D}^{(q)}_\mu \Phi_{(q)} = \left(\partial_\mu + \frac{i}{2} g W^a_\mu \tau^a + \frac{i}{2}\alpha_{(q)} A_\mu + \frac{i}{2}\beta_{(q)} B_\mu\right)\Phi_{(q)}, \qquad \text{q=1,2} \tag{2}$$

where g is the $SU(2)$ coupling constant while $\alpha_{(q)}$ and $\beta_{(q)}$ represents the different couplings of $\Phi_{(q)}$ with A_μ and B_μ. A minimal $N=1$ supersymmetric extension of this model is given by an action which in superspace reads:

$$\begin{aligned} \mathcal{S}_{N=1} &= \frac{1}{2}\int d^3x d^2\theta \Big[\overline{\Omega}_A \Omega_A + \overline{\Omega}_B \Omega_B + \overline{\Omega}^a_{\vec{W}} \Omega^a_{\vec{W}} - \overline{\mathcal{D}\mathcal{A}}\mathcal{D}\mathcal{A} - \overline{\mathcal{D}\mathcal{B}}\mathcal{D}\mathcal{B} - \overline{\mathcal{D}\mathcal{W}}^a \mathcal{D}\mathcal{W}^a + \xi_1 \mathcal{A} + \xi_2 \mathcal{B} \\ &+ \frac{1}{2}\sum_{q=1}^{2}\left[(\overline{\nabla^{(q)}\Upsilon_{(q)}})^a(\nabla^{(q)}\Upsilon_{(q)})^a + i\Upsilon^\dagger_{(q)}\left(\sqrt{2\lambda^{(q)}_1}\mathcal{A} + \sqrt{2\lambda^{(q)}_2}\mathcal{B} + \sqrt{2\lambda_3}\mathcal{W}^a\tau^a\right)\Upsilon_{(q)}\right]\Big] \end{aligned} \tag{3}$$

where

$$\nabla^{(q)}\Upsilon_{(q)} = \left(\mathcal{D} + \frac{i}{2} g[\Gamma_{\vec{W}},] - \frac{i}{2}\alpha_{(q)}[\Gamma_A,] + \frac{i}{2}\beta_{(q)}[\Gamma_B,]\right)\Upsilon_{(q)}. \tag{4}$$

$\Upsilon_{(q)} \equiv (\Phi_{(q)}, \Psi_{(q)})$ are a couple of complex doublet superfields, $\mathcal{A} \equiv (A, \chi_A)$, $\mathcal{B} \equiv (B, \chi_B)$ and $\mathcal{W} \equiv (W^a, \chi^a_{\vec{W}})\tau^a$ are real scalar superfields and $\Gamma_A \equiv (A_\mu, \rho_A)$, $\Gamma_B \equiv (B_\mu, \rho_B)$ and $\Gamma_{\vec{W}} \equiv \Gamma^a_{\vec{W}}\tau^a = (W^a_\mu, \lambda^a)\tau^a$ are three spinor gauge superfields in the Wess-Zumino gauge. Ω_A, Ω_B and $\Omega^a_{\vec{W}}$, are the corresponding superfield strengths. Concerning $\lambda^{(q)}_1$, $\lambda^{(q)}_2$, λ_3, ξ_1 and ξ_2, they are real constants whose significance will be clear below. Finally, $\mathcal{D}$ is the usual supercovariant derivative, $\mathcal{D} = \partial_{\overline{\theta}} + i\overline{\theta}\,\partial\!\!\!/$, while the γ-matrices are represented by $\gamma^0 = \tau^3$, $\gamma^1 = i\tau^1$ and $\gamma^2 = -i\tau^2$. In the sake of simplicity, we shall consider configurations with vanishing A, B and $\vec{W}^1$. Then, the Higgs potential in (3) takes the form:

$$V(\Phi_{(1)}, \Phi_{(2)}) = \left(\sum_{q=1}^{2}\sqrt{\lambda^{(q)}_1}\Phi^\dagger_{(q)}\Phi_{(q)} - \frac{\xi_1}{\sqrt{2}}\right)^2 + \left(\sum_{q=1}^{2}\sqrt{\lambda^{(q)}_2}\Phi^\dagger_{(q)}\Phi_{(q)} - \frac{\xi_2}{\sqrt{2}}\right)^2 + \lambda_3\left(\sum_{q=1}^{2}\Phi^\dagger_{(q)}\tau^a\Phi_{(q)}\right)^2. \tag{5}$$

In order to extend the supersymmetric invariance of the theory to $N=2$, we consider transformations with a complex parameter [9, 11]. We first combine all the spinors into Dirac fermions as:

$$\Sigma_A \equiv \chi_A - i\rho_A \quad , \quad \Sigma_B \equiv \chi_B - i\rho_B \quad \text{and} \quad \Xi^a \equiv \chi^a_{\vec{W}} - i\Lambda^a. \tag{6}$$

The fermionic contribution to the (non-minimal part of the) interaction lagrangian can be written as

$$\begin{aligned} L_{Fer,int} &= \sum_{q=1}^{2}\Bigg[\frac{\alpha_{(q)} + \sqrt{8\lambda^{(q)}_1}}{4}\overline{\Psi}_{(q)}\Sigma_A\Phi_{(q)} + \frac{\beta_{(q)} + \sqrt{8\lambda^{(q)}_2}}{4}\overline{\Psi}_{(q)}\Sigma_B\Phi_{(q)} + \frac{g + \sqrt{8\lambda_3}}{4}\overline{\Psi}_{(q)}\Xi^a\tau^a\Phi_{(q)} \\ &- \frac{\alpha_{(q)} - \sqrt{8\lambda^{(q)}_1}}{4}\overline{\Psi}_{(q)}\tilde{\Sigma}_A\Phi_{(q)} - \frac{\beta_{(q)} - \sqrt{8\lambda^{(q)}_2}}{4}\overline{\Psi}_{(q)}\tilde{\Sigma}_B\Phi_{(q)} - \frac{g - \sqrt{8\lambda_3}}{4}\overline{\Psi}_{(q)}\tilde{\Xi}^a\tau^a\Phi_{(q)}\Bigg], \end{aligned} \tag{7}$$

where $\tilde{\Xi}^a$, $\tilde{\Sigma}_A$ and $\tilde{\Sigma}_B$ are the charge conjugates of Ξ^a, Σ_A and Σ_B respectively. Now, transformations with complex parameter η are equivalent to transformations with a real parameter followed by a phase transformation for fermions, $\{\Xi^a, \Sigma_A, \Sigma_B, \Psi_{(q)}\} \longrightarrow e^{i\alpha}\{\Xi^a, \Sigma_A, \Sigma_B, \Psi_{(q)}\}$. Then, $N=2$ supersymmetry requires invariance under this fermion rotation. One can easily see from (7) that fermion phase rotation invariance is achieved if and only if:

$$\lambda_3 = \frac{g^2}{8} \quad , \quad \lambda^{(q)}_1 = \frac{\alpha^2_{(q)}}{8} \quad \text{and} \quad \lambda^{(q)}_2 = \frac{\beta^2_{(q)}}{8}. \tag{8}$$

[1] The $SU(2) \times U(1)_Y \times U(1)_{Y'}$ theory with its full field content is considered in Ref.[12].

These conditions, imposed by the requirement of extended supersymmetry, fix the coupling constants exactly as they appear in [5]. Thus, we have shown that the potential and the coupling constants of the $SU(2)\times U(1)_Y\times U(1)_{Y'}$ model are simply dictated by $N=2$ supersymmetry. This result is analogous to that recently found in the Abelian Higgs model [11].

We shall now analyse the $N=2$ algebra of supercharges for our model. To construct the conserved charges we follow the Noether method and obtain $\mathcal{Q}[\eta]\equiv\overline{\eta}Q+\overline{Q}\eta$, with

$$\begin{aligned}\overline{Q} &= -\frac{i}{2}\int d^2x\left\{\Sigma_A^\dagger\left[\frac{1}{2}\epsilon^{\mu\nu\lambda}F_{\mu\nu}\gamma_\lambda+\sum_{q=1}^{2}\frac{\alpha_{(q)}}{2}\Phi_{(q)}^\dagger\Phi_{(q)}-\xi_1\right]+\Sigma_B^\dagger\left[\frac{1}{2}\epsilon^{\mu\nu\lambda}G_{\mu\nu}\gamma_\lambda+\sum_{q=1}^{2}\frac{\beta_{(q)}}{2}\Phi_{(q)}^\dagger\Phi_{(q)}\right.\right.\\ &- \left.\left.\xi_2\right]+\Xi^{\dagger a}\left[\frac{1}{2}\epsilon^{\mu\nu\lambda}W_{\mu\nu}^a\gamma_\lambda+\frac{g}{2}\sum_{q=1}^{2}\Phi_{(q)}^\dagger\tau^a\Phi_{(q)}\right]-i\sum_{q=1}^{2}\Psi_{(q)}^\dagger\gamma^\mu\mathcal{D}_\mu^{(q)}\Phi_{(q)}\right\} \qquad (9)\end{aligned}$$

Since we are interested in connecting the $N=2$ supercharge algebra with Bogomol'nyi equations and bound, we impose static configurations with $A_0=B_0=W_0^a=0$, and we restrict ourselves to the bosonic sector of the theory after computing the algebra. We obtain, after some calculations

$$\{\bar{\mathcal{Q}},\mathcal{Q}\}=2\overline{\eta}\gamma_0\eta P^0+\overline{\eta}\eta Z \qquad (10)$$

where

$$P^0=E=\frac{1}{2}\int d^2x\left[\frac{1}{2}(W_{ij}^a)^2+\frac{1}{2}(F_{ij})^2+\frac{1}{2}G_{ij}^2+\sum_{q=1}^{2}|\mathcal{D}_i^{(q)}\Phi_{(q)}|^2+V(\Phi_{(1)},\Phi_{(2)})\right] \qquad (11)$$

while the central charge is given by:

$$\begin{aligned}Z &= -\int d^2x\left[\frac{1}{2}\epsilon^{ij}F_{ij}\left(\sum_{q=1}^{2}\frac{\alpha_{(q)}}{2}\Phi_{(q)}^\dagger\Phi_{(q)}-\xi_1\right)+\frac{1}{2}\epsilon^{ij}G_{ij}\left(\sum_{q=1}^{2}\frac{\beta_{(q)}}{2}\Phi_{(q)}^\dagger\Phi_{(q)}-\xi_2\right)\right.\\ &+ \left.\frac{g}{4}\epsilon^{ij}W_{ij}^a\sum_{q=1}^{2}\Phi_{(q)}^\dagger\Phi_{(q)}+i\epsilon^{ij}\sum_{q=1}^{2}(\mathcal{D}_i^{(q)}\Phi_{(q)})(\mathcal{D}_j^{(q)}\Phi_{(q)})^*\right]=\frac{1}{2}\oint\mathcal{V}_i dx^i, \qquad (12)\end{aligned}$$

where $\mathcal{V}^i$ is given by

$$\mathcal{V}^i=\left(\xi_1A_j+\xi_2B_j+i\sum_{q=1}^{2}\Phi_{(q)}^\dagger\mathcal{D}_j^{(q)}\Phi_{(q)}\right)\epsilon^{ij}. \qquad (13)$$

Finite energy dictates the following asymptotic behaviour for the Higgs doublets [5]

$$\Phi_{(1)\infty}=\frac{\phi_0}{\sqrt{2}}\begin{pmatrix}0\\ \exp in_{(1)}\varphi\end{pmatrix}\quad,\quad\Phi_{(2)\infty}=\frac{\phi_0}{\sqrt{2}}\begin{pmatrix}\exp in_{(2)}\varphi\\ 0\end{pmatrix}, \qquad (14)$$

where $n_{(1)}$ and $n_{(2)}$ are integers that sum up to the topological charge of the configuration m,

$$m\equiv n_{(1)}+n_{(2)}. \qquad (15)$$

Then, coming back to eq.(12) for the central charge, after Stokes' theorem, we see that

$$Z=\oint(\xi_1A_i+\xi_2B_i)dx^i=-4\pi\phi_0^2m \qquad (16)$$

that is, the central charge of the $N=2$ algebra equals the topological charge of the configuration. It is now easy to find the Bogomol'nyi bound from the supersymmetry algebra (10). Indeed,

$$\{\bar{\mathcal{Q}},\mathcal{Q}\}=\int d^2x\left[(\delta\Xi^a)^\dagger(\delta\Xi^a)+(\delta\Sigma_A)^\dagger(\delta\Sigma_A)+(\delta\Sigma_B)^\dagger(\delta\Sigma_B)+\sum_{q=1}^{2}(\delta\Psi_{(q)})^\dagger(\delta\Psi_{(q)})\right]\geq 0. \qquad (17)$$

the lower bound being saturated if and only if $\delta\Xi^a = \delta\Sigma_A = \delta\Sigma_B = \delta\Psi_{(q)} = 0$. Non-trivial solutions to these equations force us to choose a parameter with definite chirality, say η_+. Now, conditions

$$\delta_{\eta_+}\Xi^a = \delta_{\eta_+}\Sigma_A = \delta_{\eta_+}\Sigma_B = \delta_{\eta_+}\Psi_{(q)} = 0 \tag{18}$$

are nothing but the Bogomol'nyi equations of the theory:

$$\frac{1}{2}\epsilon^{ij}F_{ij} + \sum_{q=1}^{2}\frac{\alpha_{(q)}}{2}\Phi^\dagger_{(q)}\Phi_{(q)} - \xi_1 = 0 \quad , \quad \frac{1}{2}\epsilon^{ij}G_{ij} + \sum_{q=1}^{2}\frac{\beta_{(q)}}{2}\Phi^\dagger_{(q)}\Phi_{(q)} - \xi_2 = 0 \tag{19}$$

$$\epsilon^{ij}W_{ij}{}^a + g\sum_{q=1}^{2}\Phi^\dagger_{(q)}\tau^a\Phi_{(q)} = 0 \quad \text{and} \quad (\mathcal{D}_i^{(q)} - i\epsilon_{ij}\mathcal{D}_j^{(q)})\Phi_{(q)} = 0. \tag{20}$$

Note that, for this chiral parameter, eq.(17) implies the Bogomol'nyi bound of our model,

$$M \geq 2\pi\phi_0^2 m. \tag{21}$$

Let us remark on the fact that field configurations solving Bogomol'nyi equations break half of the supersymmetries (those generated by η_-), a common feature in all models presenting Bogomol'nyi bounds with supersymmetric extension [13]. Were we faced with an antichiral parameter, we would have obtained antisoliton solutions with broken of the supersymmetry transformation generated by η_+.

The connection of our model with realistic supersymmetric extensions of the Standard model, and its coupling with supergravity (the possible existence of string-like solutions in this last theory) remain open problems. We hope to report on these issues in a forthcoming work.

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