ON THE SUTHERLAND'S INTEGRABILITY CONDITION FOR TWO-DIMENSIONAL N-PARTICLE SYSTEMS.

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Abstract

Following Sutherland's work on one-dimensional integrable systems we formulate and study its two-dimensional version.Physically it expresses the absence of true 3-body forces among an assembly of N particles leaving exclusively effective 2-body interactions.This criterion may be a suitable candidate for an integrability condition.

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1 Introduction

In the last twenty years, the quantum N-body problem has been chiefly studied in one space dimension with great success thanks to the discovery of integrable systems. Among those, one can count systems soluble with the Bethe ansatz wave function as well as others systems discovered originally by Calogero and Sutherland at the beginning of the 70's. The last topic has evolved considerably and connections to many algebraic structure have been established [1].

One main ingredient to the solubility is the fact that the dynamics of the N-particle system is only due to pair potential and eventually external applied fields. This abscence of other type of N-body forces renders the problem completly soluble whether it is relativistic or not.

Few breakthroughs have been made in 2 or 3 space dimensions. Due to more "room" available, particles can go around each other instead of being confined on the line where they must scatter inevitably. The resulting effect is that it is difficult to give a concept of "integrable" systems. Yet Calogero and Machioro [5] have discovered that if one includes some type of 3-body forces which are highly dependent on direction, one may obtain exactly soluble multiparticle dynamics in three space dimensions.

The interest in recent years in two-dimensional quantum systems stems from the discovery of the fractional quantum Hall effect which indirectly revived an older work of Leinass and Myrheim [2] on quantum theory of many-particles in two-dimensions as well as quantum dots and electronic plasmas [19]. At the center of this topic are new objects, the anyons which seems to be responsible for some physical phenomenona (such as fractional quantum Hall effect)[15] or realizations of "exotic" statistics which are not excluded in two-dimensions. Hence different many-particle wavefunctions have been proposed in this spirit such as the known Laughlin wavefunction for the fractional quantum Hall effect [16, 17].

In this paper we shall not discuss the physical phenomena, but instead pose the following problem: under which condition, in two-dimensions can one have only pair interactions? And more generally when does one obtain an integrable system? We shall see then that the emergence of fractional statistics appears to be natural and consistent with the existence of pair-potentials among the N-particles, only for a specific class of pair-potentials.

Let us first recall that in non-relativistic quantum mechanics of a single particle the wavefunction depends on two real variables x and y i.e. $\psi(x, y)$. One may equivalently use the complex combinations:

$$z = x + iy \qquad \text{and} \qquad \overline{z} = x - iy \tag{1}$$

and consider instead $\psi(z, \overline{z})$; this last wave function is in fact a restriction of a fonction of two complex variables $\psi(z, z')$, such that $z' = \overline{z}$. To account for manifest correlations among particles of the systems it is natural to postulate the Bijl-Dingle-Jastrow wavefunction [8, 18] which is generally proposed in one-dimension to describe the ground state of a system of N particles.

$$\Psi = \Psi(z_1, \overline{z}_1; \ldots; z_N, \overline{z}_N) = \prod_{i<j} \psi(z_{ij}, \overline{z}_{ij}) \tag{2}$$

where $z_{ij} = z_i - z_j$. Excited states may be constructed from the ground state using a standard method [9].

In this wavefunction, the order of pairs of particles is single out by the pair wavefunction $\psi(z_{ij}, \overline{z}_{ij})$. This is a general feature in all soluble N-particle systems in one-dimension, and becomes thus a valuable starting point for the study of two-dimensional systems.

In section I, we review the situation in one-dimension to establish the grounds for such procedure. There, the central object is the Sutherland's condition for the solubility of the problem. This condition merely states that the 3-body potential arising from a state described by the Bijl-Dingle-Jastrow wavefunction, may be recast into a sum of pair-potentials so that the whole system behaves pratically under pure effective pair potentials. This is our main argument and this may be related to what is known in integrable quantum field theory in one-space dimension. In fact when the S-matrix of such a quantum field theory is factorizable or can be written as product of S-matrices for pairs of particles, subjected to the usual conditions of unitarity and analyticity, does one have an infinite number of concerved quantities in the theory. What is more interesting is that A.B. Zamolodchikov and Al.B. Zamolodchikov [4] have shown that in the non-relativistic limit, these pair S-matrices reproduce the scattering phase shifts of the soluble pair potentials in one-dimension. It is then tempting to identify the Sutherland's condition as the integrability condition in one-dimension. In fact this is really so because such a condition contains all the known soluble pair potentials.

Our objective is thus to explore the Sutherland's condition in two-dimensions. In section II, we shall derive the condition of Sutherland in two dimensions and study its properties in details. As we shall see, many new aspects emerge as compared to those in one-dimension: the inclusion of fractional statistics, the nature of the effective pair-potentials, and the triviality of the attractive harmonic potential in two-dimensions.

In section III, we shall raise the question whether there exists a repulsive interaction among pairs which might insure stability against the collapse due to the harmonic attractive potential . In fact we shall present a two-dimensional version of the Sutherland elliptic potential as a concrete example. The model has the merit of providing an example of Wigner solid in two dimensions. In the scaling limit of this potential we shall see how the fact that particles can never "over take" each other in two-dimensions is the key to the solubility of the problem. In fact the particles are locked inside two-dimensional rectangular cells and remain "impenetrable".

We conclude by presenting some ideas on how one may generalize the previous ideas to find non-trivial integrable systems in two-dimensions as well as to seek limits to construct integrable relativistic quantum field theories in two-space dimensions.

2 The one-dimensionnal case revisited

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This has been worked out in the seventies by B.Sutherland [8, 9] and F.Calogero [12, 13]. The N-particule wavefunction is assumed to be of the Dingle-Bijl-Jastrow form:

$$\Psi = \prod_{i<j} |\psi(x_i - x_j)|^\lambda \tag{3}$$

where $\psi(x)$ is the pair wavefunction and λ a real constant.

Introducing the associated functions ϕ and φ by:

$$\psi = \exp\phi \qquad\qquad \varphi = \frac{d\phi}{dx} \tag{4}$$

and applying the kinetic energy operator on Ψ, one obtains:

$$\begin{aligned} -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}\Psi = \{ & -2\lambda \sum_{j<i} [\varphi'(x_i - x_j) + \lambda\varphi^2(x_i - x_j)] \\ & +2\lambda^2 \textstyle\sum_{i,j,k} [\varphi(x_k - x_i)\varphi(x_i - x_j) + \varphi(x_i - x_j)\varphi(x_j - x_k) \\ & +\varphi(x_j - x_k)\varphi(x_k - x_i)]\}\Psi. \end{aligned} \tag{5}$$

The first term of the r.h.s. of (5) represents the pair potentiel between pairs of particles due to the choice of $\psi(x)$. the second term is an induced 3-body potential beetwen any triplet i,j,k of the N-particles.

Sutherland proposed then to choose ψ such that this 3-body potential breaks up into additional pair potentials, namely that:

$$\varphi(x)\varphi(y) + \varphi(y)\varphi(z) + \varphi(z)\varphi(x) = f(x) + f(y) + f(z) \tag{6}$$

for
$$x + y + z = 0$$

The form of $f(x)$ being essentially due to the choice of $\psi(x)$. If such a choice is made, then there exists among the N-particles only an effective pair potential:

$$V(x) = \lambda[\varphi'(x) + \lambda\varphi^2(x) - \lambda f(x)]. \tag{7}$$

The philosophy of this statement is analoguous to the one adopted in the theory of integrable quantized fields in 1+1-dimensions. There it is stated that the N-body S-matrices are factorized into two-body S-matrices. In fact Zamolodchikov and Zamolodchikov [4] have shown that in one of these theories, the non-relativistic limit of such 2-body S-matrix is precisely the phase-shift in a relative pair potential of the type (7). In this sense, one may say that the Sutherland's condition (6), is in fact an integrability condition for an N-body problem in one-dimension.

To get more insight we may advantageously replaced the condition (6) by the following one, using elementary algebra:

$$(\varphi(x) + \varphi(y) + \varphi(z))^2 = g(x) + g(y) + g(z) \tag{8}$$

with $\quad x + y + z = 0 \quad$ and $\quad g(x) = \varphi^2(x) + 2f(x).$

The form of (8) turns out to be exactly a relation satisfied by the Weierstrassian elliptic functions $\zeta(x)$ and P(x) namely:

$$(\zeta(x) + \zeta(y) + \zeta(z))^2 = P(x) + P(y) + P(z) \tag{9}$$

with $\quad x + y + z = 0$

Sutherland who discovered this connection identified then $\psi(x)$ to $\sigma(x)$, the Weierstrassian σ-function . The periodicity of $\sigma(x)$ is instrumental in exhibiting an example of Wigner solid in one-dimension [6, 7].

Some remarks on the properties of $\varphi(x)$ are now in order:

a) $\varphi(x)$ is in fact defined up to a linear term: the substitution $\varphi(x) \rightarrow \varphi(x) + ax + b$ leaves relation (6) invariant.The linear term $ax + b$ induces a pair potential:

$$V(x) = \lambda\{a + \frac{\lambda}{2}(ax + b)^2 - \lambda x\} \tag{10}$$

which is essentially a shifted harmonic oscillator potential. In this sense, the harmonic oscillator pair potential is simply a trivial one.

b) As shown by Sutherland [8], particular limits of the $\zeta(x)$ potential reproduce all the non-singular pair-potential known in one-dimension.

c) We note also that the pair δ-function is also contained in (6) which in this case the r.h.s of [7] is simply constant and the Ψ take up the form of a Bethe ansatz wavefunction.

3 Two-dimensional case

As pointed out before, the pair wavefunction is in fact a restricted wavefunction of 2 complex variables $\psi(z, z')$ with $z' = \overline{z}$. In analogy to section 2 we shall introduce the notations:

$$\begin{array}{lclclcl} \psi & = & \exp \phi & \qquad & \phi & = & \phi(z, \overline{z}) \\ \varphi & = & \dfrac{d\phi}{dz} & & \overline{\varphi} & = & \dfrac{d\phi}{d\overline{z}} \end{array} \tag{11}$$

The application of the kinetic energy operator, assuming that particles are of unit mass.

$$\sum_{i=1}^{N} \frac{\partial^2}{\partial \overline{z}_j \partial z_i} = -\frac{1}{2} \sum_{i=1}^{N} (\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2})$$

on a N-particle Dingle-Bijl-Jastrow wavefunction yields a sum of pair-potentials and an induced 3-body potential as in one-dimension. Therefore following Sutherland we are led to the generalized condition:

$$\begin{array}{lcl} \{ \ \varphi(x, \overline{x})\overline{\varphi}(y, \overline{y}) & + & \varphi(y, \overline{y})\overline{\varphi}(x, \overline{x}) \\ + \ \varphi(y, \overline{y})\overline{\varphi}(z, \overline{z}) & + & \varphi(z, \overline{z})\overline{\varphi}(y, \overline{y}) \\ + \ \varphi(z, \overline{z})\overline{\varphi}(x, \overline{x}) & + & \varphi(x, \overline{x})\overline{\varphi}(z.\overline{z}) \ \ \} \\ & & = [f(x, \overline{x}) + f(y, \overline{y}) + f(z, \overline{z})] \end{array} \tag{12}$$

with:

$$x + y + z = 0 \qquad \text{and} \qquad \overline{x} + \overline{y} + \overline{z} = 0$$

which evidently states that the 3-body potential between any triplet of particles will be break up into a sum of 2-body potentials.

Unfortunatly, there is up to now no theory of factorized S-matrices in (2+1) dimensions. Yet, we may call this condition an integrability condition if non-trivial interesting solutions can be found. In the sequel we shall seek to construct some solutions based on experience in one-dimension.

But before doing so let us mention that an alternative way for formulating (12) would be, in analogy to eq (8)

$$\begin{array}{r} \{\varphi(x, \overline{x}) + \varphi(y, \overline{y}) + \varphi(z, \overline{z})\}\{\overline{\varphi}(x, \overline{x}) + \overline{\varphi}(y, \overline{y}) + \overline{\varphi}(z, \overline{z})\} \\ = g(x, \overline{x}) + g(y, \overline{y}) + g(z, \overline{z}) \end{array} \tag{13}$$

whenever:

$$x+y+z=0 \qquad \text{and} \qquad \overline{x}+\overline{y}+\overline{z}=0$$

Under this form, eq.(13) remains obviously invariant under the double substitution:

$$\varphi(x,\overline{x}) \rightarrow \varphi(x,\overline{x}) + ax + a'\overline{x} + b \tag{14}$$
$$\overline{\varphi}(x,\overline{x}) \rightarrow \overline{\varphi}(x,\overline{x}) + \overline{a'}x + \overline{a}\overline{x} + \overline{b} \tag{15}$$

Again this linear part is responsible for the shifted harmonic oscillator pair-potential between particles. Thus such a potential is of trivial nature and has tendency to cause a collapse to the center of mass of the system of N-particles. Morever,

$$\phi \rightarrow \phi + a'z\overline{z} + \frac{1}{2}(az^2 + \overline{a}\overline{z}^2) + bz + \overline{b}\overline{z} + \text{const.}$$

and the new potential is:

$$V \rightarrow (V + a') + \varphi(x,\overline{x}) + \overline{\varphi}(x,\overline{x}) + (b+\overline{b}). \tag{16}$$

This transfomation may be used to generate new pair potential from a known one.The two-dimensional effective pair-potential in general has the form:

$$V = \lambda(\lambda-1)\frac{\partial\phi}{\partial z}\frac{\partial\phi}{\partial\overline{z}} + \lambda^2\frac{\partial^2\phi}{\partial z\partial\overline{z}} - \frac{\lambda^2}{2}f(z,\overline{z})$$

We are now in a position to study some particular situations:

a) If ϕ is a solution of the Laplace operator

$$\frac{\partial^2}{\partial z\partial\overline{z}}\phi = 0$$

Then $\phi = f(z) + g(\overline{z})$, here f and g are independent functions not related to those of eq.(12) or (13). Sutherland's condition becomes

$$\{f'(x)+f'(y)+f'(z)\}\{g'(\overline{x})+g'(\overline{y})+g'(\overline{z})\} = h(x,\overline{x})+h(y,\overline{y})+h(z,\overline{z}) \tag{17}$$

for

$$x+y+z=0.$$

We note that if g' =const (or resp. f' =const) the condition (17) is automatically satisfied: $\psi = \exp f(z)$ or $\psi = \exp g(\overline{z})$ is a solution.

b) If $\phi = \phi(z\overline{z})$, function of the distance $z\overline{z}$, this represents a physically reasonable situation for which the Sutherland 's condition is:

$$\begin{aligned} \phi'(y\overline{y})\phi'(x\overline{x})(\overline{x}y + \overline{y}x) + \phi'(y\overline{y})\phi'(z\overline{z})(y\overline{z} + \overline{y}z) + \phi'(z\overline{z})\phi'(x\overline{x})(x\overline{z} + z\overline{x}) \\ = h(x,\overline{x}) + h(y,\overline{y}) + h(z,\overline{z}) \end{aligned} \tag{18}$$

Note that for ϕ' =const one recovers the harmonic oscillator pair-potential, which has been recently investigated by Mushkevich et al [11]. We observe that if the pair wavefunction is taken az $\psi(z,\overline{z}) \cong z^\alpha \exp(-z\overline{z})$ for example, the effective pair-potential is

$$V = (\frac{3}{2}\lambda^2 - \lambda)z\overline{z} - \lambda + \alpha(\frac{\lambda^2}{2} - \lambda)$$

Thus the anyonic factor z^α simply shifts V by a constant amount. In fact this is the only known instance where exotic statistics seem to be consistent with the Sutherland condition.

Since:

$$\overline{x}y + \overline{y}x = 2\vec{x}.\vec{y}$$

represents the scalar product of the vector $\vec{x}$ and $\vec{y}$ in the plane, the Sutherland's condition takes a new vector form as a scalar product:

$$[\phi'(|\vec{x}|)\vec{x} + \phi'(|\vec{y}|)\vec{y} + \phi'(|\vec{z}|)\vec{z}]^2 = h(\vec{x}) + h(\vec{y}) + h(\vec{z}) \tag{19}$$

with

$$\vec{x} + \vec{y} + \vec{z} = \vec{0}$$

This has the same structure as eq.(6). Calogero and Machioro [5] have treated in three-dimensions, in the same spirit, the problem of N-particles only with potential dependent on the interparticle distance. In other words the wave function is of the Dingle-Bijl-Jastrow type. However they kept the induced three-body potential and have not look at the possibility it may decompose into pair-potentials.

4 Two dimensional Sutherland's model

Inspired by the work of Sutherland we may extend his model in two-dimensions obtaining a Wigner solid in the plane. The wavefunction Ψ is taken as :

$$\Psi = c\prod_{i<j}\Theta_1(\frac{\Pi}{L}(x_i - x_j))\Theta_1(\frac{\Pi}{L'}(y_i - y_j)) \tag{20}$$

where Θ_1 is the Jacobian odd-theta function and L, L' are lengths in the x and y directions.

Applying the kinetic energy operator on the wavefunction Ψ yields :

$$\begin{aligned}\frac{1}{\Psi}(\sum_{i=1}^{N}(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2})\Psi) = &\sum_{i<j}[N\frac{\Theta_1''(\frac{\Pi}{L}(x_i - x_j);\frac{ir}{L})}{\Theta_1(\frac{\Pi}{L}(x_i - x_j);\frac{ir}{L})} + 2\zeta(\frac{L}{2})(N-2)\frac{1}{L}] \\ + &\sum_{i<j}[N\frac{\Theta_1''(\frac{\Pi}{L}(y_i - y_j);\frac{ir'}{L'})}{\Theta_1(\frac{\Pi}{L'}(y_i - y_j);\frac{ir'}{L'})} + 2\zeta(\frac{L'}{2})(N-2)\frac{1}{L'}]\end{aligned} \tag{21}$$

In this case the integrability condition (13) will appears as being derived by a combination of (9).In fact we have :

$$\Phi = \ln\Theta_1(\frac{\Pi}{L}x) + \ln\Theta_1(\frac{\Pi}{L'}x) \tag{22}$$

consequently

$$\varphi = \frac{d\Phi}{dz} = \frac{1}{2}\frac{d\Phi}{dx} + \frac{1}{2i}\frac{d\Phi}{dy} \tag{23}$$

or

$$\varphi = \frac{1}{2}(\zeta(x) - \zeta(w)\frac{x}{w}) - \frac{i}{2}(\zeta(y) - \zeta(w')\frac{y}{w'}) \tag{24}$$

where w and w' are the periods of the $\zeta(z)$-function of Weierstrass.

Similarly we obtain

$$\overline{\varphi}(z,\overline{z}) = \frac{1}{2}(\zeta(x) - \zeta(w)\frac{x}{w}) + \frac{i}{2}(\zeta(y) - \zeta(w')\frac{y}{w'}) \tag{25}$$

As observed before the parts in φ and $\overline{\varphi}$ linear in x, y and constants will not affect the integrability condition the l.h.s. of (13) which in fact becomes

$$\begin{array}{l} \{[\zeta(x_1)+\zeta(x_2)+\zeta(x_3)]-i[\zeta(y_1)+\zeta(y_2)+\zeta(y_3)]\} \\ \{[\zeta(x_1)+\zeta(x_2)+\zeta(x_3)]+i[\zeta(y_1)+\zeta(y_2)+\zeta(y_3)]\} \\ =[\zeta(x_1)+\zeta(x_2)+\zeta(x_3)]^2+[\zeta(y_1)+\zeta(y_2)+\zeta(y_3)]^2 \end{array} \tag{26}$$

But this is precisely, because of the identity [10]

$$[P(x_1)+P(x_2)+P(x_3)]+[P(y_1)+P(y_2)+P(y_3)]$$

if
$$x_1+x_2+x_3=0 \qquad y_1+y_2+y_3=0$$

One may thus says that the double Sutherland's model fulfills an integrability condition which is merely the sum of the separate integrability conditions in the x and the y directions.

To get a further insight on the integrability of this two dimensional problem one may consider one of its limiting case where the pair potential is essentially

$$V(\vec{x}-\vec{x'})=\frac{g}{(x-x')^2}+\frac{g}{(y-y')^2} \tag{27}$$

Here g is a coupling constant.

There one see that, relative to particle $\vec{x}$, particle $\vec{x'}$ remains constantly in one of the quadrants centred at point $\vec{x}$, and conversely, this is the two-dimensional form of the non- overtaking aspect of the dynamics a particle can never get out of the quadrant in which it is located with respect to its neighbor.(see fig.1)

Thus after a scattering the two particles will fly away from each other but continue to remain in these quadrant sectors for ever.

We can generalise this picture to an ensemble of N-particles and understand why these conservation laws are behind the integrability of the model.

Although the periodical aspect leads to a two dimensional Wigner crystal, the particle dynamics in the particular limit of the pair- potential is rather artificial due to the presence of "forbidden lines" parallel to the axis attached to each particle.

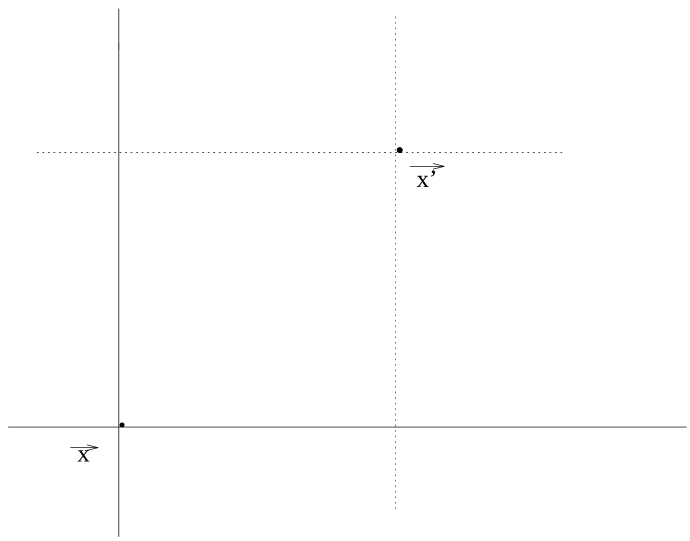

Figure 1: Particle $\vec{x}$ is in third quadrant of particle $\vec{x'}$ and particle $\vec{x'}$ is in the first quadrant of particle $\vec{x}$.

5 Conclusion and outlook.

In this article we have tried to find a generalized version of the condition found by Sutherland,for one dimensional integrable systems. Such systems admits for ground state wavefunction as an N-particle wavefunction of the Dingle-Bijl-Jastrow form.

The Sutherland's condition merely states that for such a system there exists no true three-body potential but only effective pair-potentials. Sutherland was able to find the most general solution in one dimension, it is a periodic one and its describes a Wigner crystal. In two dimensions the generalized Sutherland's condition does not admit obvious repulsive pair-potentials as in one dimension nor local δ-function pair-potentials. But it does not exclude also other type of pair-potentials which remain to be discovered. We have constructed a simple example of two-dimensional solution fulfilling the generalized Sutherland's condition, this study seems to suggest that a more general solution should be given by a 2-variable theta functions depending on three modulu. Work is in progres in this direction. It is expected that special limits would lead to new form of soluble pair-potentials in two-dimensions.

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