Black Hole Entropy and Renormalization

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Abstract. Using a new regulator, we examine 't Hooft's approach for evaluating black hole entropy through a statistical-mechanical counting of states for a scalar field propagating outside the event horizon. We find that this calculation yields precisely the one-loop renormalization of the standard Bekenstein-Hawking formula, $S = \mathcal{A}/(4G)$. Thus our result provides evidence confirming a suggestion by Susskind and Uglum regarding black hole entropy.

1 Introduction

It is now over twenty years since Bekenstein introduced the idea that black holes carry an intrinsic entropy proportional to the surface area of the event horizon measured in Planck units, *i.e.*, $\mathcal{A}/\ell_p{}^2$ (Bekenstein [1972]). Hawking's discovery (Hawking [1975]) that, in quantum field theory, black holes actually generate thermal radiation allowed the determination of a precise formula for this entropy: $S = \mathcal{A}/(4G) = \mathcal{A}/(4\ell_p{}^2)$. (We adopt the standard conventions of setting $\hbar = c = k_B = 1$, but we will explicitly retain Newton's constant, G, in our analysis.) This Bekenstein-Hawking formula is applicable for any black hole solution of Einstein's equations. In more general gravitational theories, black hole entropy is no longer given by the Bekenstein-Hawking formula (Wald [1993]; Visser [1993]; Jacobson, Kang, Myers [1994]; Iyer, Wald [1994]). However, the entropy is given by the integral of a geometric density over a cross section of the horizon. For example, in a theory described by the effective gravitational action

$$I = \int d^4x \sqrt{-g}\left[\frac{R}{16\pi G} + \frac{\gamma}{4\pi}R^{abcd}R_{abcd}\right] , \qquad (1.1)$$

the entropy of a black hole is

$$S = \oint_H d^2x \sqrt{h}\left[\frac{1}{4G} - \frac{\gamma}{4\pi}R^{abcd}\hat{\epsilon}_{ab}\hat{\epsilon}_{cd}\right] . \qquad (1.2)$$

Here H is a space-like cross section of the horizon, h_{ab} is the induced metric on H, and $\hat{\epsilon}_{ab}$ is the binormal to H.

Our present understanding of black hole entropy is limited to a thermodynamic framework. Many attempts have been made to define black hole entropy within a statistical mechanical context, but despite a great deal of effort, such a microphysical interpretation is still lacking. A common feature of many of the attempts is that they yield a black hole entropy proportional to the area, but with a divergent coefficient. A recent suggestion by Susskind and Uglum [1994] is that these divergences can be absorbed in the Bekenstein-Hawking formula as a renormalization of Newton's constant. We investigate this possibility by using a common regulator for a calculation of the renormalization of the gravitational coupling constants, and a statistical calculation of black hole entropy. This allows a precise comparison of the divergences appearing in these two calculations. Our results support the suggestion of Susskind and Uglum [1994]. A more complete description of this work appears in Demers, Lafrance, Myers [1995a].

2 Renormalization of the gravitational action

In the study of the one-loop effective action (see *e.g.* Birrell, Davies [1982]), one may start with the gravitational action

$$I_g = \int d^4x \sqrt{-g}\left[-\frac{\Lambda_B}{8\pi G_B} + \frac{R}{16\pi G_B} + \frac{\alpha_B}{4\pi}R^2 + \frac{\beta_B}{4\pi}R_{ab}R^{ab} + \frac{\gamma_B}{4\pi}R_{abcd}R^{abcd} + \dots\right]$$

(2.1)

where Λ_B is the cosmological constant, G_B is Newton's constant, while α_B, β_B and γ_B are dimensionless coupling constants for the interactions quadratic in the curvature. The subscript B indicates that these are all bare coupling constants. The ellipsis indicates that the action may also include other covariant, higher-derivative interactions, but no terms beyond those shown will be of interest in the present analysis. We also include the action for a minimally coupled neutral scalar field,

$$I_m = -\frac{1}{2}\int d^4x \sqrt{-g}\left[g^{ab}\nabla_a\phi\nabla_b\phi + m^2\phi^2\right].$$

(2.2)

We wish to determine the effective action for the metric which results when in the path integral the scalar field is integrated out. This integration is simply gaussian, yielding the square root of the determinant of the propagator. Thus the contribution to the effective gravitational action, which is the logarithm of this result, is given by $W(g) = -\frac{i}{2}\text{Tr}\log[-G_F(g, m^2)]$ (Birrell, Davies [1982]). Of course, as it stands, this expression is divergent and it must first be regulated to be properly defined. The divergences of this one-loop effective action, as well as its metric dependence, are then easily identified using an adiabatic expansion for the DeWitt-Schwinger proper time representation of the propagator. To regulate our calculations, we choose the Pauli-Villars method. In this scheme, one introduces fictitious regulator fields with very large masses. Some of these fields are also quantized with the "wrong" statistics. For the present four-dimensional scalar field theory, one needs five regulator fields, and the total action for the matter fields becomes

$$I_m = -\frac{1}{2}\sum_{i=0}^{5}\int d^4x \sqrt{-g}\left[g^{ab}\nabla_a\phi_i\nabla_b\phi_i + m_i^2\phi_i^2\right].$$

(2.3)

Here, $\phi_0 = \phi$ is the physical scalar with mass $m_0 = m$, ϕ_1 and ϕ_2 are two anticommuting fields with mass $m_{1,2} = \sqrt{\mu^2 + m^2}$, ϕ_3 and ϕ_4 are two commuting fields with mass $m_{3,4} = \sqrt{3\mu^2 + m^2}$, and ϕ_5 is an anticommuting field with mass $m_5 = \sqrt{4\mu^2 + m^2}$. Now, each field makes a contribution to the effective action as discussed above, except that as a result of their anticommuting statistics, the contributions of ϕ_2, ϕ_3 and ϕ_5 have the opposite sign, $i.e.$, $W(g) \simeq +\frac{i}{2}\mathrm{Tr}\log[-G_F(g, m_i^2)]$. We will focus on the divergent terms of the one-loop effective action:

$$W_{div} = \frac{1}{32\pi^2}\int d^4x\,\sqrt{-g}\left[-C\,a_0(x) + B\,a_1(x) + Aa_2(x)\right]\,. \qquad (2.4)$$

In this expression, the coefficients a_0, a_1, a_2 are functionals of the local geometry

$$a_0 = 1 \qquad\qquad a_1 = \frac{1}{6}R$$
$$a_2 = \frac{1}{180}R^{abcd}R_{abcd} - \frac{1}{180}R^{ab}R_{ab} + \frac{1}{30}\Box R + \frac{1}{72}R^2 \qquad (2.5)$$

and A, B and C are constants which depend on the masses, m and μ, and which diverge for $\mu \to \infty$:

$$A = \ln\frac{4\mu^2 + m^2}{m^2} + 2\ln\frac{\mu^2 + m^2}{3\mu^2 + m^2}$$

$$B = \mu^2\left[2\ln\frac{3\mu^2 + m^2}{\mu^2 + m^2} + 4\ln\frac{3\mu^2 + m^2}{4\mu^2 + m^2}\right] + m^2\left[\ln\frac{m^2}{4\mu^2 + m^2} + 2\ln\frac{3\mu^2 + m^2}{\mu^2 + m^2}\right]$$

$$C = \mu^4\left[8\ln\frac{3\mu^2 + m^2}{4\mu^2 + m^2} + \ln\frac{3\mu^2 + m^2}{\mu^2 + m^2}\right] + 2m^2\mu^2\left[\ln\frac{3\mu^2 + m^2}{\mu^2 + m^2} + 2\ln\frac{3\mu^2 + m^2}{4\mu^2 + m^2}\right]$$
$$+ \frac{m^4}{2}\left[\ln\frac{m^2}{4\mu^2 + m^2} + 2\ln\frac{3\mu^2 + m^2}{\mu^2 + m^2}\right]\,.$$
$$(2.6)$$

For large values of μ, the constants A, B and C grow to leading order as $\ln(\mu/m)$, μ^2 and μ^4, respectively, but they also contain subleading and finite contributions. Combining the scalar one-loop action with the original bare action in eq. (2.1), we can identify the renormalized coupling constants in the effective gravitational action

$$I_{eff} = I_g + W$$
$$= \int d^4x\sqrt{-g}\left[-\frac{1}{8\pi}\left(\frac{\Lambda_B}{G_B} + \frac{C}{4\pi}\right) + \frac{R}{16\pi}\left(\frac{1}{G_B} + \frac{B}{12\pi}\right) + \frac{R^2}{4\pi}\left(\alpha_B + \frac{A}{576\pi}\right)\right.$$
$$\left. + \frac{1}{4\pi}R_{ab}R^{ab}\left(\beta_B - \frac{A}{1440\pi}\right) + \frac{1}{4\pi}R_{abcd}R^{abcd}\left(\gamma_B + \frac{A}{1440\pi}\right) + \cdots\right]$$
$$(2.7)$$

where in the action we discard the total derivative term $\Box R$ occurring in a_2. In particular from eq. (2.7), we obtain the renormalized Newton's constant:

$$\frac{1}{G_R} = \frac{1}{G_B} + \frac{B}{12\pi}\,. \qquad (2.8)$$

In eq. (2.7), divergent renormalizations also occur for the cosmological constant Λ_B and the quadratic-curvature coupling constants α_B, β_B and γ_B. The higher order bare coupling constants (beyond those explicitly shown) would receive finite renormalizations but they will play no role in the present analysis.

3 Statistical entropy

't Hooft's statistical mechanical calculation of black hole entropy ('t Hooft [1985]) involves counting the states for a thermal ensemble of a scalar field propagating just outside a black hole. Following 't Hooft's calculation, we introduce a fixed background metric, which is a Schwarzchild black hole

$$ds^2 = -\left(1 - \frac{r_0}{r}\right)dt^2 + \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\ . \qquad (3.1)$$

In this background, we consider a minimally coupled scalar field satisfying the Klein-Gordon equation

$$(\Box - m^2)\phi(x) = 0\ . \qquad (3.2)$$

Because of the infinite blue-shift at the horizon, 't Hooft introduced a "brick wall" cut-off: $\phi(x) = 0$ for $r \le r_0 + h$, with $h \ll r_0$. To eliminate infrared divergences, a second cut-off is introduced at a large radius $L \gg r_0$: $\phi(x) = 0$ for $r \ge L$.

To calculate the entropy, we consider the free energy of a thermal ensemble of scalar particles at inverse temperature β

$$\beta F = \int_0^\infty dE\, \frac{dg}{dE}\, \ln(1 - e^{-\beta E})\ , \qquad (3.3)$$

where $dg(E)/dE$ is the density of modes with energy E. In the WKB appproximation, the number of states $g(E)$ is given by

$$g(E) = \frac{2}{3\pi} \int_{r_0+h}^L dr\, r^2 \left(1 - \frac{r_0}{r}\right)^{-2} \left[E^2 - \left(1 - \frac{r_0}{r}\right)m^2\right]^{3/2}\ . \qquad (3.4)$$

From eq. (3.4), it is apparent that F will diverge if one takes the limit $h \to 0$. However, the divergences are removed if we add the same Pauli-Villars fields that were introduced in section 2. With the regulator fields and $h \to 0$, the free energy becomes

$$\bar{F} = -\frac{2}{3\pi} \int_0^\infty \frac{dE}{e^{\beta E} - 1} \int_0^L dr\, r^2 \left(1 - \frac{r_0}{r}\right)^{-2} \sum_{i=0}^5 \Delta_i \left[E^2 - \left(1 - \frac{r_0}{r}\right)m_i^2\right]^{3/2}\ , \qquad (3.5)$$

where $\Delta_0 = \Delta_3 = \Delta_4 = 1$ for the commuting fields and $\Delta_1 = \Delta_2 = \Delta_5 = -1$ for the anticommuting fields. Now the divergent contributions at the horizon are:

$$\bar{F} = -r_0^3 \left[\frac{\pi}{6\beta^2} B + \frac{8\pi^3}{45\beta^4} A\right]\ , \qquad (3.6)$$

where A and B are the same constants given by Eq. (2.6).

The corresponding entropy is then:

$$S_q = \beta^2 \frac{\partial F}{\partial \beta}\bigg|_{\beta = 4\pi r_0} = \frac{\mathcal{A}}{4}\frac{B}{12\pi} + \frac{A}{90}\ . \qquad (3.7)$$

Using eq. (1.2), the bare entropy for the Schwarzchild black hole related to the gravitational action (2.1) is

$$S_B = \frac{\mathcal{A}}{4G_B} + 16\pi\gamma_B\ . \qquad (3.8)$$

Adding this result to S_q yields:

$$S = S_B + S_q = \frac{\mathcal{A}}{4}\left(\frac{1}{G_B} + \frac{B}{12\pi}\right) + 16\pi\left(\gamma_B + \frac{A}{1440}\right)$$
$$= \frac{\mathcal{A}}{4G_R} + 16\pi\gamma_R \ . \tag{3.9}$$

Thus this final entropy expression contains precisely the renormalized coupling constants appearing in eq. (2.7).

3 Conclusion

Through the use of a Pauli-Villars regulator, we have found evidence for the suggestion by Susskind and Uglum [1994] — *i.e.,* the divergences appearing in 't Hooft's statistical mechanical calculation of black hole entropy can be absorbed by a renormalization of the gravitational coupling constants. (Similar results were found by Fursaev, Solodukhin [1994] and Solodukhin [1995].) We have extended these calculations to a Reissner-Nordström background, and the same conclusions follow (Demers, Lafrance, Myers [1995a]). In the case of a non-minimally coupled scalar field, naively it appears that the divergences appearing in the statistical mechanical entropy will be different from those in the renormalization of Newton's constant. However, one obtains the correct renormalization after taking into account the appropriate degrees of freedom at the horizon (Demers, Lafrance, Myers [1995b]).

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