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An elegant solution of the n-body Toda problem

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Abstract

The solution of the classical open-chain n-body Toda problem is derived from an ansatz and is found to have a highly symmetric form. The proof requires an unusual identity involving Vandermonde determinants. The explicit transformation to action-angle variables is exhibited.

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The Toda chain is one of the paradigmatic examples of an integrable many-body system of interacting particles. The discovery of its conserved integrals of motion[1, 2] and its subsequent solution[3, 4, 5] were important steps in the development of the theory of integrable systems[6]. An almost universal feature of analytical studies of the Toda system is the use of the Lax pair formalism. In this paper, an alternative derivation of the solution of the classical open-chain n-body Toda system is given.

The derivation proceeds essentially from an ansatz about the form of the solution and therefore lacks the power and generality of the Lax pair treatment. The solution however has an elegant structure which is not evident in previous representations. More, it can be interpreted as the classical canonical transformation from the Toda system to a free theory. This is an important clue to constructing the classical and quantum solutions by a sequence of elementary canonical transformations[7]. Following the successful solution of the three-body Toda problem with this approach[8], work is in progress on the classical and quantum open-chain n-body problems.

The Hamiltonian for the (n+1)-body open chain Toda system is

$$H = \frac{1}{2} \sum_{k=1}^{n+1} p_k^2 + \sum_{k=1}^{n} e^{q_k - q_{k+1}}. \tag{1}$$

The arguments of the exponential potentials can be interpreted as expressions for the root vectors of A_n in the Cartan basis[5]. A coordinate transformation will put the root vectors into the Chevalley basis and separate out the motion of the center of mass. The transformation is given by

$$\begin{aligned}
q_1 &\mapsto q_1 + \frac{q_{n+1}}{n+1}, \\
q_k &\mapsto -q_{k-1} + q_k + \frac{q_{n+1}}{n+1}, \qquad (2 \le k \le n) \\
q_{n+1} &\mapsto -q_n + \frac{q_{n+1}}{n+1}, \\
p_k &\mapsto \frac{1}{n+1}\Big(\sum_{j=k}^{n}(n+1-j)p_j - \sum_{j=1}^{k-1} jp_j\Big) + p_{n+1}, \qquad (1 \le k \le n). \\
p_{n+1} &\mapsto \frac{-1}{n+1}\Big(\sum_{j=1}^{k-1} jp_j\Big) + p_{n+1}.
\end{aligned} \tag{2}$$

The transformed Hamiltonian is

$$\begin{aligned}
H^a &= \frac{1}{2(n+1)}\Big(\sum_{k=1}^{n} k(n+1-k)p_k^2 + \sum_{k=2}^{n}\sum_{j=1}^{k-1} 2j(n+1-k)p_jp_k\Big) + \frac{n+1}{2}p_{n+1}^2 \\
&\quad + e^{2q_1-q_2} + \sum_{k=2}^{n-1} e^{2q_k-q_{k-1}-q_{k+1}} + e^{2q_n-q_{n-1}}.
\end{aligned} \tag{3}$$

This leads to the equations of motion

$$\begin{aligned}
\ddot{q}_1 &= -e^{2q_1-q_2} \\
\ddot{q}_k &= -e^{2q_k-q_{k-1}-q_{k+1}} \qquad (2 \le k \le n-1) \\
\ddot{q}_n &= -e^{2q_n-q_{n-1}}.
\end{aligned} \tag{4}$$

The solution of these equations has the remarkably simple form

$$e^{-q_m} = \sum_{j_1<\cdots<j_m}^{n+1} f_{j_1}\cdots f_{j_m}\Delta^2(j_1,\ldots,j_m)e^{(\mu_{j_1}+\cdots+\mu_{j_m})t}, \tag{5}$$

where $\Delta^2(j_1,\ldots,j_m)$ is the square of the Vandermonde determinant

$$\Delta^2(j_1,\ldots,j_m) = \prod_{j_i<j_k}(\mu_{j_i}-\mu_{j_k})^2, \tag{6}$$

and f_k and μ_k are arbitrary constants, satisfying

$$\begin{aligned} \prod_{k=1}^{n+1} f_k &= \Delta^{-2}(1,\ldots,n+1), \\ \sum_{k=1}^{n+1} \mu_k &= 0. \end{aligned} \tag{7}$$

(There are additional constraints on the range of the f_k if one requires the q_m be real.) The solution has $2n$ free parameters as required. The solution in the original variables is determined from the transformation (2) to be composed of ratios of these solutions times a factor for the center of mass motion.

To derive the solution, make the ansatz

$$e^{-q_m} = \sum_{j_1<\cdots<j_m}^{n+1} f_{j_1\cdots j_m} e^{(\mu_{j_1}+\cdots+\mu_{j_m})t}, \tag{8}$$

where the μ_k are arbitrary real numbers. Note that this ansatz defines a variable

$$e^{-q_{n+1}} = f_{1\cdots n+1} e^{(\mu_1+\cdots+\mu_{n+1})t}. \tag{9}$$

Such a variable might naturally appear in the final equation of (4) to give $\ddot{q}_n = -e^{2q_n-q_{n-1}-q_{n+1}}$. That this variable does not appear can be interpreted as meaning that one has set $q_{n+1}=0$. This ultimately is the origin of the restrictions (7) on the f_k and μ_k. The equation for $\ddot{q}_1$ is also of the form of the others if there is a $q_0=0$. The open-chain Toda system thus has fixed endpoints in this sense. The ansatz and solution are compatible with the slightly more general problem where $e^{-q_{n+1}} = ce^{\kappa t}$. Then, in the solution, one would have $\prod_{k=1}^{n+1} f_k = c\Delta^{-2}(1,\ldots,n+1)$ and $\sum_{k=1}^{n+1}\mu_k = \kappa$.

Consider e^{-q_m}. Differentiating twice and multiplying by e^{-q_m} leads to

$$-\ddot{q}_m e^{-2q_m} = e^{-q_m}\partial_t^2 e^{-q_m} - (\partial_t e^{-q_m})^2. \tag{10}$$

But from the equations of motion $-\ddot{q}_m e^{-2q_m} = e^{-q_{m-1}-q_{m+1}}$ (using $q_0 = 0 = q_{n+1}$). Substituting the ansatz into the resulting equation gives $(2 \le m \le n-1)$

$$\sum_{\substack{j_1<\cdots<j_m \\ k_1<\cdots<k_m \\ j_1<k_1}}^{n+1} f_{j_1\cdots j_m} f_{k_1\cdots k_m} (\sum_{i=1}^{m}\mu_{j_i} - \sum_{i=1}^{m}\mu_{k_i})^2 e^{(\sum_{i=1}^{m}\mu_{j_i}+\sum_{i=1}^{m}\mu_{k_i})t} = \tag{11}$$

$$= \sum_{\substack{j_1<\cdots<j_{m-1} \\ k_1<\cdots<k_{m+1}}}^{n+1} f_{j_1\cdots j_{m-1}} f_{k_1\cdots k_{m+1}} e^{(\sum_{i=1}^{m-1}\mu_{j_i}+\sum_{i=1}^{m+1}\mu_{k_i})t}.$$

The equation for $m = 1$ is

$$\sum_{j_1<j_2}^{n+1} f_{j_1} f_{j_2} (\mu_{j_1} - \mu_{j_2})^2 e^{(\mu_{j_1}+\mu_{j_2})t} = \sum_{j_1<j_2}^{n+1} f_{j_1 j_2} e^{(\mu_{j_1}+\mu_{j_2})t}. \tag{12}$$

The equation for $m = n$ involves $f_{j_1\cdots j_n}$ where $1 \le j_1 < \cdots < j_n \le n+1$. As one is choosing n integers out of $n+1$, this is more succinctly labelled by $f_{\hat{r}}$ where r is the integer which is not in the set. Similarly $f_{\widehat{rs}}$ means the two integers $r \neq s$ do not appear, and the indices of f are the remaining $n-1$ integers. With this notation, the equation for $m = n$ is

$$\sum_{r<s}^{n+1} f_{\hat{r}} f_{\hat{s}} (-\mu_r + \mu_s)^2 e^{(-\mu_r-\mu_s+2\sum_{k=1}^{n+1}\mu_k)t} = \sum_{r<s}^{n+1} f_{\widehat{rs}} e^{(-\mu_r-\mu_s+\sum_{k=1}^{n+1}\mu_k)t}. \tag{13}$$

Assume the μ_k are all distinct and that they have no accidental degeneracies in their linear combinations, such as $\mu_{j_1} + \mu_{j_2} = \mu_{j_3} + \mu_{j_4}$. The asymptotic behavior of the exponentials can be used to equate like terms in the sums. The degenerate cases can be recovered later by continuity in the μ_k. Let

$$f_{j_1 \cdots j_m} = f_{j_1} \cdots f_{j_m} \Delta^2(j_1, \ldots, j_m), \tag{14}$$

where the f_{j_k} are (so far) arbitrary constants and $\Delta^2(j_1, \ldots, j_m)$ is the square of the Vandermonde determinant (6). With this definition, the $m = 1$ equation (12) is easily verified. The $m = n$ equation (13) is satisfied if the constraints (7) on the f_k and μ_k are imposed. The proof that Eq. (11) is satisfied reduces to a heirarchy of identities for Vandermonde determinants.

On the left hand side of (11), there are two sets of indices $\{j_\alpha\}$ and $\{k_\beta\}$. Since $j_1 < k_1$, at most they can have $m - 1$ indices in common. The asymptotic behavior of the exponential is given by a sum over the μ_i indexed by the combined set $S = \{j_\alpha, k_\beta\}$. Different partitions of S into sets $\{j_\alpha\}$ and $\{k_\beta\}$ $(j_1 < k_1)$ will have the same asymptotic behavior. The number of such terms will depend on the number of distinct indices between the two sets, and these constitute separate cases. Let $2r$ denote the number of distinct indices.

Consider the case $r = 1$, labelling the common indices $s_1, \ldots, s_{m-1}$ and the distinct ones j_1 and k $(j_1 < k)$. There is a unique term on both sides of (11) with the asymptotic behavior given by this set of indices, and one has

$$f_{j_1 s_1 \cdots s_{m-1}} f_{\{k s_1 \cdots s_{m-1}\}} (\mu_{j_1} - \mu_k)^2 = f_{s_1 \cdots s_{m-1}} f_{\{j_1 k s_1 \cdots s_{m-1}\}}, \tag{15}$$

where the curly brackets indicate the indices should be arranged in increasing order. Using (14), the constant factors f_i cancel and one has a relation between

Vandermonde determinants

$$\Delta^2(j_1, s_1, \ldots, s_{m-1})\Delta^2(k, s_1, \ldots, s_{m-1})(\mu_{j_1} - \mu_k)^2 = \tag{16}$$
$$= \Delta^2(s_1, \ldots, s_{m-1})\Delta^2(j_1, k, s_1, \ldots, s_{m-1}).$$

Using the relation

$$\Delta^2(k, s_1, \ldots, s_{m-1}) = \Delta^2(s_1, \ldots, s_{m-1}) \prod_{i=1}^{m-1} (\mu_k - \mu_{s_i})^2 \tag{17}$$

and its relatives, the dependence on the common indices is seen to cancel and one is left with the identity

$$(\mu_{j_1} - \mu_k)^2 = \Delta^2(j_1, k). \tag{18}$$

It is a general feature for all r that the dependence on the constant factors and the common indices cancels on both sides, so without loss of generality one can focus on the distinct indices alone. Reindex the set S of distinct indices by the integers 1 to $2r$. Partition S into two sets $\alpha = \{1, \alpha_2, \ldots, \alpha_r\}$ and $\beta = \{\beta_1, \ldots, \beta_r\}$ and denote the collection of such partitions $P_{\alpha\beta}$. Separately partition S into sets $\gamma = \{\gamma_1, \ldots, \gamma_{r-1}\}$ and $\delta = \{\delta_1, \ldots, \delta_{r+1}\}$, calling the collection of partitions $P_{\gamma,\delta}$. Denote $\Delta^2(\alpha; r) = \Delta^2(1, \alpha_2, \ldots, \alpha_r)$ and similarly for the rest. The number r of indices involved in the Vandermonde determinant is made explicit to reduce confusion. Both sides of Eq. (11) will be equal if the following identity between Vandermonde determinants holds

$$\sum_{P_{\alpha\beta}} \Delta^2(\alpha; r)\Delta^2(\beta; r)(\sum_\alpha \mu_\alpha - \sum_\beta \mu_\beta)^2 = \sum_{P_{\gamma\delta}} \Delta^2(\gamma; r-1)\Delta^2(\delta; r+1). \tag{19}$$

It seems likely that this identity has a group theoretical interpretation, but in its absence, the identity can be proved inductively as follows[9]. Divide both sides by $\Delta^2(S;2r)$. This gives the equation

$$\sum_{P_{\alpha\beta}} \frac{(\sum_\alpha \mu_\alpha - \sum_\beta \mu_\beta)^2}{\prod_{\alpha,\beta}(\mu_\alpha - \mu_\beta)^2} = \sum_{P_{\gamma\delta}} \frac{1}{\prod_{\gamma,\delta}(\mu_\gamma - \mu_\delta)^2}. \tag{20}$$

Denote the left hand side of the equation by L_r and the right hand side by R_r. The equation $L_1 = R_1$ holds trivially. Assume that $L_{r-1} = R_{r-1}$. The inductive step will be made by considering the pole structure of L_r and R_r. Since L_r and R_r are analytic functions of the μ_i without zeroes, if they can be shown to have the same residue at all of their poles, they must be equal.

Choose two indices from the set S, neither equal to 1, and let their associated μ_i be labelled z and a. (The index 1 is special because it has a preferred role in the partitioning. Which of the original μ_i is associated to the index 1 is however arbitrary, so one can investigate the pole structure at the μ_i missed here by reindexing the set S.) Let S' denote the set S with these two indices removed, and let $\alpha', \beta', \gamma', \delta'$ denote partitions of S' as defined above with r replaced by $r-1$.

Consider the residue of L_r at $z = a$. L_r has a double pole at $z = a$ if $z \in \alpha$ and $a \in \beta$ or *vice versa.* In the former case, the residue is computed to be

$$\operatorname*{Res}_{z=a} L_r \Big|_{z\in\alpha, a\in\beta} = \tag{21}$$
$$= \frac{2}{\prod_{S'}(a-\mu_{S'})^2} \sum_{P_{\alpha'\beta'}} \left(\frac{1}{\sum_{\alpha'} \mu_{\alpha'} - \sum_{\beta'} \mu_{\beta'}} - \sum_{\beta'} \frac{1}{a - \mu_{\beta'}} \right) \frac{(\sum_{\alpha'} \mu_{\alpha'} - \sum_{\beta'} \mu_{\beta'})^2}{\prod_{\alpha',\beta'}(\mu_{\alpha'} - \mu_{\beta'})^2}.$$

In the alternative case $z \in \beta$, $a \in \alpha$, the residue is

$$\operatorname*{Res}_{z=a} L_r \Big|_{z\in\beta, a\in\alpha} = \tag{22}$$

$$= \frac{2}{\prod_{S'}(a-\mu_{S'})^2} \sum_{P_{\alpha'\beta'}} \Big(- \frac{1}{\sum_{\alpha'}\mu_{\alpha'} - \sum_{\beta'}\mu_{\beta'}} - \sum_{\alpha'} \frac{1}{a-\mu_{\alpha'}} \Big) \frac{(\sum_{\alpha'}\mu_{\alpha'} - \sum_{\beta'}\mu_{\beta'})^2}{\prod_{\alpha',\beta'}(\mu_{\alpha'}-\mu_{\beta'})^2}.$$

Adding these, the residue of L_r at $z = a$ is

$$\operatorname*{Res}_{z=a} L_r = -\frac{2L_{r-1}}{\prod_{S'}(a-\mu_{S'})^2} \sum_{S'} \frac{1}{a-\mu_{S'}}. \tag{23}$$

The residue at $z = a$ of R_r is similarly composed of terms where $z \in \gamma$, $a \in \delta$ and *vice versa.* The full residue is

$$\operatorname*{Res}_{z=a} R_r = -\frac{2R_{r-1}}{\prod_{S'}(a-\mu_{S'})^2} \sum_{S'} \frac{1}{a-\mu_{S'}}. \tag{24}$$

This is seen to equal the residue of L_r at $z = a$, given $L_{r-1} = R_{r-1}$. Since this result holds for all pairs of the original μ_i, one concludes that $L_r = R_r$ and the induction is complete.

To exhibit the solution (5) as a canonical transformation from (3) to a Hamiltonian independent of coordinates, one must introduce final coordinates and momenta and find a relation between them and the f_j and μ_k so that the transformation is canonical. It is clear that one can redefine f_j by an overall constant,

$$f_j = e^{\bar{x}_j} \tilde{f}_j. \tag{25}$$

The arguments of the exponentials then define the final coordinates

$$x_j = \mu_j t + \bar{x}_j. \tag{26}$$

There should only be n independent degrees of freedom, and the coordinate $x_{n+1} = -\sum_{i=1}^{n} x_i$ is not independent because it is related to the others by the constraints (7). It is useful however to introduce a temporary form of the final Hamiltonian

$$\tilde{H} = \frac{1}{2}\sum_{i=1}^{n+1}\mu_i^2. \tag{27}$$

The μ_j are not the momenta conjugate to x_j because if the the constraint $\mu_{n+1} = -\sum_{i=1}^{n}\mu_i$ were eliminated, the wrong $\dot{x}_j$ would follow from Hamilton's equations. It is necessary to introduce n momenta k_j conjugate to the x_j, so that $\dot{x}_j = \frac{\partial\tilde{H}}{\partial k_j} = \mu_j$. The relation between k_j and μ_j is found to be

$$k_j = \mu_j - \mu_{n+1} = \mu_j + \sum_{i=1}^{n}\mu_i \tag{28}$$

or in reverse ($j \neq n+1$)

$$\begin{aligned} \mu_j &= k_j - \frac{1}{n+1}\sum_{i=1}^{n} k_i, \\ \mu_{n+1} &= -\frac{1}{n+1}\sum_{i=1}^{n} k_i \end{aligned} \tag{29}$$

The final Hamiltonian is then

$$\tilde{H} = \frac{n}{2(n+1)}\sum_{j=1}^{n} k_j^2 - \frac{1}{n+1}\sum_{i<j} k_i k_j. \tag{30}$$

The next step is to find an equation for the evolution of the original momenta. This is easily done by taking a time derivative of the solution (5)

$$e^{-q_m} = \sum_{j_1<\cdots<j_m}^{n+1} \tilde{f}_{j_1}\cdots\tilde{f}_{j_m}\Delta^2(j_1,\ldots,j_m)e^{x_{j_1}+\cdots+x_{j_m}} \tag{31}$$

to find

$$-\dot{q}_m e^{-q_m} = \sum_{j_1<\cdots<j_m}^{n+1} \tilde{f}_{j_1}\cdots\tilde{f}_{j_m}\Delta^2(j_1,\ldots,j_m)(\mu_{j_1}+\cdots+\mu_{j_m})e^{x_{j_1}+\cdots+x_{j_m}}. \quad (32)$$

Using Hamilton's equations with the Hamiltonian (3), one can express $\dot{q}_m$ in terms of the momenta as

$$\dot{q}_m = \frac{1}{(n+1)}[m(n+1-m)p_m + \sum_{i=1}^{m-1} i(n+1-m)p_i + \sum_{i=m+1}^{n} m(n+1-i)p_i], \quad (33)$$

The result is

$$\frac{-1}{(n+1)}[m(n+1-m)p_m + \sum_{i=1}^{m-1} i(n+1-m)p_i + \sum_{i=m+1}^{n} m(n+1-i)p_i]e^{-q_m} =$$
$$= \sum_{j_1<\cdots<j_m}^{n+1} \tilde{f}_{j_1}\cdots\tilde{f}_{j_m}\Delta^2(j_1,\ldots,j_m)(\mu_{j_1}+\cdots+\mu_{j_m})e^{x_{j_1}+\cdots+x_{j_m}}. \quad (34)$$

Finally, by requiring that the Poisson brackets be preserved under the transformation, one can determine the $\tilde{f}_j$ in terms of the k_i's. The result is that $(j \neq n+1)$

$$\tilde{f}_j = (-1)^{j-1}k_j^{-1}\prod_{i\neq j}^{n}(k_j-k_i)^{-1}, \quad (35)$$
$$\tilde{f}_{n+1} = \prod_{i=1}^{n} k_i^{-1}$$

One confirms that the f_j satisfy the constraint (7). (Note that the maximal symmetry is evident in terms of the μ_i's since $k_j = \mu_j - \mu_{n+1}$ and $k_j - k_i = \mu_j - \mu_i$.) The proof that this is the correct form for the f_j follows by constructing the Poisson brackets and collecting like exponentials. Conditions are quickly found that the f_j must be particular products of differences between momenta. It is then seen that there are no additional requirements.

Using (29) and (35) in (31) and (34) gives the explicit canonical transformation between the open-chain n-body Toda Hamiltonian in the Chevalley basis (3) and a Hamiltonian (30) which is independent of coordinates. The reduction to action-angle variables is essentially complete. From this point, one can attempt to construct a product of elementary canonical transformations which produces this full transformation. This has been done for the 3-body system [8] and work is in progress on the n-body system. The value of such a product is that, when it is found in the quantum system, it allows the construction of integral representations of the eigenfunctions of the system.

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References

[1] M. Toda, J. Phys. Soc. Japan **22**, 431 (1975); **23**, 501 (1967).

[2] M. Henon, Phys. Rev. **B 9**, 1921 (1974); H. Flaschka, Phys. Rev. **B 9**, 1924 (1974).

[3] M. Kac and van Moerbeke, Proc. Nat. Acad. Sci. (USA) **72**, 1627, 2879 (1975).

[4] J. Moser (ed.), in Dynamical Systems, Theory and Application, Lecture Notes in Physics, vol. 38 (Springer: Berlin, 1975), p. 467; Adv. Math. **16**, 197 (1975).

[5] R. Farwell and M. Minami, Prog. Theo. Phys. **69**, 1091 (1983).

[6] M. Toda, *Nonlinear Waves and Solitons*, (Kluwer Academic Publishing: Dordrecht, 1983); *Theory of Nonlinear Waves, 2nd enlarged ed.*, (Springer-Verlag: Berlin, 1989, 1981).

[7] A. Anderson, Ann. Phys. (NY) **232**, 292 (1994).

[8] A. Anderson, B. Nilsson, C. Pope and K.S. Stelle, Nucl. Phys. B**430**, 107 (1994).

[9] The method of proof was suggested by P. Goddard.