# CONSERVATION OF STATISTICS AND GENERALIZED GRASSMANN NUMBERS

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## Abstract

Conservation of statistics requires that fermions be coupled to Grassmann external sources. Correspondingly, conservation of statistics requires that parabosons, parafermions and quons be coupled to external sources that are the appropriate generalizations of Grassmann numbers.

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## 1. Introduction

The classic constraints of conservation of statistics in theories with bosons and fermions are that all terms in the Hamiltonian must have an even number of Fermi fields, and composites of bosons and fermions are bosons, unless they contain an odd number of fermions, in which case they are fermions. Thus an even number of fermions must participate in any reaction and no reaction can involve only one fermion.

With the introduction of new kinds of particle statistics, such as parabosons and parafermions[1, 2] and quons[3, 4, 5, 6, 7, 8, 9, 10] it is relevant to consider possible Hamiltonian densities, including couplings to external sources, that involve fields obeying the new statistics, rather than the familiar Bose and Fermi statistics. One is tempted to carry over constructions used for Bose and Fermi fields to the new types of fields. The purpose of this paper is to point out that care must be exercised to ensure that the Hamiltonian density is an effective Bose operator in the sense that

$$[\mathcal{H}(x), \phi(y)] = 0, \;\; |\mathbf{x} - \mathbf{y}| \rightarrow \infty. \tag{1}$$

for *all* fields $\phi$, regardless of whether $\phi$ is Bose, Fermi, parabose, parafermi or quon. This requirement, which is necessary in order that the energy of widely separately particles is the sum of the energies of the individual particles, leads to the conservation of statistics discussed above. Sudbery[11] pointed out the implications of this constraint for particles with anomalous statistics. In the case of couplings to external sources where the particle number is not conserved, the additivity of energy requirement is replaced by additivity of transition matrix elements. The simplest extension of conservation of statistics is that a single parabose, parafermi or quon particle cannot couple to "normal" (Bose or Fermi) particles. To couple these "anomalous" particles to external sources, I introduce parabose, parafermi and quon analogs of Grassmann numbers. Their external sources must be coupled to the quantized fields in such a way that the term in the Hamiltonian is an effective Bose operator; otherwise additivity of transition matrix elements for widely separated subsystems would be violated. Since qualitative issues concerning statistics should be the same for noninteracting particles as for particles whose interactions vanish for large space separation, I give the discussion in terms of noninteracting particles. In this case, using discrete notation, the condition for an effective Bose operator is $[n_i, a_j^\dagger]_- = \delta_{ij} a_j^\dagger$ without external sources and $[s_i, a_j^\dagger]_- = \delta_{ij} g_j^\star$ with an external source, where $s_i$ is the external source term and $g_j^\star$ is the appropriate generalization of a Grassmann number.

As an example of what can go wrong, consider a collection of free identical

Fermi particles with annihilation and creation operators, $a_i$ and $a_i^\dagger$, labelled by quantum numbers $i$. In all cases, we want an external source to contribute equally to each of these particles. We must couple the external source to the Fermi particles using Grassmann numbers, $f_i$, that obey $[f_i, f_j^\star]_+ = 0$, and take as the external Hamiltonian,

$$H_{ext} = \sum_i (f_i^\star a_i + a_i^\dagger f_i). \tag{2}$$

Then,

$$[H_{ext}, a_k^\dagger]_- = f_k^\star, \tag{3}$$

and, acting on a state of several fermions,

$$H_{ext} a_1^\dagger a_2^\dagger \cdots a_n^\dagger |0\rangle = (f_1^\star a_2^\dagger \cdots a_n^\dagger + a_1^\dagger f_2^\star \cdots a_n^\dagger + \cdots a_1^\dagger a_2^\dagger \cdots f_n^\star)|0\rangle. \tag{4}$$

Here, each fermion is treated in an equivalent way by the external source. If we had not coupled the external source using Grassmann numbers, but instead used c-nos., $j_i$, then we would have had

$$H_{ext} = \sum_i (j_i^\star a_i + a_i^\dagger j_i), \ [H_{ext}, a_k^\dagger]_- = j_k + 2\sum_i (a_i^\dagger j_i a_k^\dagger - a_k^\dagger j_i^\star a_i^\dagger) \tag{5}$$

and, acting on a state of several fermions,

$$H_{ext} a_1^\dagger a_2^\dagger \cdots a_n^\dagger |0\rangle = (j_1^\star a_2^\dagger \cdots a_n^\dagger - a_1^\dagger j_2^\star \cdots a_n^\dagger + \cdots + (-1)^{n-1} a_1^\dagger a_2^\dagger \cdots j_n^\star)|0\rangle, \tag{6}$$

so the interactions of the successive fermions with the external source would have alternated in sign. If one considers a transition matrix element between a state with $n$ particles and a state with $n \pm 1$ particles, the contribution to the transition matrix element from the Fermi particles adds in the case in which the Fermi particles are coupled to the external sources with Grassmann numbers, but the signs of the contributions from the Fermi particles alternates in the case in which the particles are coupled with c-numbers. Because equivalent particles should contribute in an equivalent way, the external sources must be Grassmann numbers in this case.

An analogous issue arises in considering the choice of Hamiltonian in a theory of noninteracting quons. The commutation relation for the quons is

$$a_i a_j^\dagger - q a_j^\dagger a_i = \delta_{ij}; \tag{7}$$

there is no relation that allows transposing two quon creation or two quon annihilation operators[3]. Consider two possibilities: (a) the number operator, Hamiltonian, etc., have their usual algebraic form,

$$n_i = a_i^\dagger a_i, \quad H = \sum_i \omega_i a_i^\dagger a_i, \quad \text{etc.} \tag{8}$$

or (b) the number operator, etc., have the usual commutators with the annihilation and creation operators,

$$[n_i, a_j^\dagger]_- = \delta_{ij}, \quad [H, a_i^\dagger]_- = \omega_i, \quad [\mathbf{P}, a_i^\dagger]_- = \mathbf{p}_i a_i^\dagger. \tag{9}$$

For case (a), the energy equation for a state of $n$ identical quons is

$$H|a_1^\dagger a_2^\dagger \cdots a_n^\dagger\rangle = \sum_j q^{j-1}\omega_j |a_1^\dagger a_2^\dagger \cdots a_n^\dagger\rangle. \tag{10}$$

In this case, the identical noninteracting quons contribute to the energy with different powers of $q$ depending on where in the state vector they appear. This is unreasonable, since identical noninteracting particles should contribute to the energy in an equivalent way. Another problem with this choice is that the algebra of the generators of space-time symmetry groups will not be satisfied. To see this, let the momentum operator be

$$\mathbf{P} = \sum_i \mathbf{p}_i a_i^\dagger a_i. \tag{11}$$

The commutator of these observables is

$$[H, P]_- = q \sum_{i,j} \omega_i \mathbf{p}_j (a_i^\dagger a_j^\dagger a_i a_j - a_j^\dagger a_i^\dagger a_j a_i). \tag{12}$$

For the Bose and Fermi cases the two terms cancel; however, for the quon case there is no commutation relation among annihilation or among creation operators and these terms do not cancel. Thus, except for $q = 0$, the energy and momentum operators cannot obey the correct algebra in case (a). In case (b), construct $n_i$ so that

$$[n_i, a_j^\dagger]_- = \delta_{ij} a_j^\dagger. \tag{13}$$

A straight-forward calculation shows that the energy and, in the external source case, the transition matrix elements of noninteracting particles are additive, and that the space-time generators obey the correct algebra. I made this choice for the special case (the Cuntz algebra[13]) of $q = 0$[14], and also made this choice for general $q$[3]. For the special case of $q = 0$, I found the exact expression for the number operator, from which the space-time symmetry operators can be constructed. In the latter case, I gave the first few terms of the number operator; the complete formula for the number operator was given by Stanciu[15]. I conclude that (b), choosing the annihilation and creation operators to have the usual commutation relations with the number operator, is the correct choice.

The corresponding error with external sources is to couple the quons to a c-number external source $j_i$ using

$$H_{ext} = \sum_i (j_i^\star a_i + a_i^\dagger j_i). \tag{14}$$

Then, acting on a state of several quons,

$$H_{ext} a_1^\dagger a_2^\dagger \cdots a_n^\dagger |0\rangle = (j_1^\star a_2^\dagger \cdots a_n^\dagger + q a_1^\dagger j_2^\star \cdots a_n^\dagger + \cdots q^{n-1} a_1^\dagger a_2^\dagger \cdots j_n^\star)|0\rangle. \tag{15}$$

Here, the powers of $q$ replace the powers of $(-1)$ in the Fermi case discussed above. The contributions to transition matrix elements acquire corresponding factors of powers of $q$. The external sources must be quon analogs of Grassmann numbers in order that the contributions to transition matrix elements of widely separated quons be additive. Because quons were coupled to external sources with c-numbers in [16], the conclusions of that paper are not reliable.

A further problem with [16] is that in Model 2 of this reference the $q$-exponential is not unitary: the unitary evolution operator does not have the form $exp(-itH)$, with $H$ time independent, but rather has this form with $H(t)$ having the time dependence implied by the peculiarities of the $q$-exponential. The repair of this nonunitarity introduces an uncontrolled time dependence in the Hamiltonian. Since the large-time dependence of the occupation number is crucial, this uncontrolled time dependence is a serious flaw.

What is true for the coupling of external sources to quons is also true for the coupling of parabosons and of parafermions to external sources: in all cases, the coupling must involve the appropriate analog of Grassmann numbers and the external Hamiltonian must be an effective Bose operator. The commutation relations for these Grassmann analogs do not seem to appear in the literature. I supply them below.

## 2. Coupling to external sources for parabosons and parafermions

Green's trilinear commutation relations for parabose and parafermi operators are

$$[n_{kl}, a_m^\dagger]_- = \delta_{lm} a_k^\dagger, \tag{16}$$

where

$$n_{kl} = \frac{1}{2}([a_k^\dagger, a_l]_\pm \mp p\delta_{kl}), \tag{17}$$

and the upper (lower) sign is for parabosons (parafermions). Since Eq.(17) is trilinear, two conditions are necessary to fix the Fock-like representation: the usual vacuum condition is

$$a_k|0\rangle = 0; \tag{18}$$

the new condition

$$a_k a_l^\dagger|0\rangle = p\delta_{kl}, \quad p \text{ integer}, \tag{19}$$

contains the integer $p$ that is the order of the parastatistics. The Hamiltonian for free particles obeying parastatistics has the same form, in terms of the number operators, as for Bose and Fermi statistics,

$$H = \sum_k \epsilon_k n_k, \text{ where, as usual } [H, a_l^\dagger]_- = \epsilon_l a_l^\dagger. \tag{20}$$

For interactions with an external source, introduce para-Grassmann numbers that make the interaction Hamiltonian an effective Bose operator. Require

$$[H_{ext}, a_l^\dagger] = c_l^\star. \tag{21}$$

This is accomplished by choosing

$$H_{ext} = \sum_k \frac{1}{2}([c_k^\star, a_k]_\pm + [a_k^\dagger, c_k]_\pm), \tag{22}$$

where the para-Grassmann numbers $c_k$ and $c_k^\dagger$ obey

$$[[c_k^\star, c_l]_\pm, c_m^\star]_- = 0, \; [[c_k^\star, a_l]_\pm, a_m^\dagger]_- = 2\delta_{lm} c_k^\star, \text{ etc.}, \tag{23}$$

and the upper (lower) sign is for parabose-Grassmann (parafermi-Grassmann) numbers. The "etc." in Eq.(23) means that when some of the $c$'s or $c^\dagger$'s are replaced by an $a$ or an $a^\dagger$, the relation retains its form, except when the $a$ and $a^\dagger$ can contract, in which case the term with the contraction appears on the right-hand-side.

### 3. Coupling to external sources for quons

The case of quons differs from all the previous cases in that the external source Hamiltonian is of infinite degree, instead of being bilinear. (The Hamiltonian for free particles is also of infinite degree[3].) Since the infinite series is simple in the special case of $q = 0$[14], I discuss this case first. In that case, the commutation relation is

$$a_k a_l^\dagger = \delta_{kl}, \tag{24}$$

with the usual vacuum condition, Eq.(18). To construct observables, we want number operators and transition operators that obey

$$[n_k, a_l^\dagger]_- = \delta_{kl} a_l^\dagger, \quad [n_{kl}, a_m^\dagger]_- = \delta_{lm} a_k^\dagger. \tag{25}$$

Once Eq.(25) holds, the Hamiltonian and other observables can be constructed in the usual way; for example,

$$H=\sum_k \epsilon_k n_k, \quad \text{etc.} \tag{26}$$

The obvious thing is to try

$$n_k = a_k^\dagger a_k. \tag{27}$$

Then

$$[n_k, a_l^\dagger]_- = a_k^\dagger a_k a_l^\dagger - a_l^\dagger a_k^\dagger a_k. \tag{28}$$

The first term in Eq.(28) is $\delta_{kl}a_k^\dagger$ as desired; however the second term is extra and must be canceled. This can be done by adding the term $\sum_t a_t^\dagger a_k^\dagger a_k a_t$ to the term in Eq.(27). This cancels the extra term, but adds a new extra term, that must be canceled by another term. This procedure yields an infinite series for the number operator and for the transition operator,

$$n_{kl} = a_k^\dagger a_l + \sum_t a_t^\dagger a_k^\dagger a_l a_t + \sum_{t_1,t_2} a_{t_2}^\dagger a_{t_1}^\dagger a_k^\dagger a_l a_{t_1} a_{t_2} + \ldots \tag{29}$$

As in the Bose case, this infinite series for the transition or number operator defines an unbounded operator whose domain includes states made by polynomials in the creation operators acting on the vacuum.

The quon-Grassmann numbers must satisfy

$$c_k c_l^\star = 0;\;\; c_k a_l^\dagger = 0;\;\; a_k c_l^\star = 0. \tag{30}$$

Then $H_{ext}$ must be chosen to obey

$$[H_{ext}, a_l^\dagger]_- = c_l^\star. \tag{31}$$

This is accomplished by choosing

$$H_{ext} = \sum_k (c_k^\star a_k + a_k^\dagger c_k) + \sum_k \sum_t a_t^\dagger (c_k^\star a_k + a_k^\dagger c_k) a_t + \cdots \tag{32}$$

in analogy with Eq.(29).

The general quon algebra[3] is

$$a_k a_l^\dagger - q a_l^\dagger a_k = \delta_{kl}, \tag{33}$$

with the usual vacuum condition, Eq.(18). For observables without an external source, one again needs a set of number operators $n_k$ such that

$$[n_k, a_l^\dagger]_- = \delta_{kl} a_l^\dagger. \tag{34}$$

Like the $q = 0$ case, the expression for $n_k$ or $n_{kl}$ is an infinite series in creation and annihilation operators; unlike the $q = 0$ case, the coefficients are complicated. The first two terms are

$$n_{kl} = a_k^\dagger a_l + (1-q^2)^{-1}\sum_t (a_t^\dagger a_k^\dagger - q a_k^\dagger a_t^\dagger)(a_l a_t - q a_t a_l) + \cdots . \tag{35}$$

Here I have given the transition number operator $n_{kl}$ for $k \to l$ since this takes no extra effort. The general formula for the number operator is given in [15] following a conjecture of Zagier [10]. As before, the Hamiltonian is

$$H = \sum_k \epsilon_k n_k, \quad \text{with} \quad [H, a_l^\dagger]_- = \epsilon_l a_l^\dagger . \tag{36}$$

For an external source, we again require that $H_{ext}$ be an effective Bose operator and again accomplish this using quon-Grassmann numbers. Now these obey

$$c_k c_l^\star - q c_l^\star c_k = 0;\ c_k a_l^\dagger - q a_l^\dagger c_k = 0;\ a_k c_l^\star - q c_l^\star a_k = 0, \tag{37}$$

and $H_{ext}$ obeys

$$[H_{ext}, a_l^\dagger]_- = c_l^\star . \tag{38}$$

For this to work, we need

$$H_{ext} = \sum_k (c_k^\star a_k + a_k^\dagger c_k) + (1-q^2)^{-1}\sum_t (a_t^\dagger c_k^\star - q c_k^\star a_t^\dagger)(a_k a_t - q a_t a_k)$$

$$+ \sum_k (1-q^2)^{-1}(a_t^\dagger a_k^\dagger - q a_k^\dagger a_t^\dagger)(c_k a_t - q a_t c_k) + \cdots \tag{39}$$

The general result for $H_{ext}$ can be gotten from the number operator of Ref.[15] by replacing some of the $a$'s and $a^\dagger$'s by $c$'s and $c^\star$'s in analogy to the change from Eq.(35) to Eq.(39). If, instead, we incorrectly choose $H_{ext} = \sum_k (j_k^\star a_k + a_k^\dagger j_k)$, where $j$ is a $c$-number, then the interactions of noninteracting systems (or of widely separated subsystems) with the external sources are not additive as illustrated in the introduction. Because this point was not recognized, the bound on laser intensities due to a small violation of Bose statistics for photons claimed in [16] cannot be taken seriously.

### 4. Difficulties in obtaining high-precision bounds on Bose statistics

There are two reasons that make it difficult to get high-precision bounds on the validity of Bose statistics for photons and other presumed bosons. (1) Stable matter is made of fermions, not bosons, so one cannot search for stable or quasi-stable states of bosons that exhibit anomalous statistics, nor can one search for transitions to such states. (2) It is difficult to make a high-precision measurement of deviations from the Bose distribution in macroscopic samples, because the effect due to a possible small concentration of anomalous states will be swamped by the much larger number of normal states. This problem also arises in the case of fermions. A general discussion of tests of Fermi and Bose statistics is given in [17]. The best bound on the Fermi statistics of electrons is due to Ramberg and Snow[18].

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## References

[1] H.S. Green, Phys. Rev. 90 (1953) 270.

[2] O.W. Greenberg and A.M.L. Messiah, Phys. Rev. 138 (1965) B1155, discuss the selection rules for parafields in detail.

[3] O.W. Greenberg, Phys. Rev. D 43 (1991) 4111. See also O.W. Greenberg, Bull. Am. Phys. Soc. 35 (1990) 981.

[4] O.W. Greenberg, in Workshop on Harmonic Oscillators, NASA Conference Publication 3197, ed. D. Han, Y.S. Kim and W.W. Zachary (National Aeronautics and Space Administration, Greenbelt, 1993), pp 5-18.

[5] O.W. Greenberg, D.M. Greenberger and T.V. Greenbergest, In Quantum Coherence and Reality, eds. J.S. Anandan and J.L. Safko, (World Scientific, Singapore, 1994), pp 301-312.

[6] O.W. Greenberg, Physica A 180 (1992) 419.

[7] D.I. Fivel, Phys. Rev. Lett. 65 (1990) 3361; Fivel's repair of an error in the proof is in the erratum, ibid 69 (1992) 2020.

[8] M. Bozejko and R. Speicher, Commun. Math. Phys. 137 (1991) 519; in Quantum Probability and Related Topics, Vol. VII (ed. L. Accardi, World Scientific 1992), p. 6; and Math. Ann. 300 (1994), 97.

[9] R. Speicher, Lett. Math. Phys. 27 (1993) 97.

[10] D. Zagier, Commun. Math. Phys. 147 (1992) 199.

[11] T. Sudbery, Nature 348 (1990) 193, remarked on the implication of conservation of statistics for the interpretation by Kekez, et al, [12] of experiments searching for gamma rays as setting bounds of violation of the exclusion principle for nucleons.

[12] D. Kekez, A. Ljubicic and B.A. Logan, Nature 348 (1990) 224.

[13] J. Cuntz, Commun. Math. Phys. 57 (1977) 173.

[14] O.W. Greenberg, Phys. Rev. Lett. 64 (1990) 705.

[15] S. Stanciu, ibid, 147 (1992) 211.

[16] D.I. Fivel, Phys. Rev. A 43 (1991) 4913.

[17] O.W. Greenberg and R.N. Mohapatra, Phys. Rev. D 39 (1989) 2032.

[18] E. Ramberg and G.A. Snow, Phys. Lett. B 238 (1990) 438.